Almost-Optimal Sublinear Additive Spanners

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Graph Sparsification

Given an **input graph *G***, find a **smaller graph *H*** with similar shortest paths

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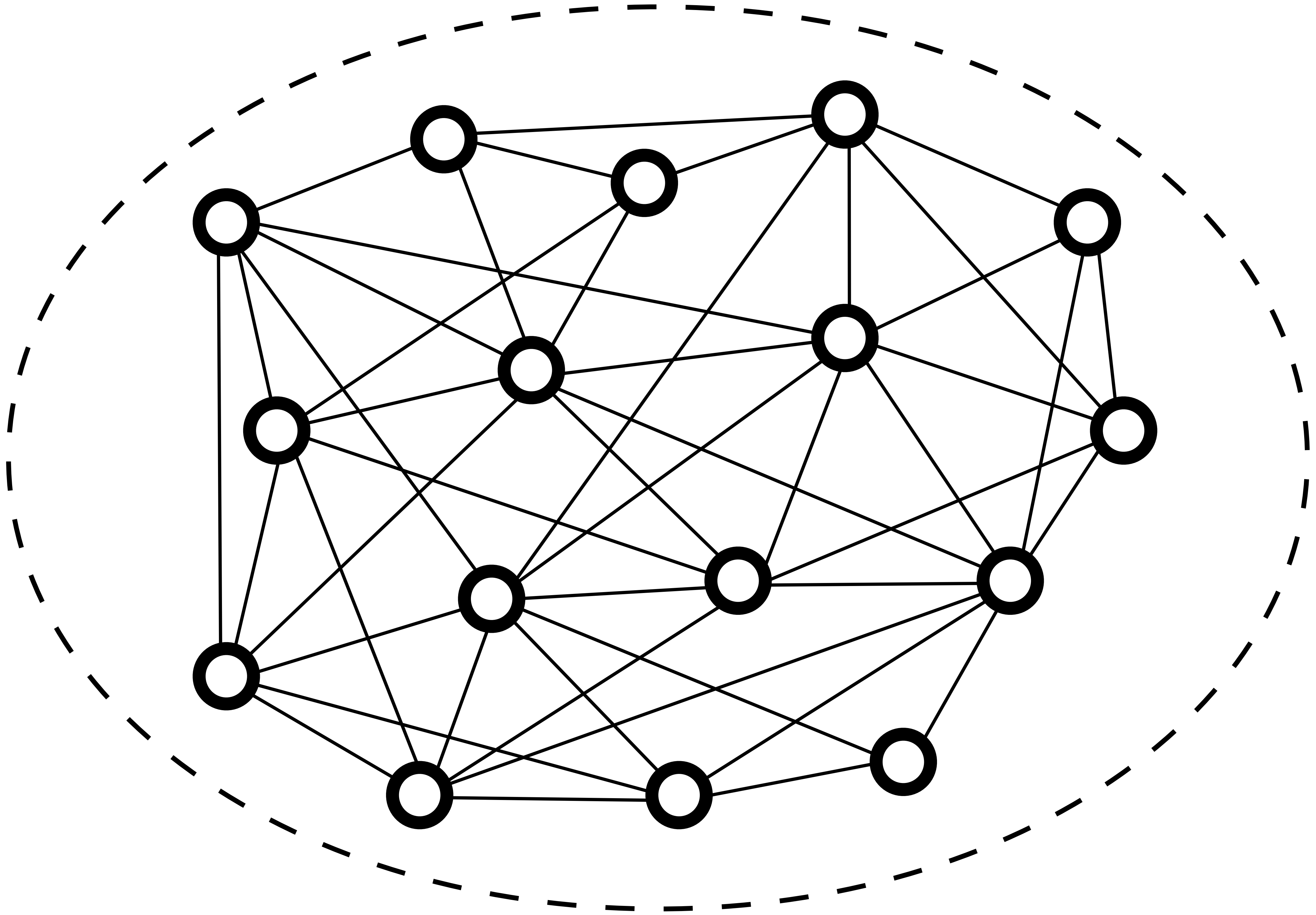

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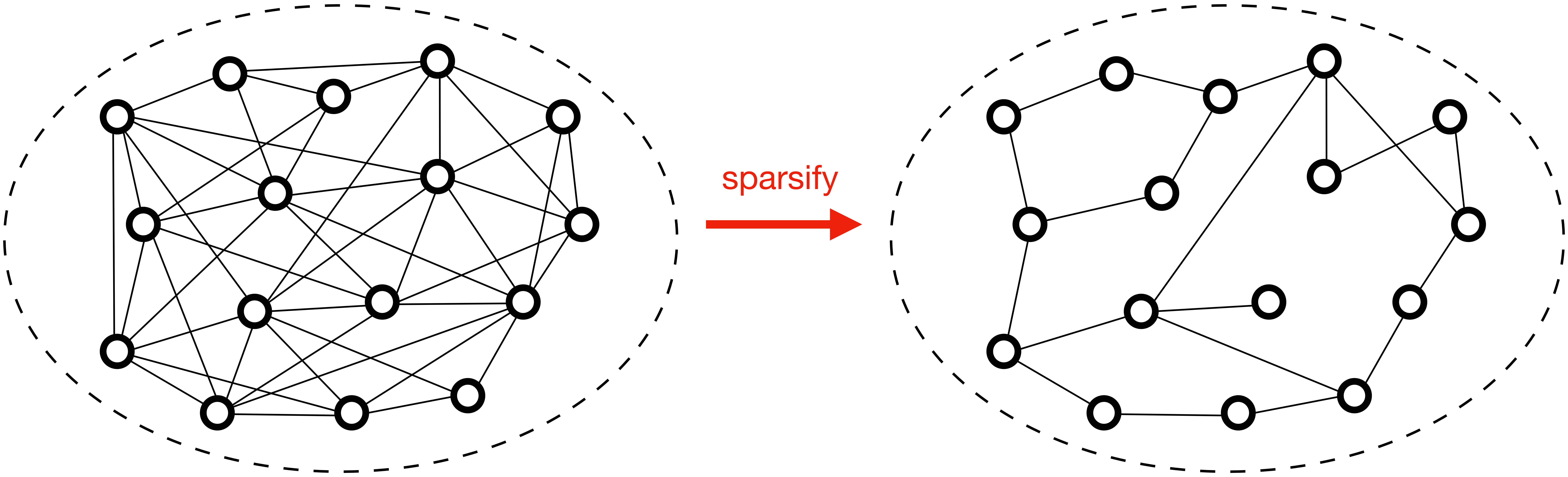

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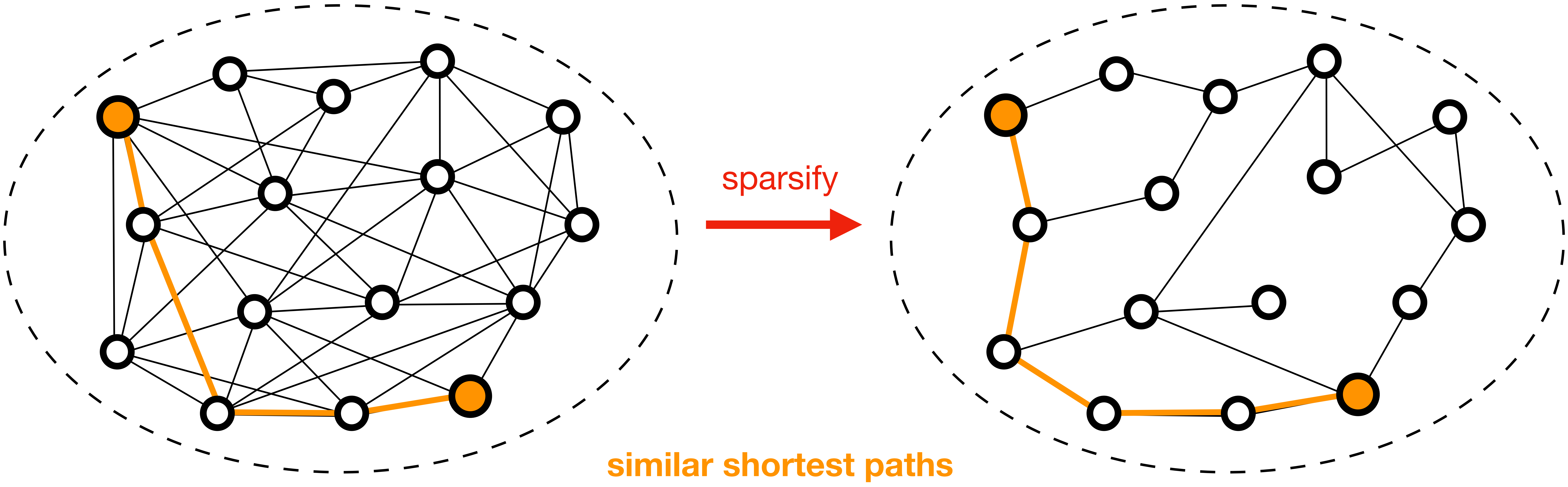
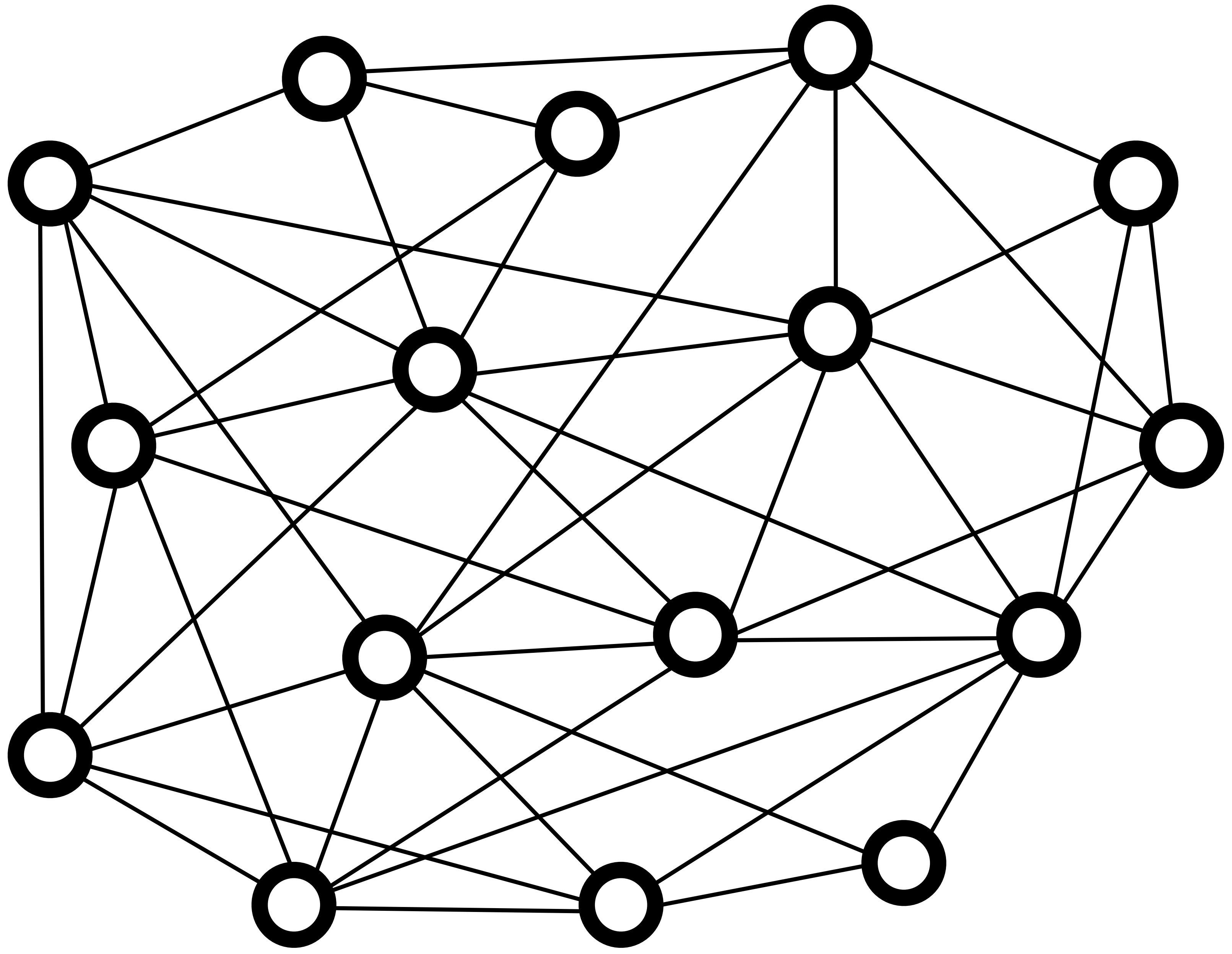


Spanners

- Given an **input graph *G***
- Find a sparse subgraph *H* with similar shortest paths



Emulators

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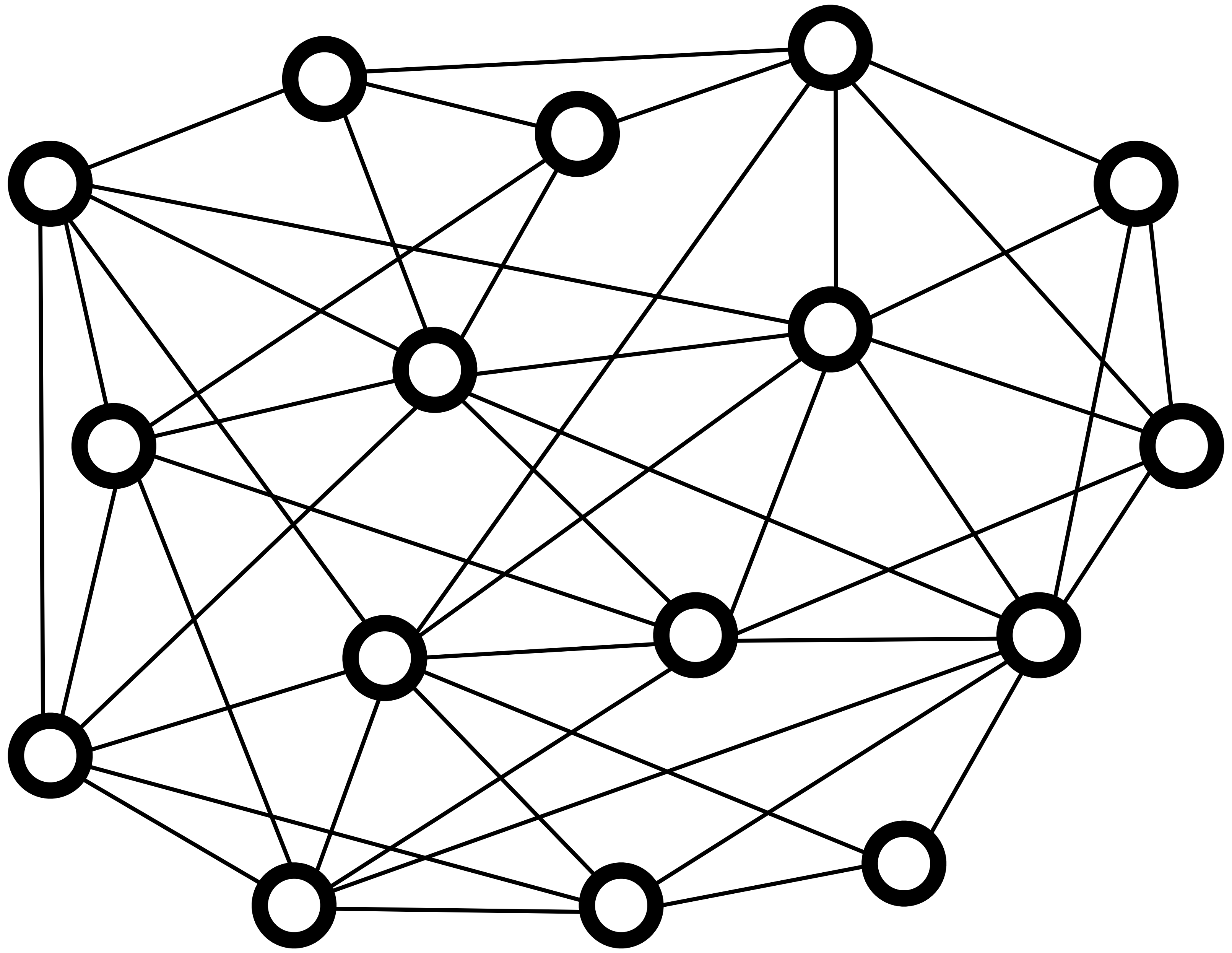

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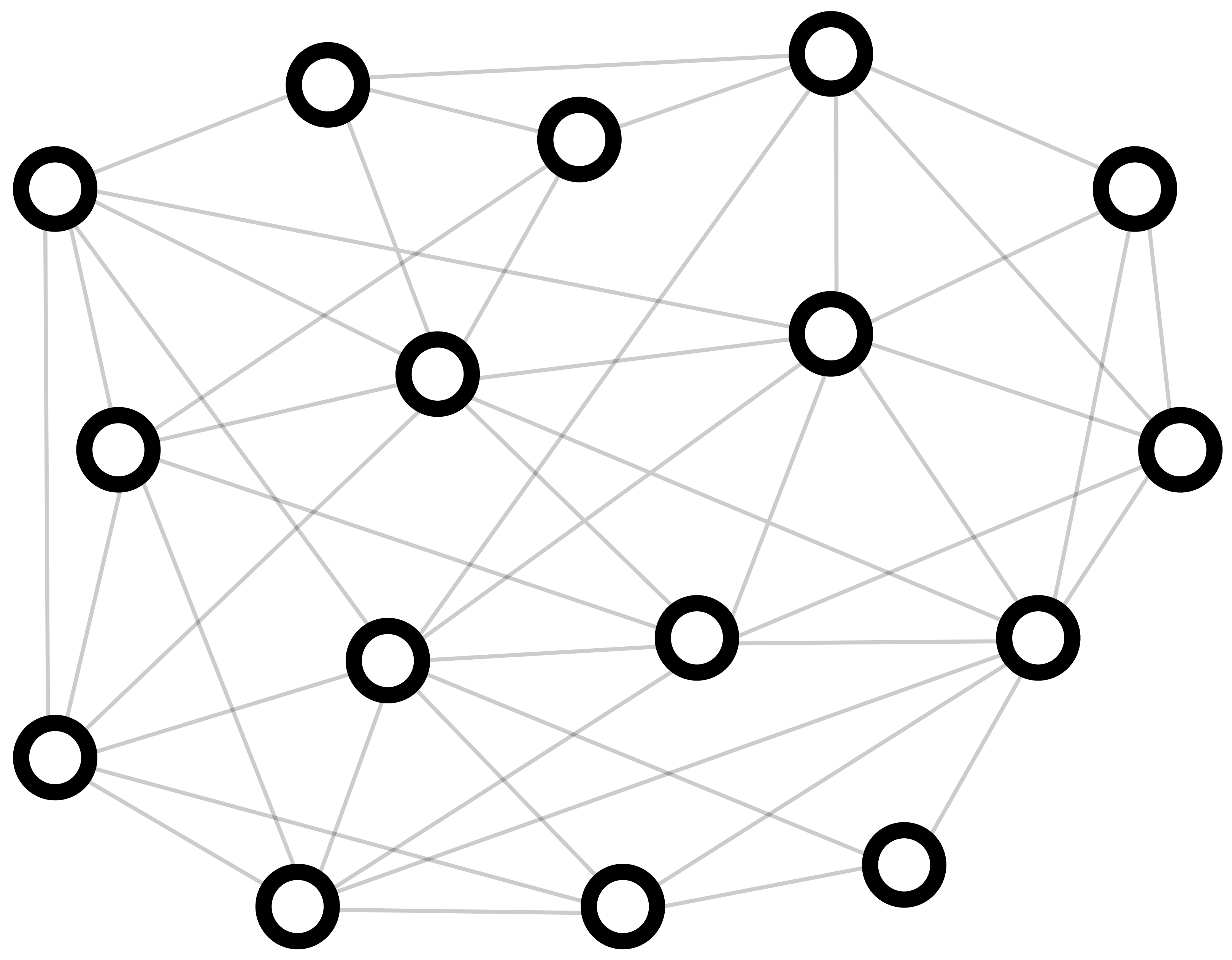

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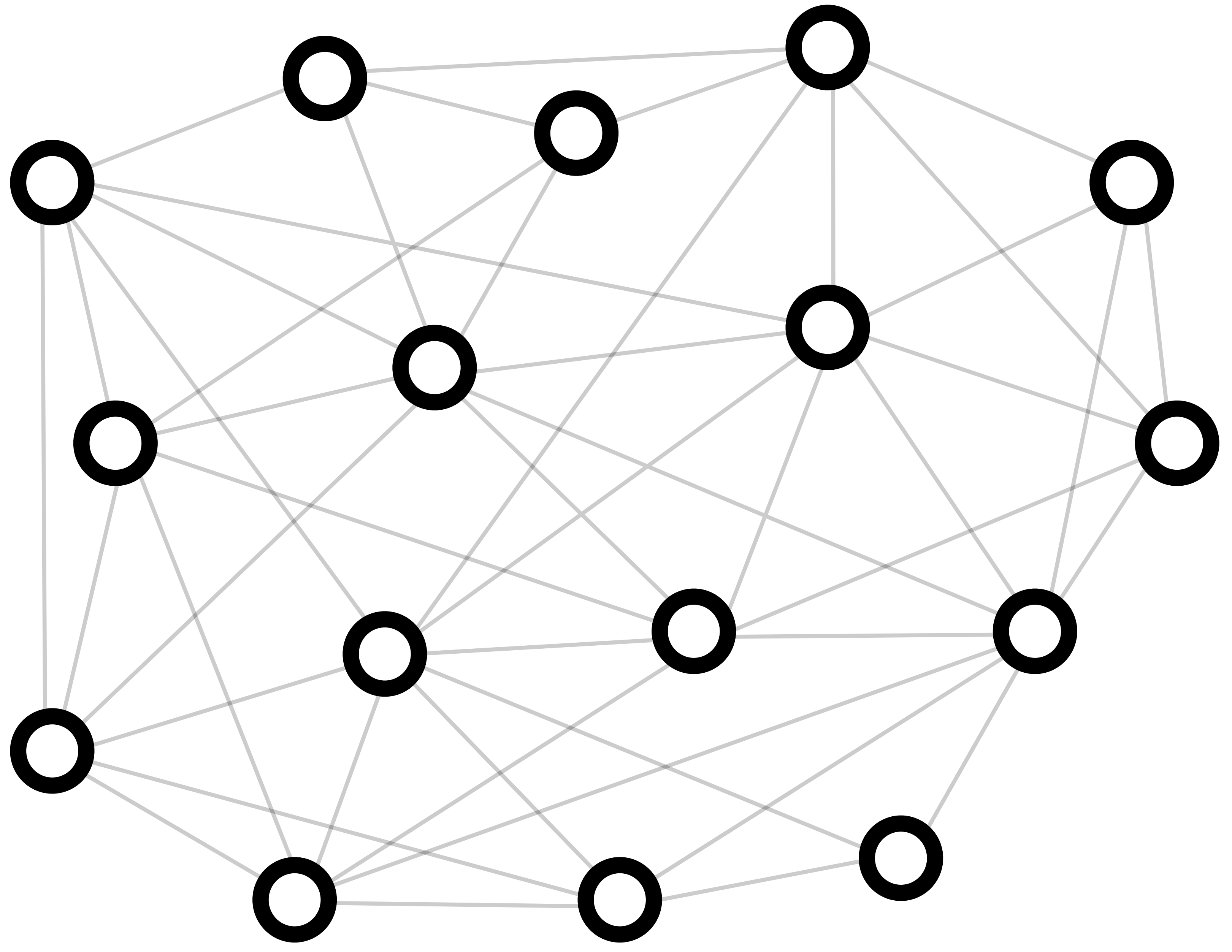

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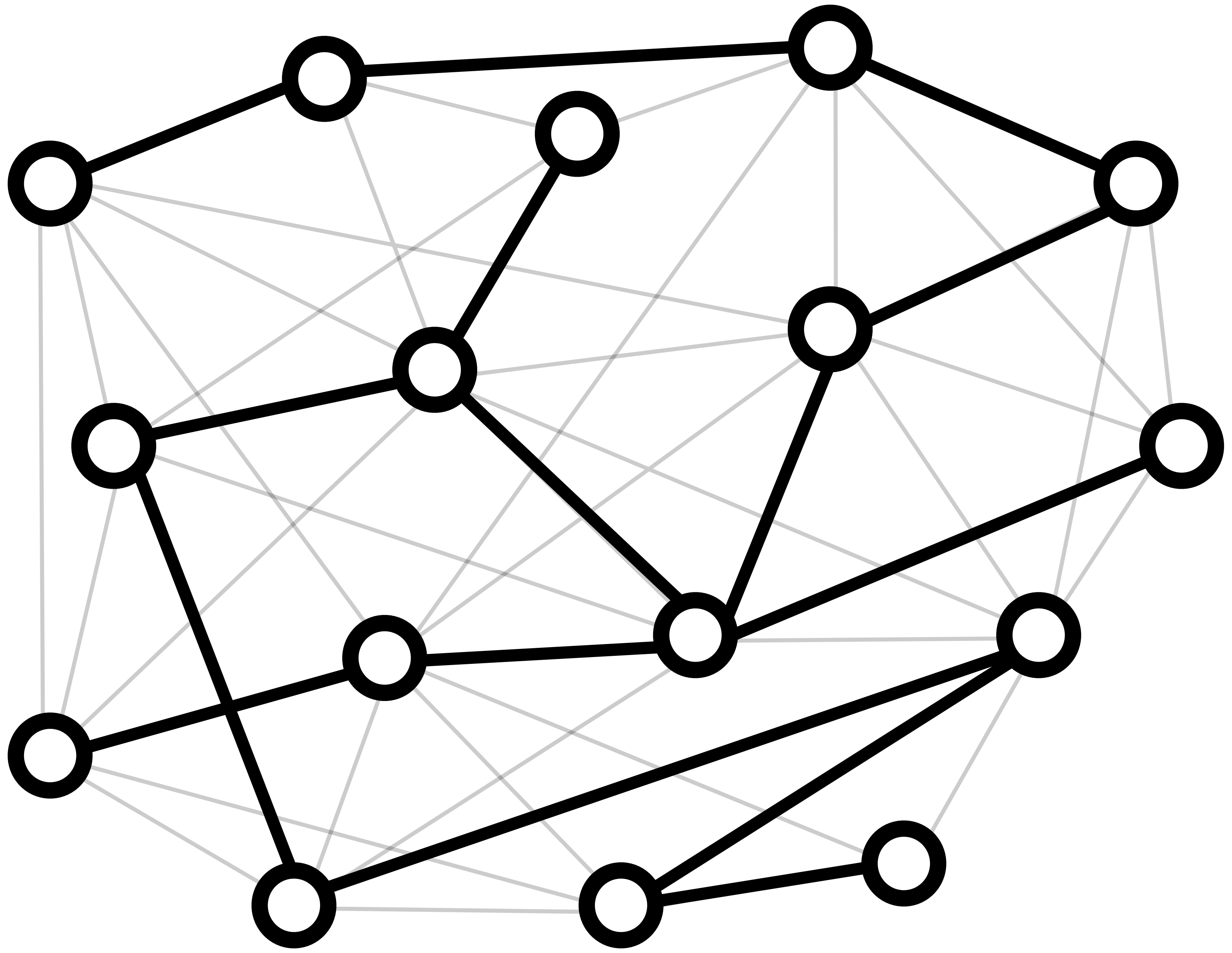

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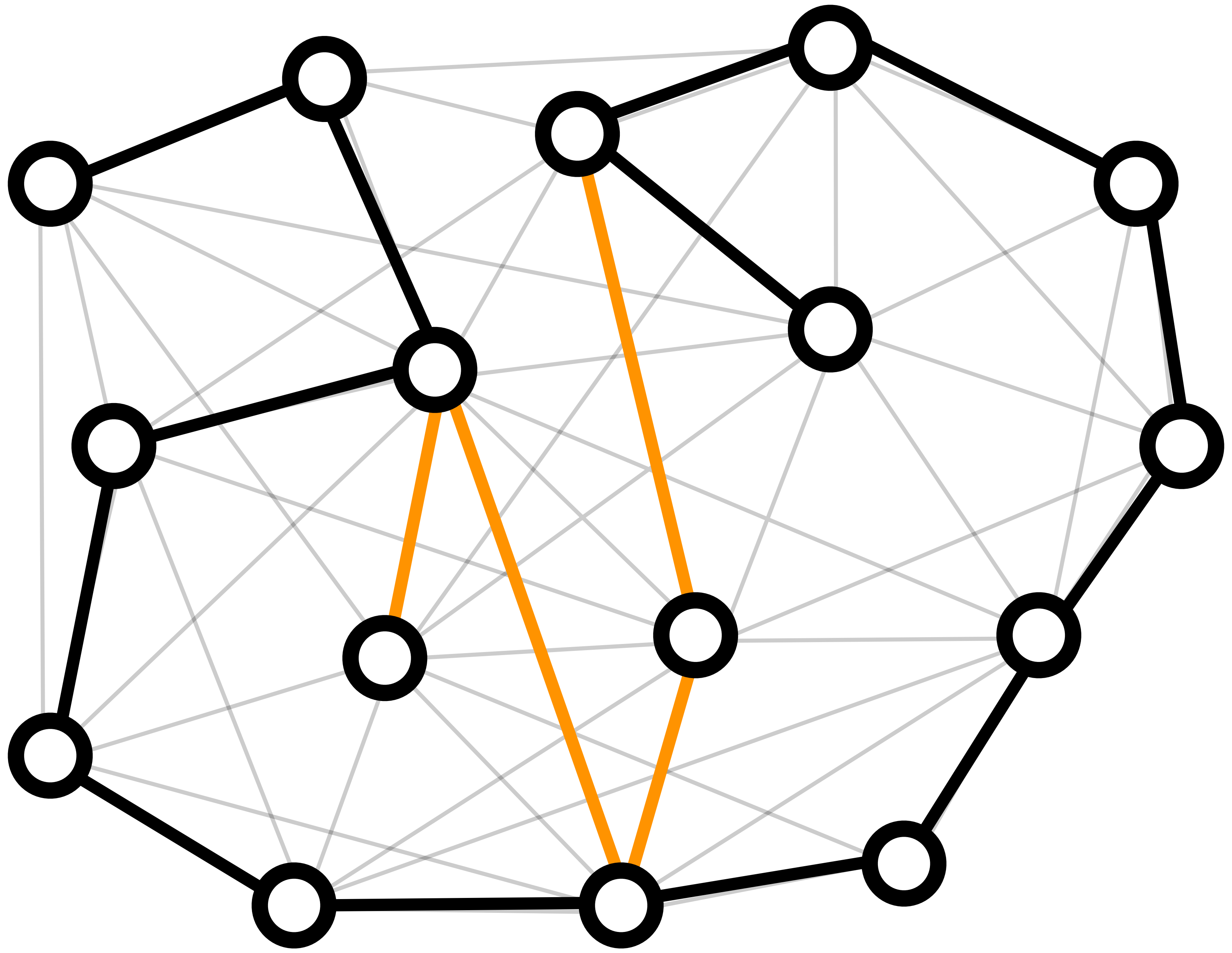

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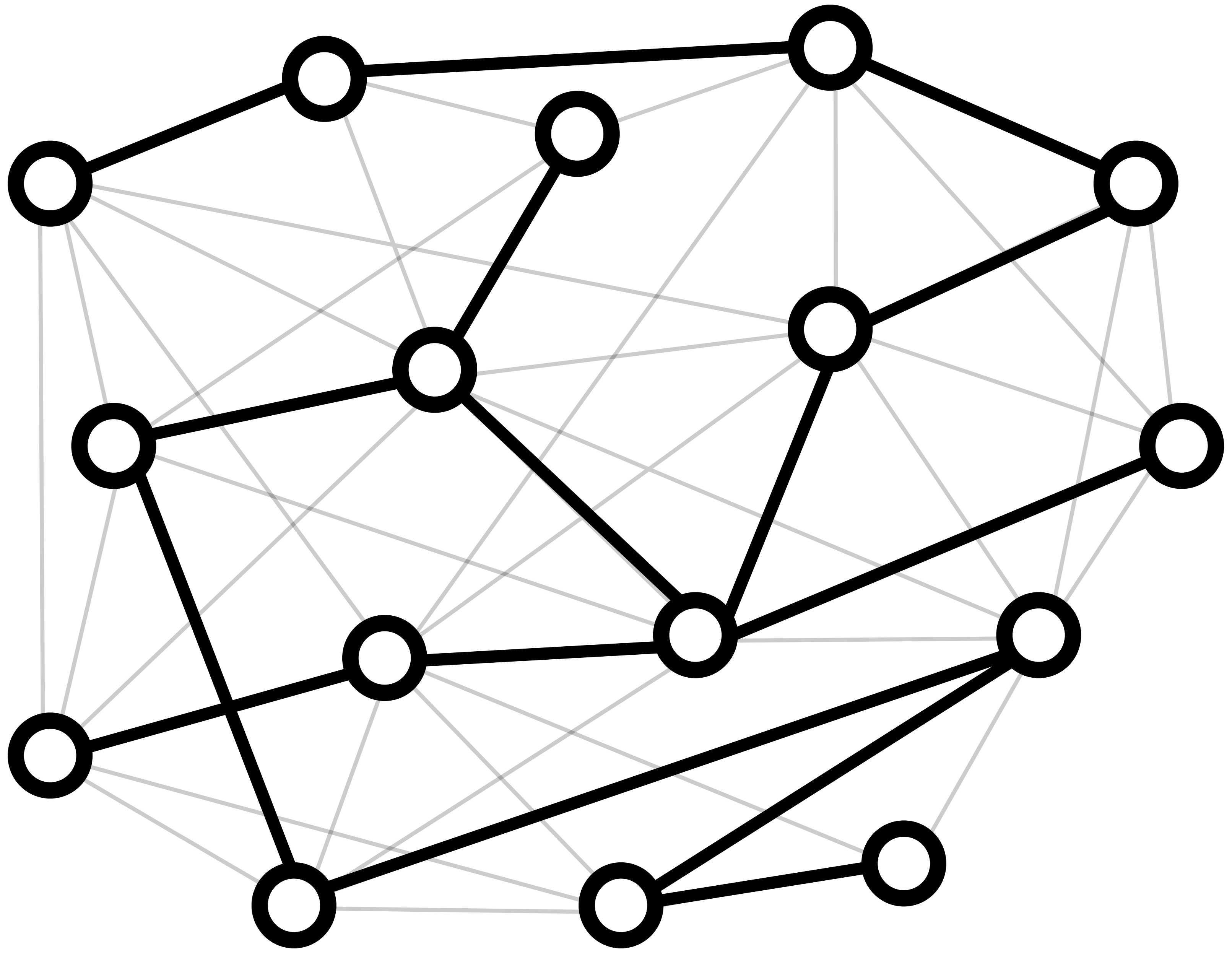

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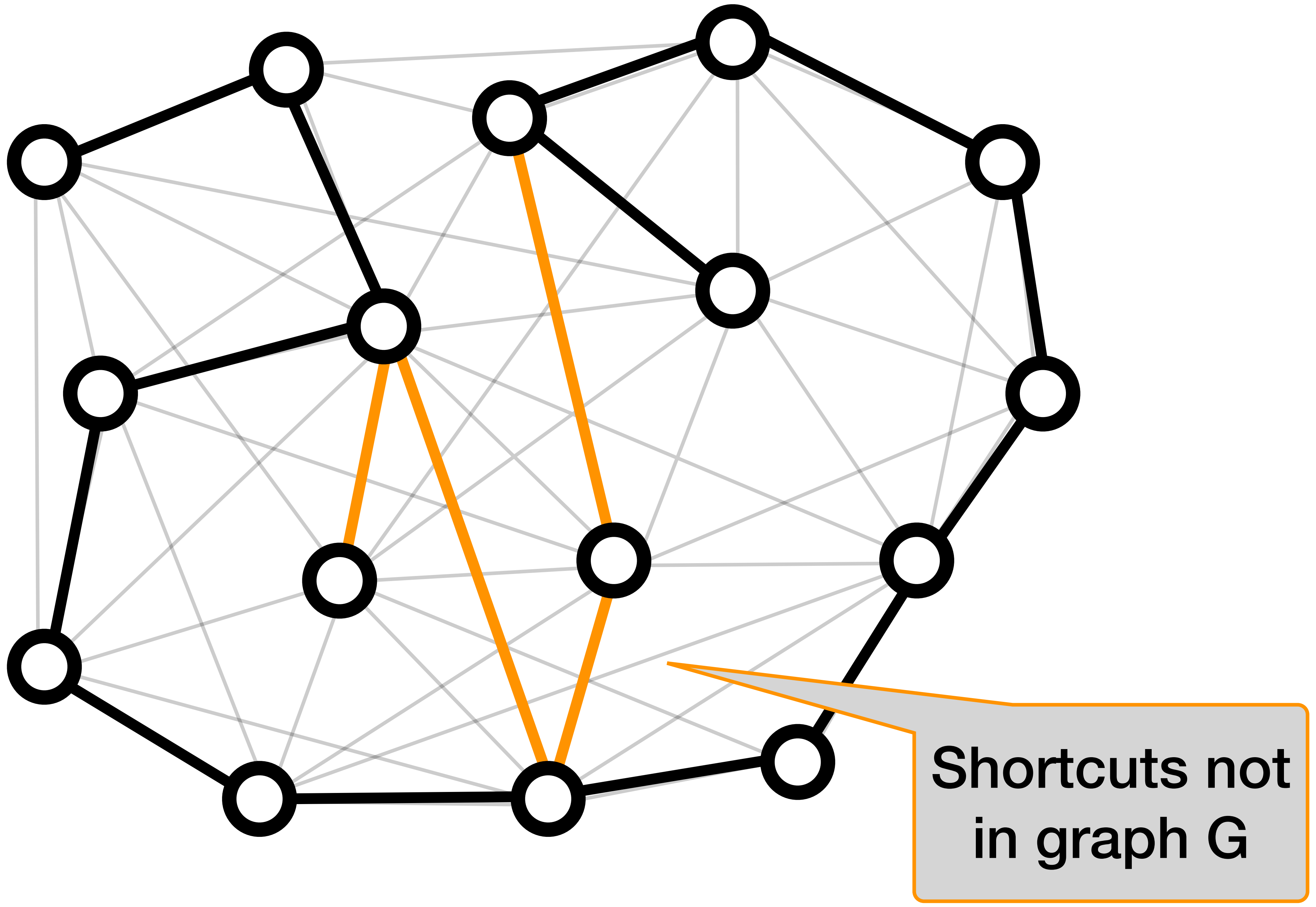

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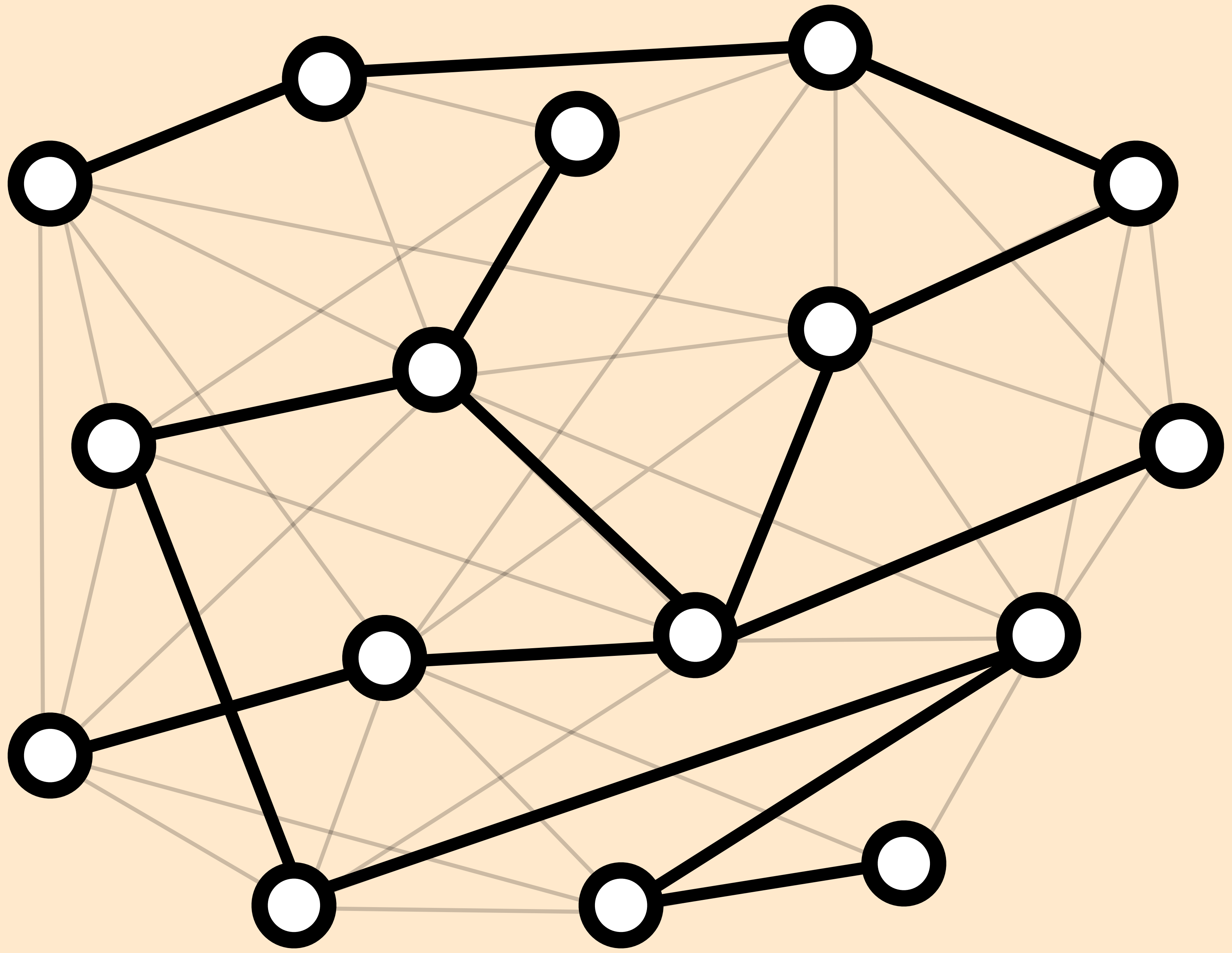

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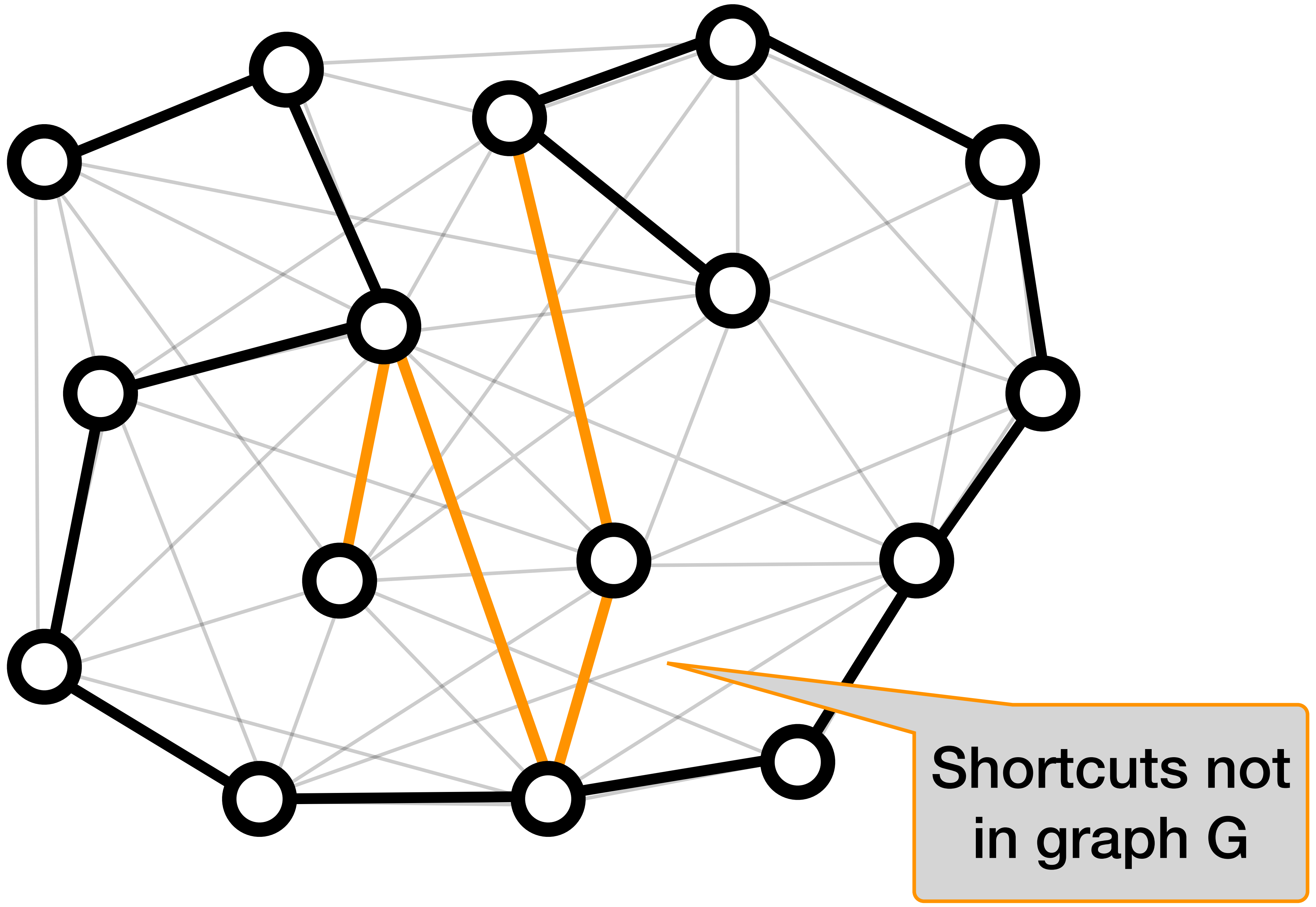


Graph Spanners

- Unweighted undirected graph $G = (V, E)$ and subgraph $H \subseteq G$
- **Stretch function *f*** for any $s, t \in V$:

$$\text{dist}_G(s,t) \leq \text{dist}_H(s,t) \leq f(\text{dist}_G(s,t))$$

- Optimal balance between sparsity $|E(H)|$ and stretch *f*

Multiplicative Stretch

- **Stretch function *f*** for any $s, t \in V$:

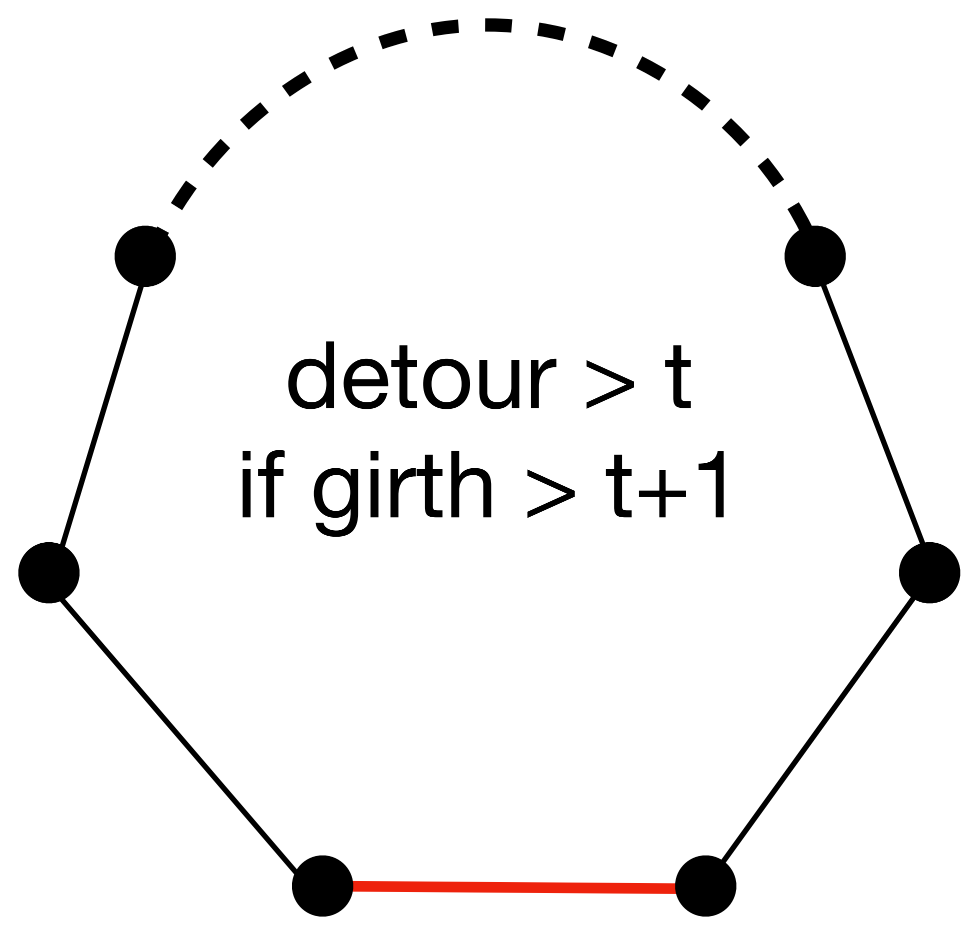
$$\text{dist}_G(s,t) \leq \text{dist}_H(s,t) \leq f(\text{dist}_G(s,t))$$

- Multiplicative stretch: $f(d) = t \cdot d$

- **<u>Definition:</u>**
  The **girth** of a graph is the length of its shortest simple cycle

- Define $\gamma(n,t)$ = largest #edges of n-vertex graph whose girth > t

- **<u>Corollary:</u>** Size of t-spanner on n-vertex graph is $\geq \gamma(n, t+1)$



Multiplicative Stretch

- Conjecture [Erdös, 1963]: $\gamma(n,t) = \Theta(n^{1+\frac{1}{\lceil t/2 \rceil}})$

- Upper bound [Althöfer et al, 1993]
  There exist spanners of size $O(n^{1+\frac{1}{k}})$ & stretch $f(d) = (2k-1)d$

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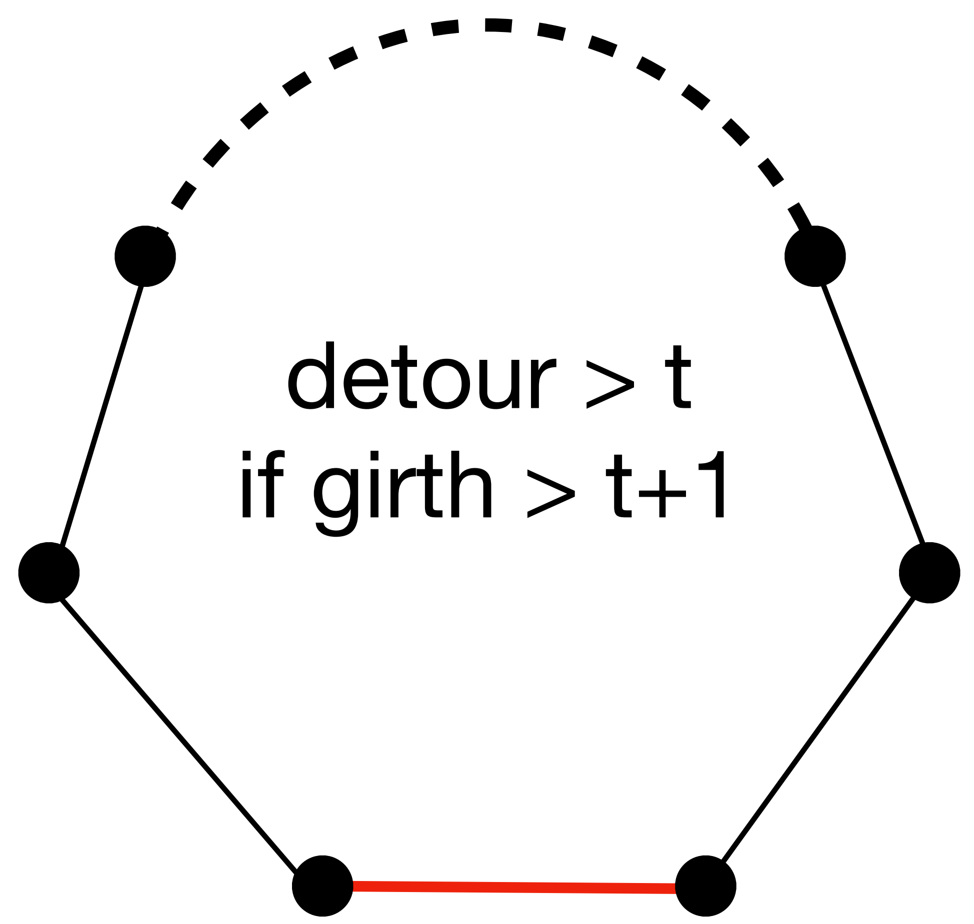

Multiplicative Stretch

What about t=2k,
and fractional values?

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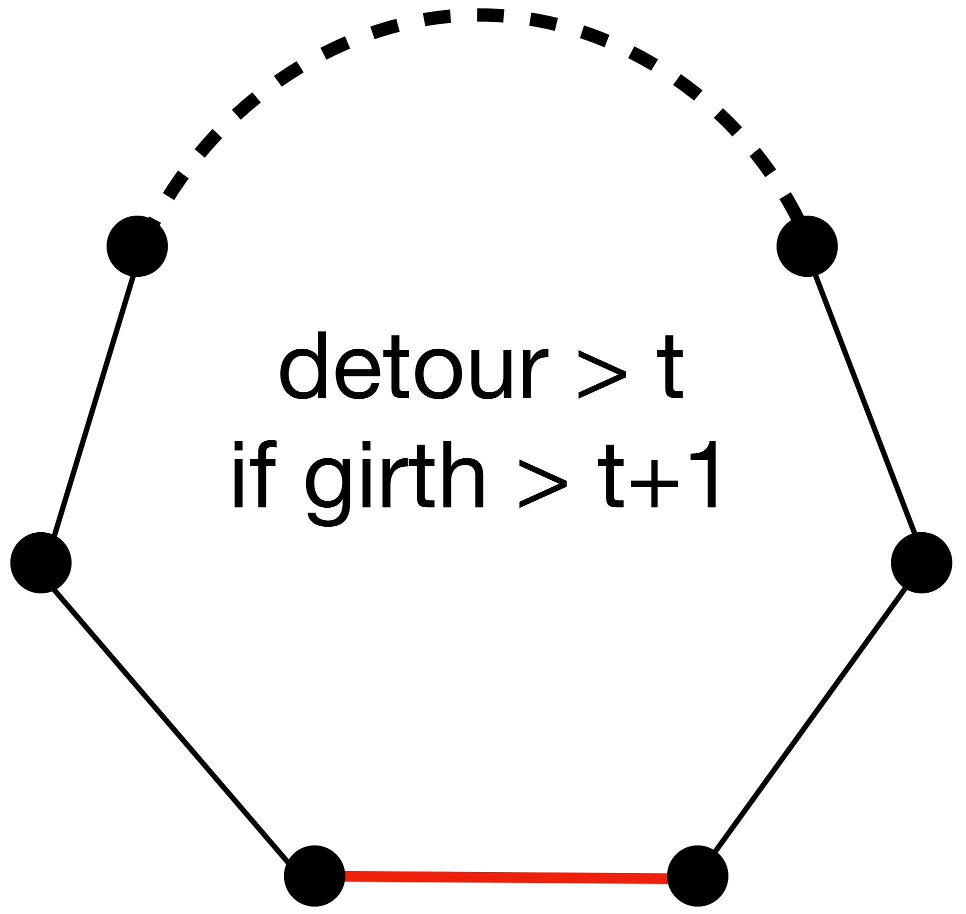

Small Additive Stretch

- **Stretch function *f*** for any $s, t \in V$:

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- Small **additive stretch**: $f(d) = d + O(1)$

Small Additive Stretch

reference	additive stretch	spanner size
[Aingworth, Chekuri, Indyk, Motwani, 1999]	d+2	$O(n^{3/2})$
[Woodruff, 2006]	d+2	$\Omega(n^{3/2})$
[Chechik, 2013]	d+4	$O(n^{7/5})$
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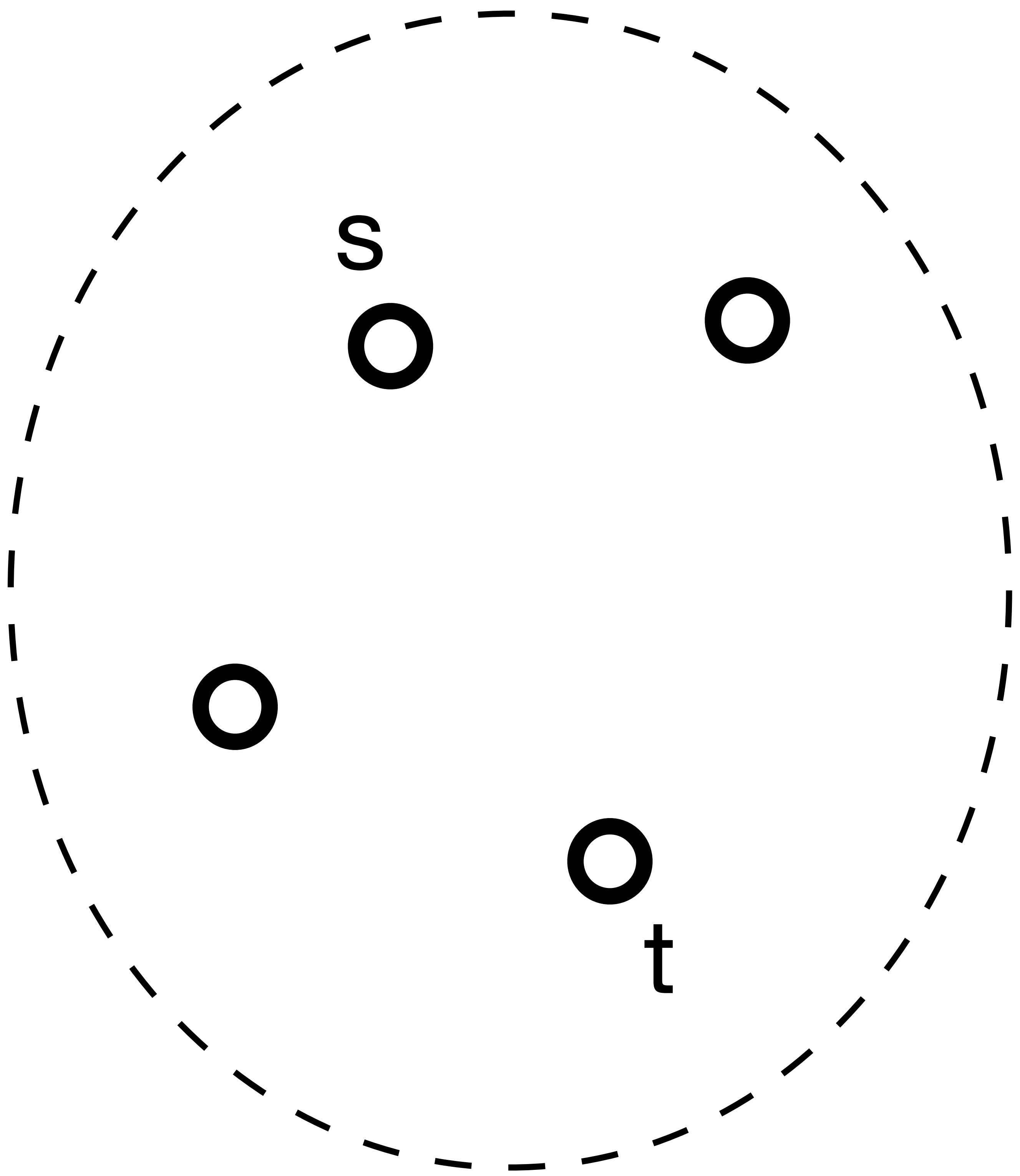
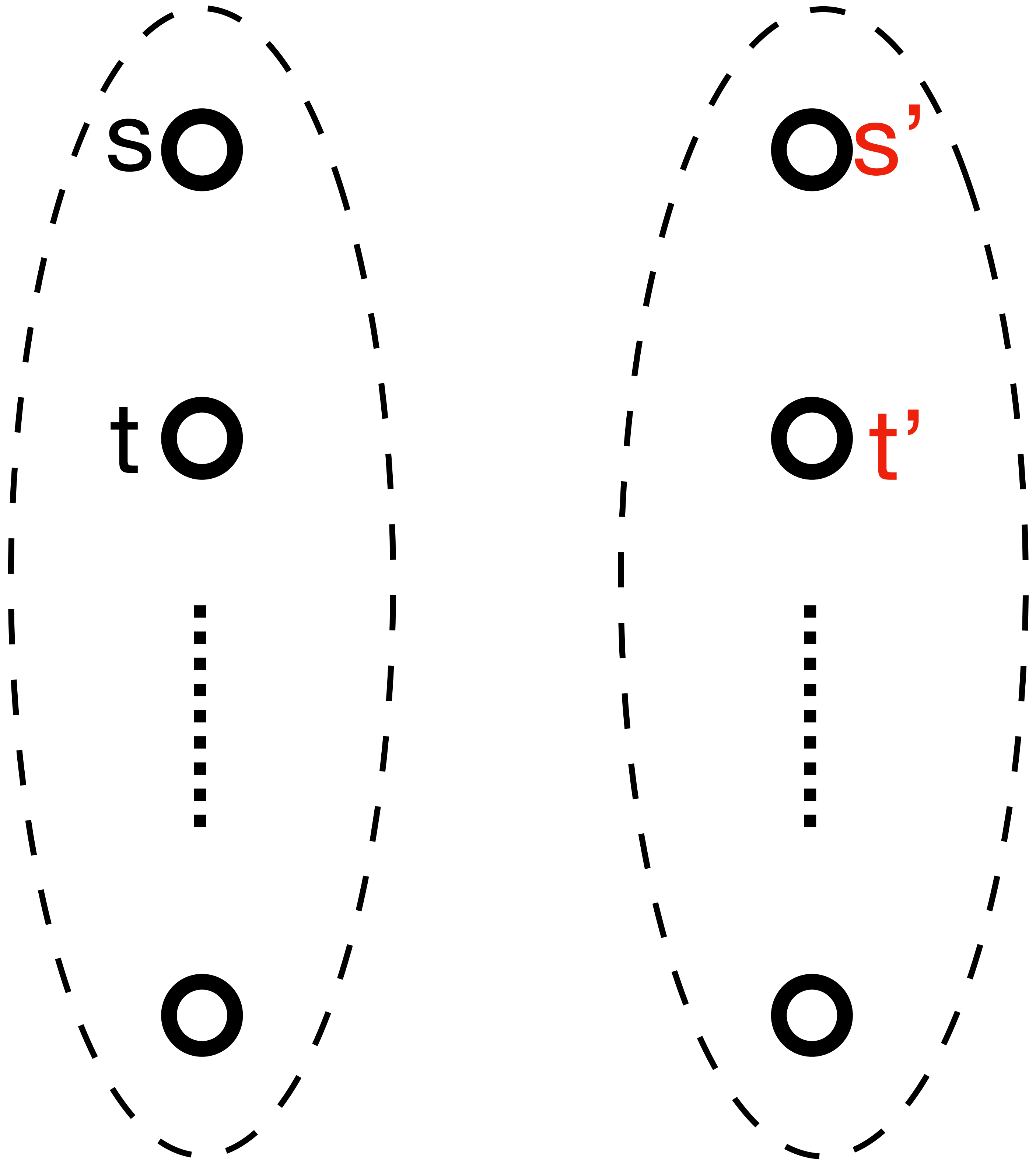
How about **odd** additive stretches?

+5 implies +4 with the same asymptotic sparsity

Small **additive stretch**: $f(d) = d + O(1)$

Reduction from even to odd





input graph $G = (V, E)$ → bipartite $G' = (V \cup V', E')$

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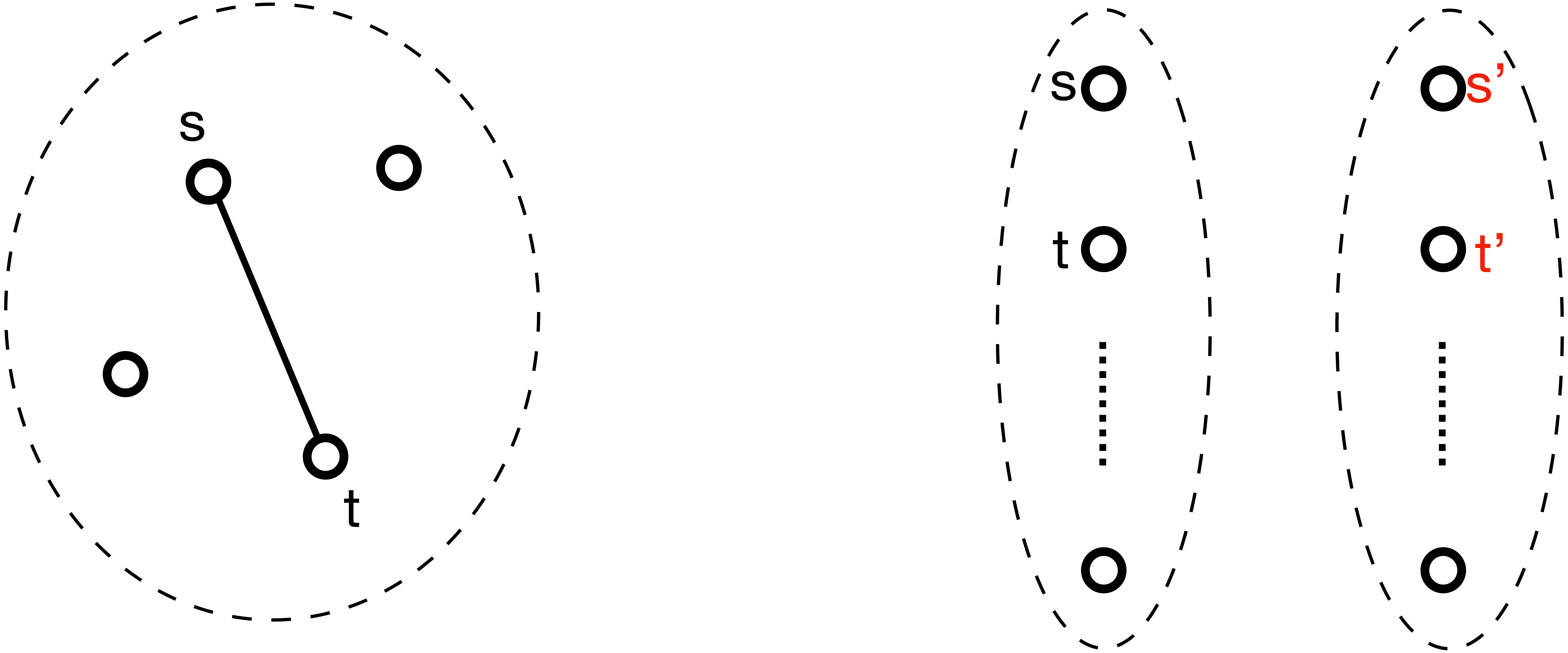

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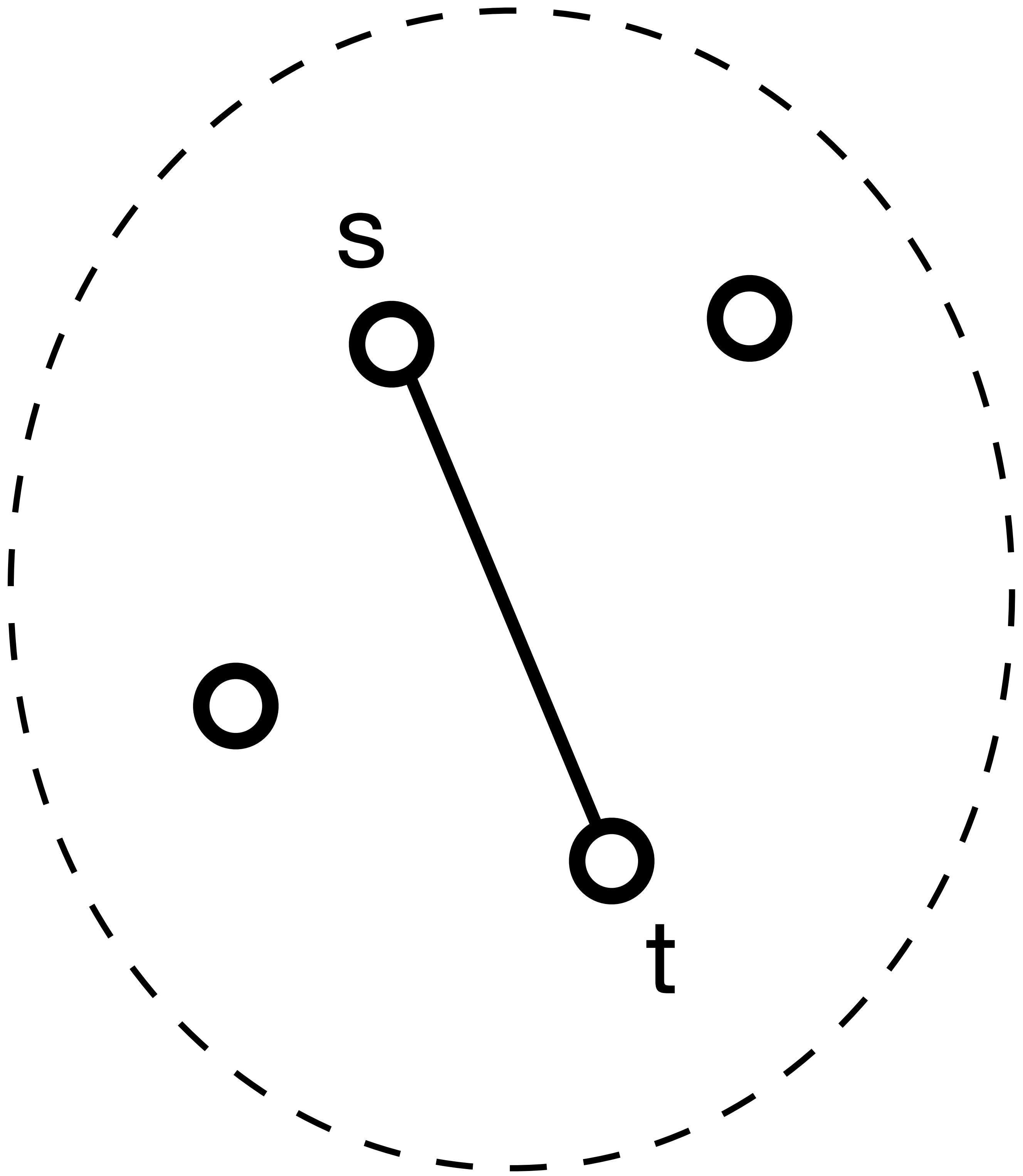

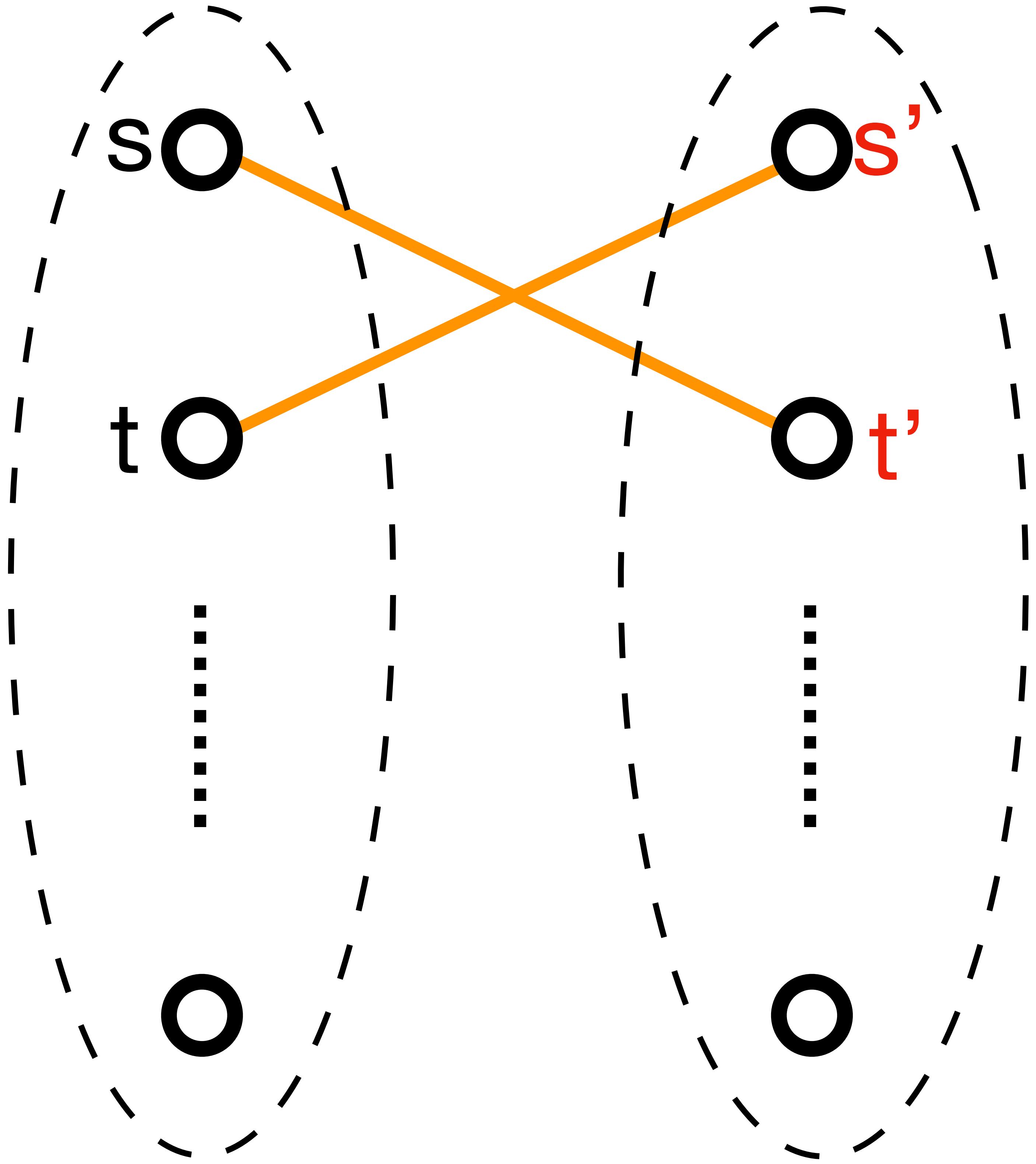

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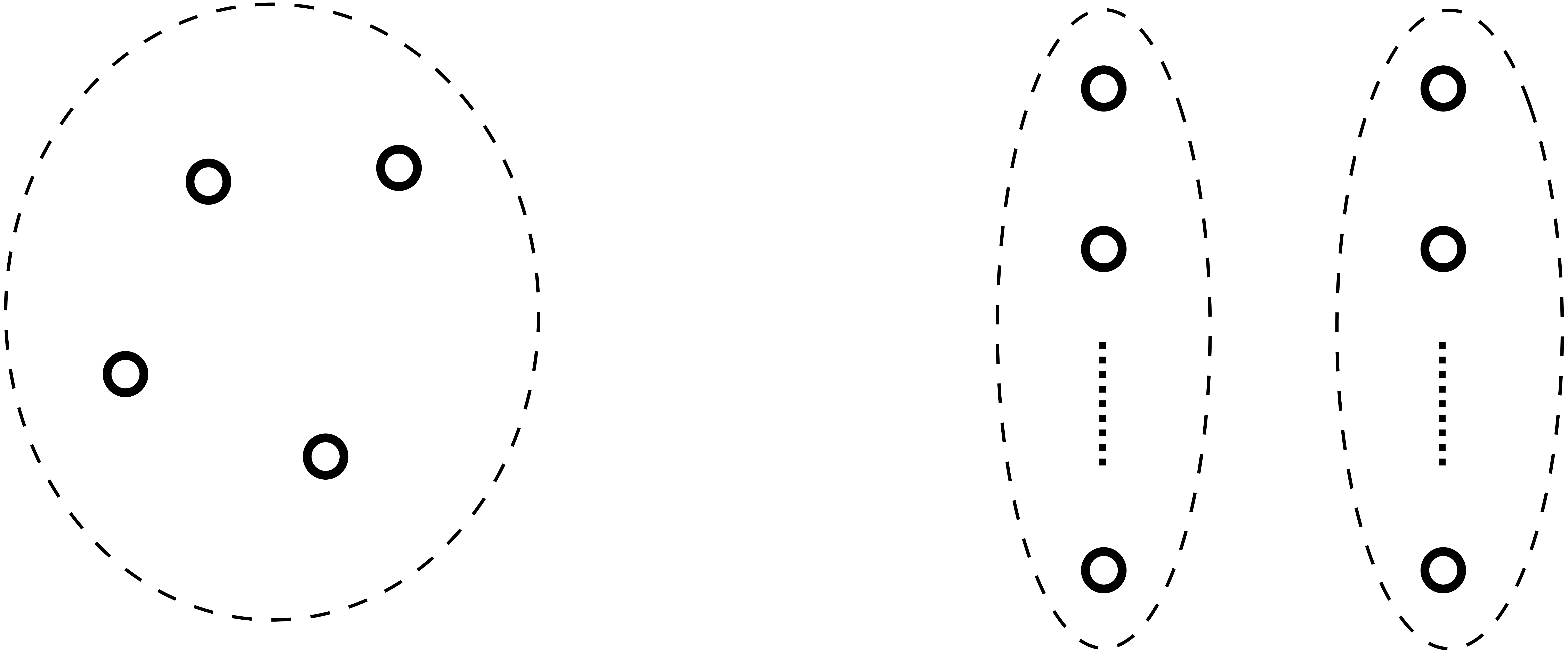

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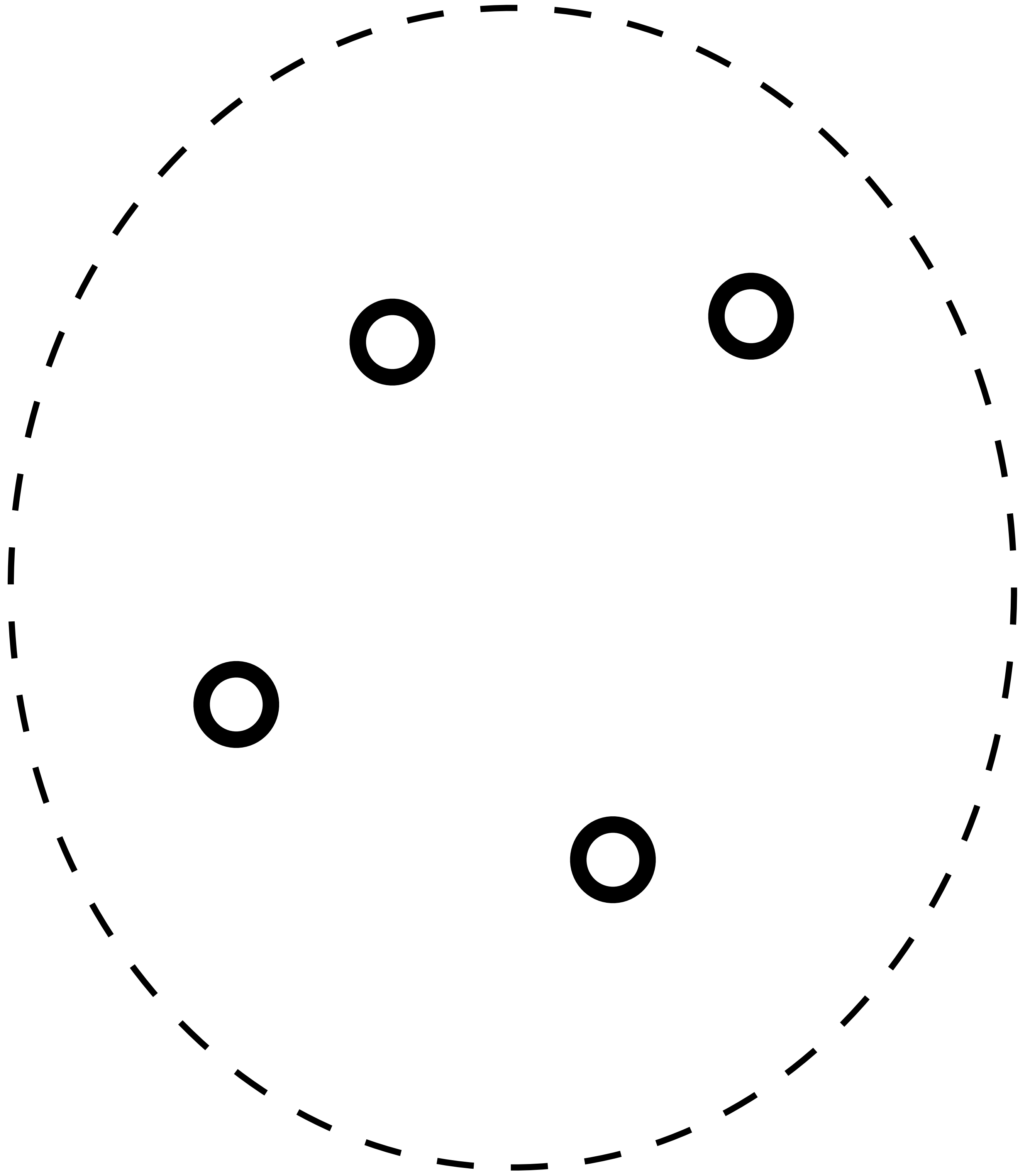

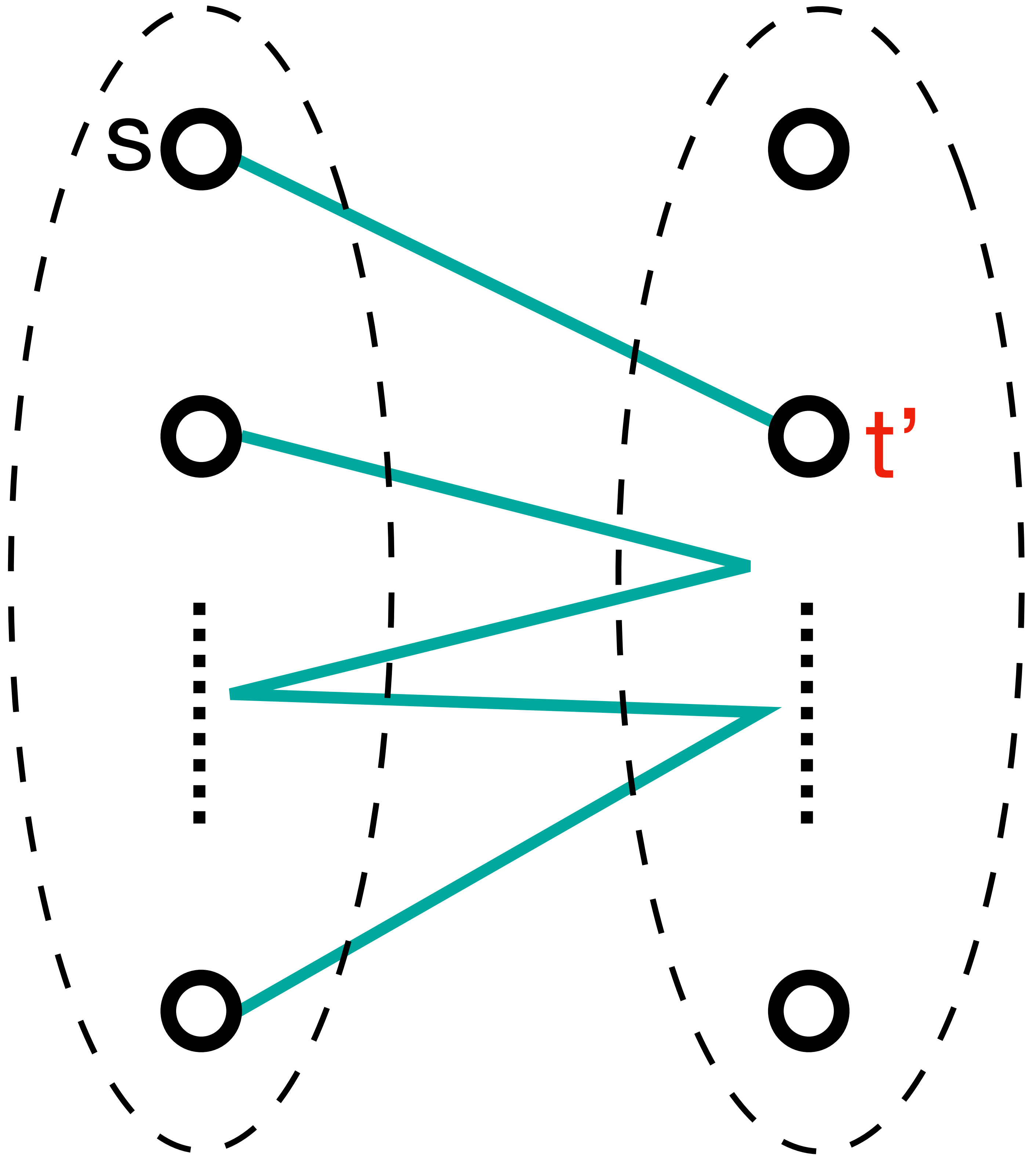

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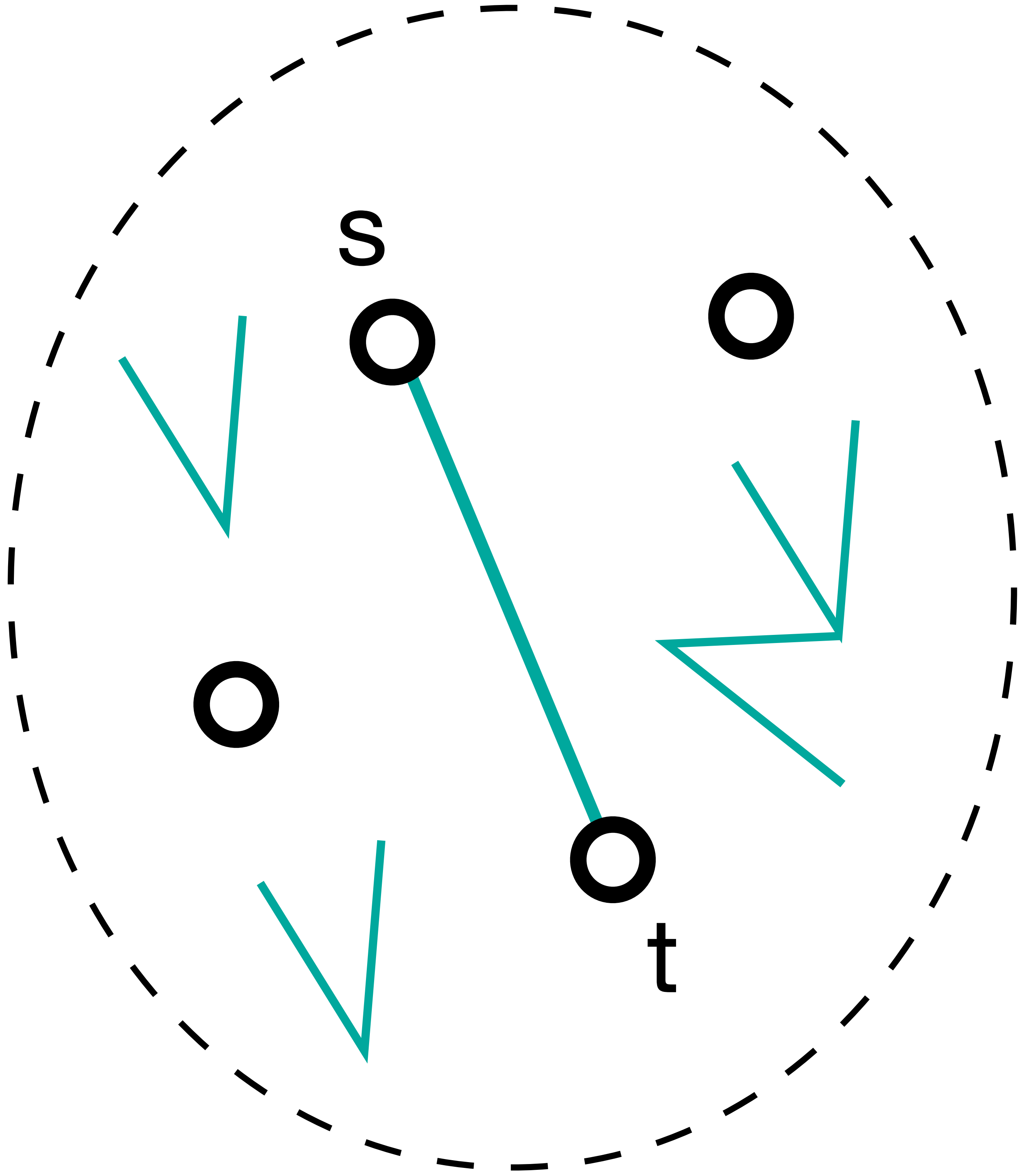
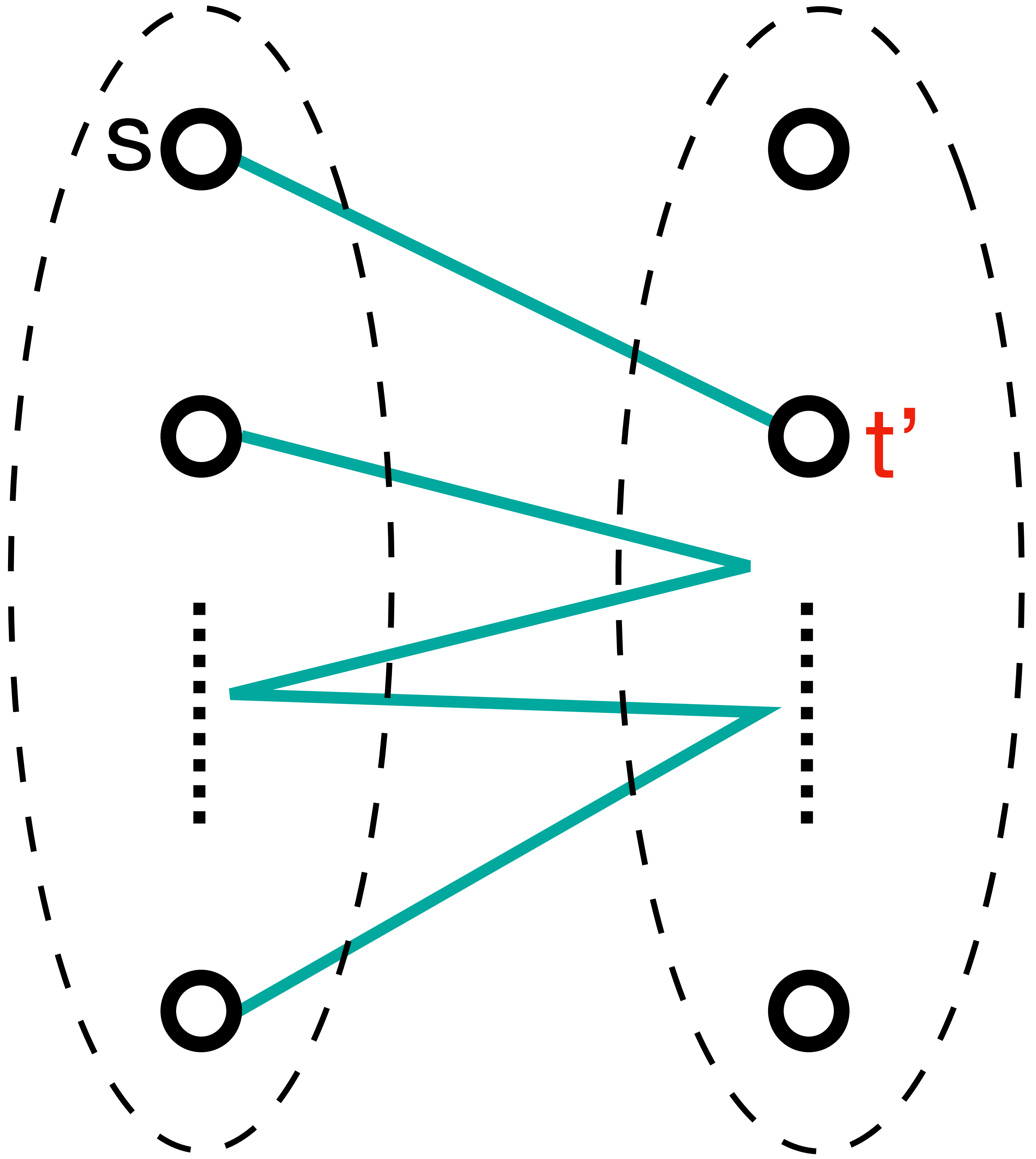
Build a **+5 spanner** of size $S(2n)$

Reduction from even to odd





input graph $G = (V, E)$ → bipartite $G' = (V \cup V', E')$

$S(2n)$-spanner of **same** stretch ← Build a **+5 spanner** of size $S(2n)$

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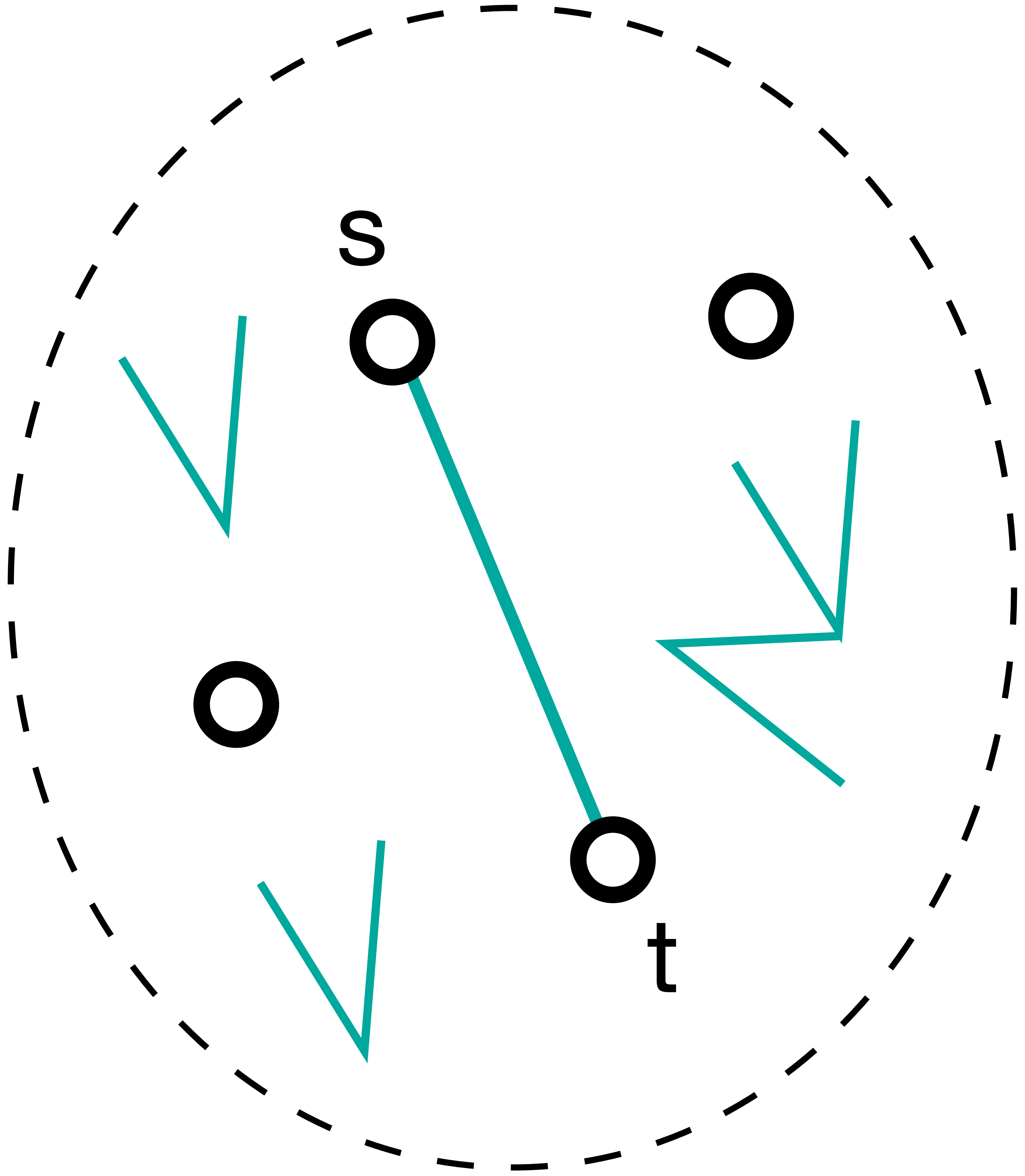

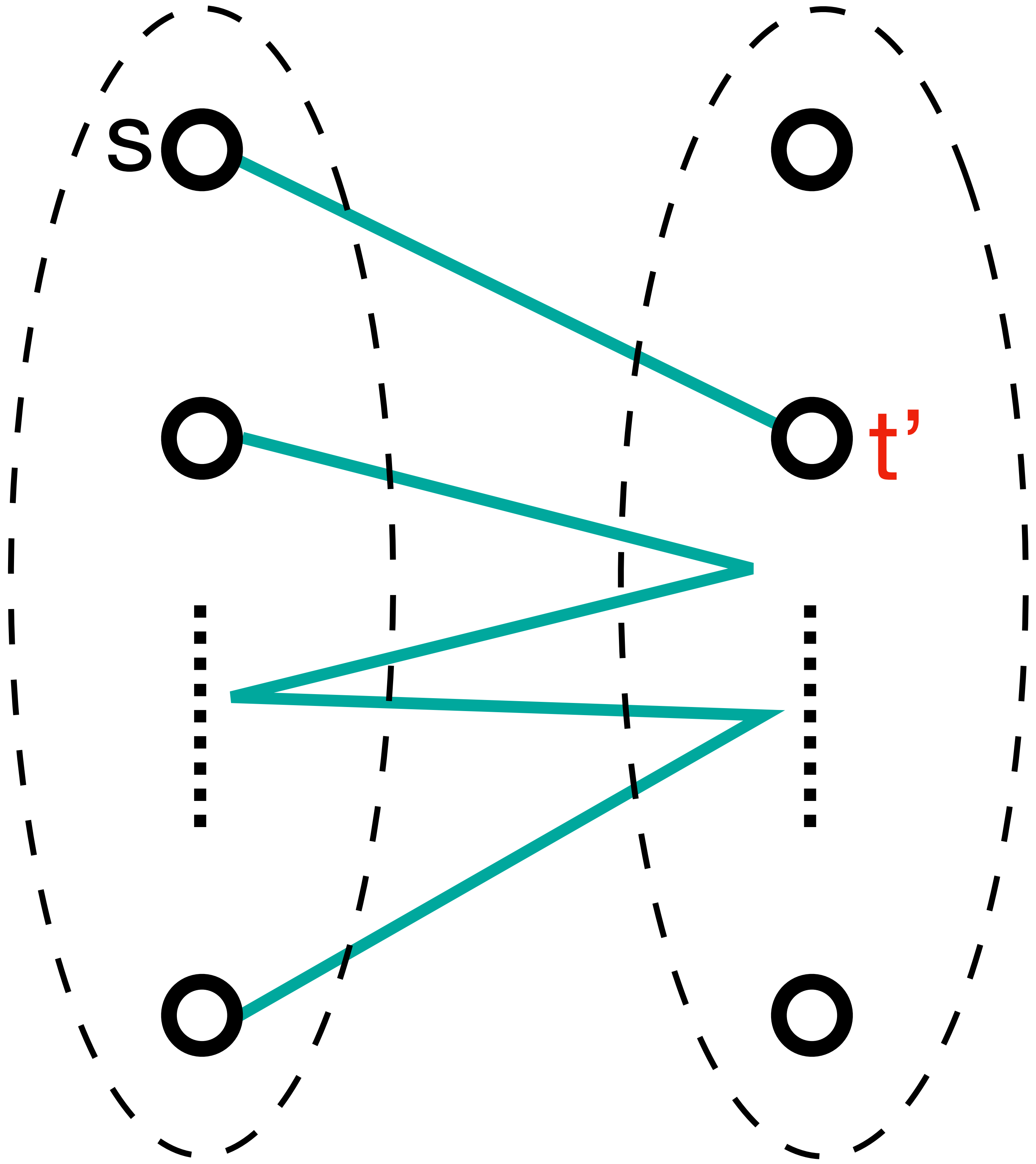

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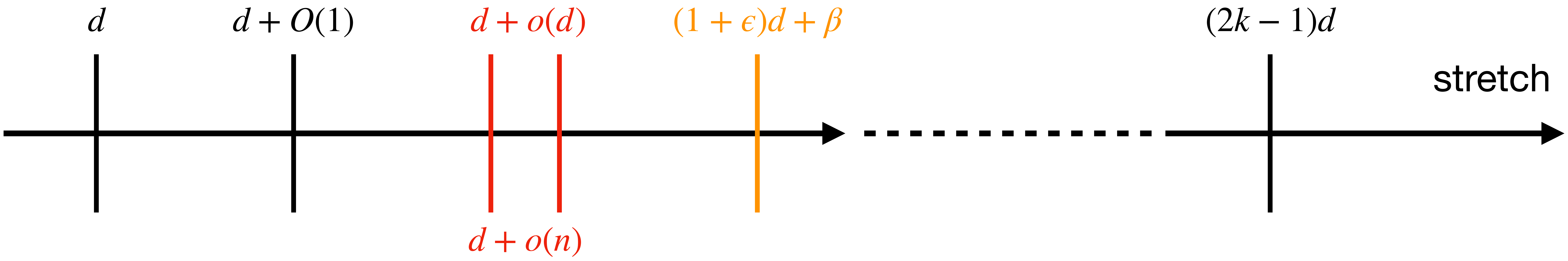
$S(2n)$-spanner of **same** stretch ⟵ Build a **+5 spanner** of size $S(2n)$

$S(2n)$-spanner of stretch **+4** ⟵ Stretch must be **even** in bipartite

Other stretch functions

Suppose G has n vertices

- Nearly additive stretch: $f(d) = (1+\epsilon)d + \beta$
- Sublinear additive stretch: $f(d) = d + o(d)$
- Purely additive stretch: $f(d) = d + o(n)$



Nearly additive

[Elkin & Peleg, 2001]

- $f(d) = (1+\epsilon)d + O(k/\epsilon)^k$
- Size = $(k/\epsilon)^{O(1)} n^{1+\frac{1}{2^{k+1}-1}}$

[Abboud, Bodwin, Pettie, 2017]

- $f(d) = (1+\epsilon)d + O(k/\epsilon)^k$
- Lower bound = $n^{1+\frac{1}{2^{k+1}-1}-o(1)}$

Sublinear additive

[Pettie, 2009]

- $f(d) = d + O(k) \cdot d^{1-1/k}$
- Size = $kn^{1+\frac{1}{7\cdot(4/3)^{k-2}-2}}$

[Chechik, 2013]

- $f(d) = d + O(1) \cdot d^{1/2}$
- Size = $n^{\frac{20}{17}}$

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What is the **right answer** for sublinear additive?

Sublinear additive stretch

[Abboud, Bodwin, Pettie, 2017]

- $f(d) = d + c_k \cdot d^{1-1/k}$
  for some **small factor** c_k

- Must have edges $n^{1+\frac{1}{2^k-1}-o(1)}$

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- $f(d) = d + O_k(d^{1-1/k})$
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Apply the lower bound for $f(d) = d + c_{k+1} \cdot d^{1-1/(k+1)}$

$n^{1+\frac{1}{2^k-1}-o(1)}$

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Our result

- $f(d) = d + 2^{k^2 2^k/\epsilon} \cdot d^{1-1/k}$
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Sublinear additive stretch

[Pettie, 2009]

- $f(d) = d$

- Size = kn

How about **other functions**?

$f(d) = d + d^{1-0.5/k-0.5/(k+1)}$

$< d + c_{k+1} \cdot d^{1-1/(k+1)}$

Must have edges $n^{1+\frac{1}{2^{k+1}-1}-o(1)}$

Subsumed by

$f(d) = d + O_k(1) \cdot d^{1-1/k}$

[Chechik, 20

- $f(d) = d + O(1) \cdot d^{1/2}$

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Linear-size additive spanners

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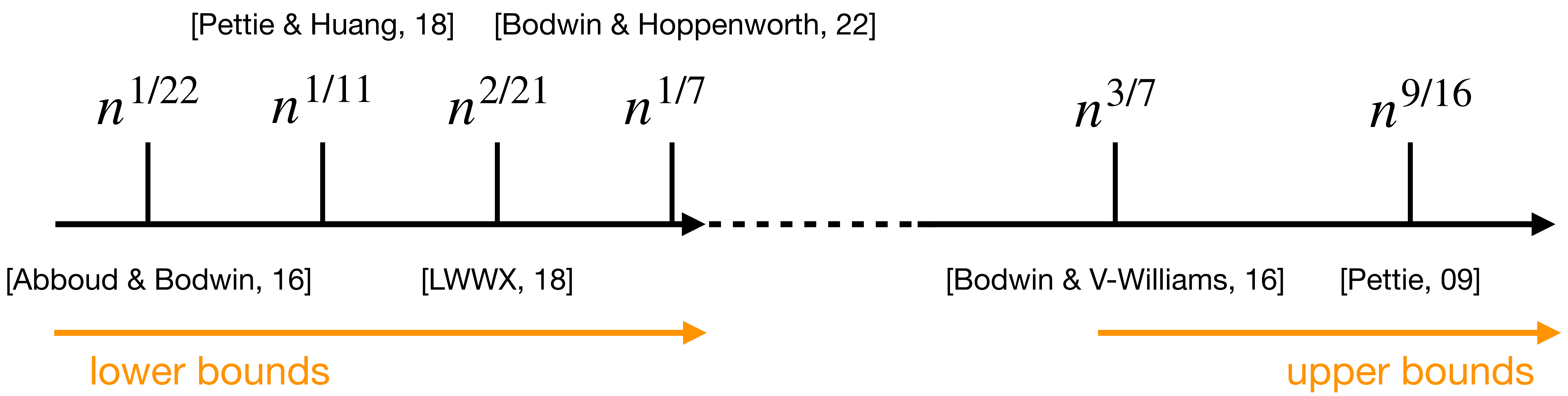
- Linear-size regime: if $E(H) = O(n)$, what is the smallest $o(n)$?



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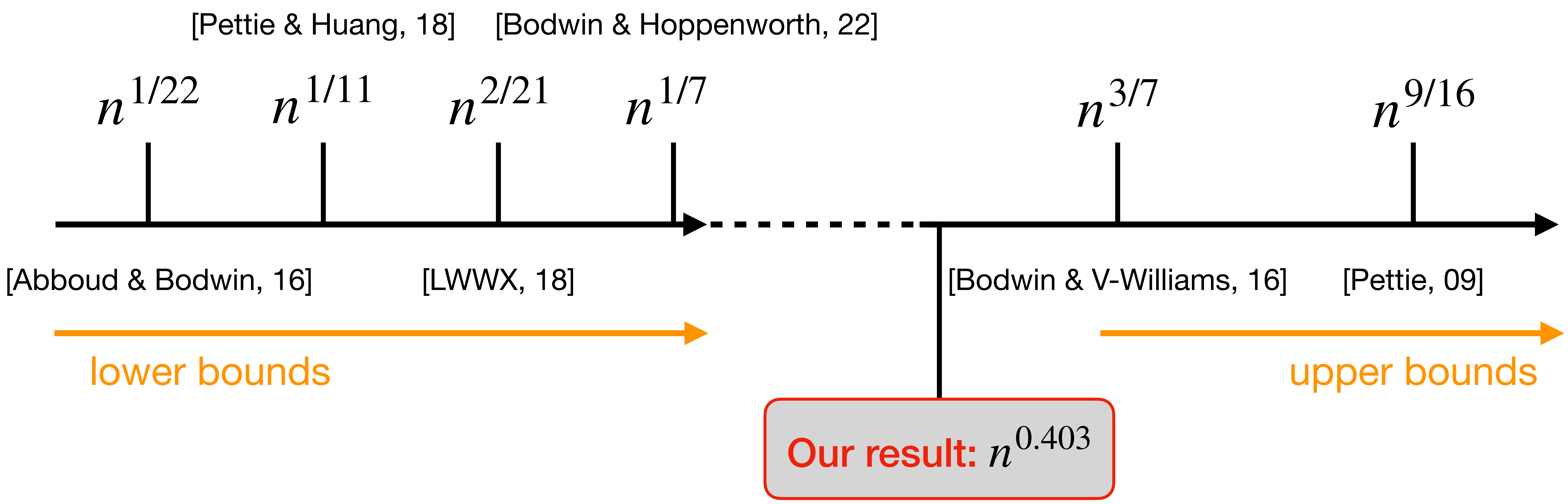

Today's plan

Sublinear additive spanners:

- Example: $f(d) = d + O(d^{1/2}), |E(H)| = n^{8/7+\epsilon}$
- Sketch: $f(d) = d + O_{k,\epsilon}(d^{1-1/k}), |E(H)| = n^{1+\frac{1+\epsilon}{2^{k+1}-1}}$

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A ball-covering lemma

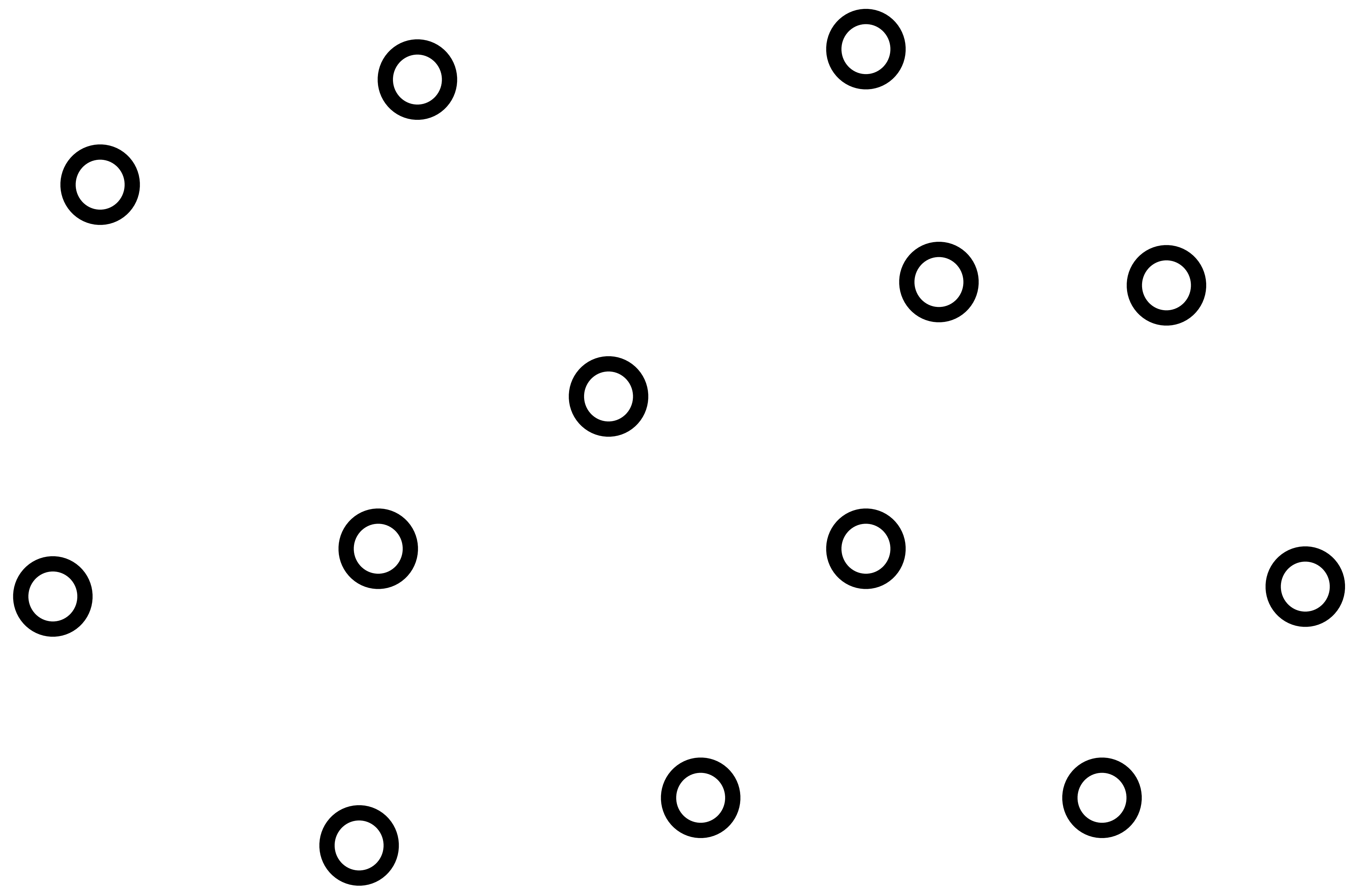
<u>Lemma [Bodwin & V-Williams, 2016]</u>

For any R, there exists all set of balls $\mathscr{B} = \{B(c,r)\}_{c \in V}$ such that

Radius: $R \leq r \leq 2^{O(1/\epsilon)}R$

Covering: $V = \bigcup_{B \in \mathscr{B}} B(c, r)$

Packing: $\sum_{B \in \mathscr{B}} |B(c, 2r)| = n^{1+\epsilon}$



A ball-covering lemma

Lemma [Bodwin & V-Williams, 2016]

For any R, there exists all set of balls $\mathscr{B} = \{B(c, r)\}_{c \in V}$ such that

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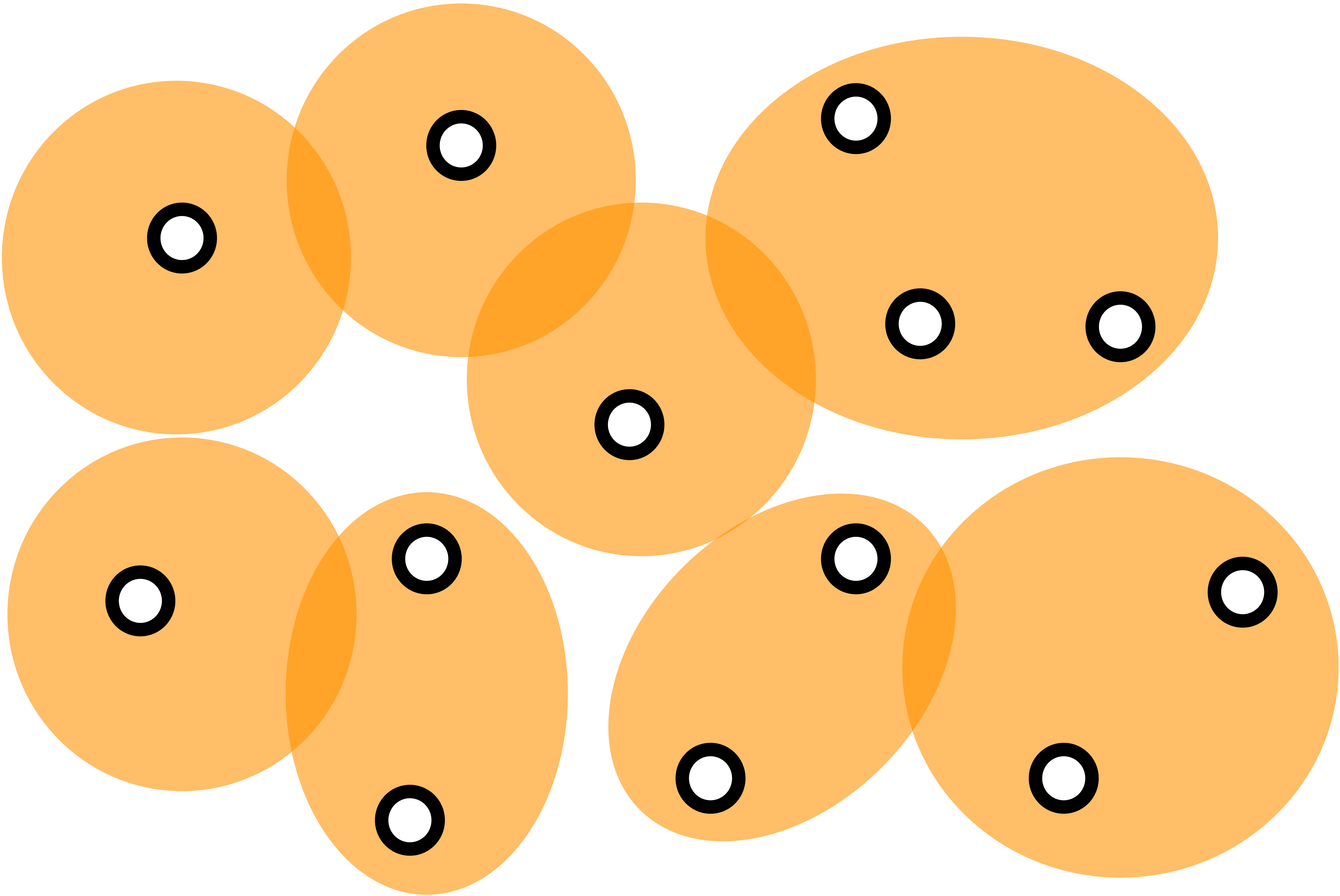

All the balls $\{B(c, r)\}$ cover the entire graph

A ball-covering lemma

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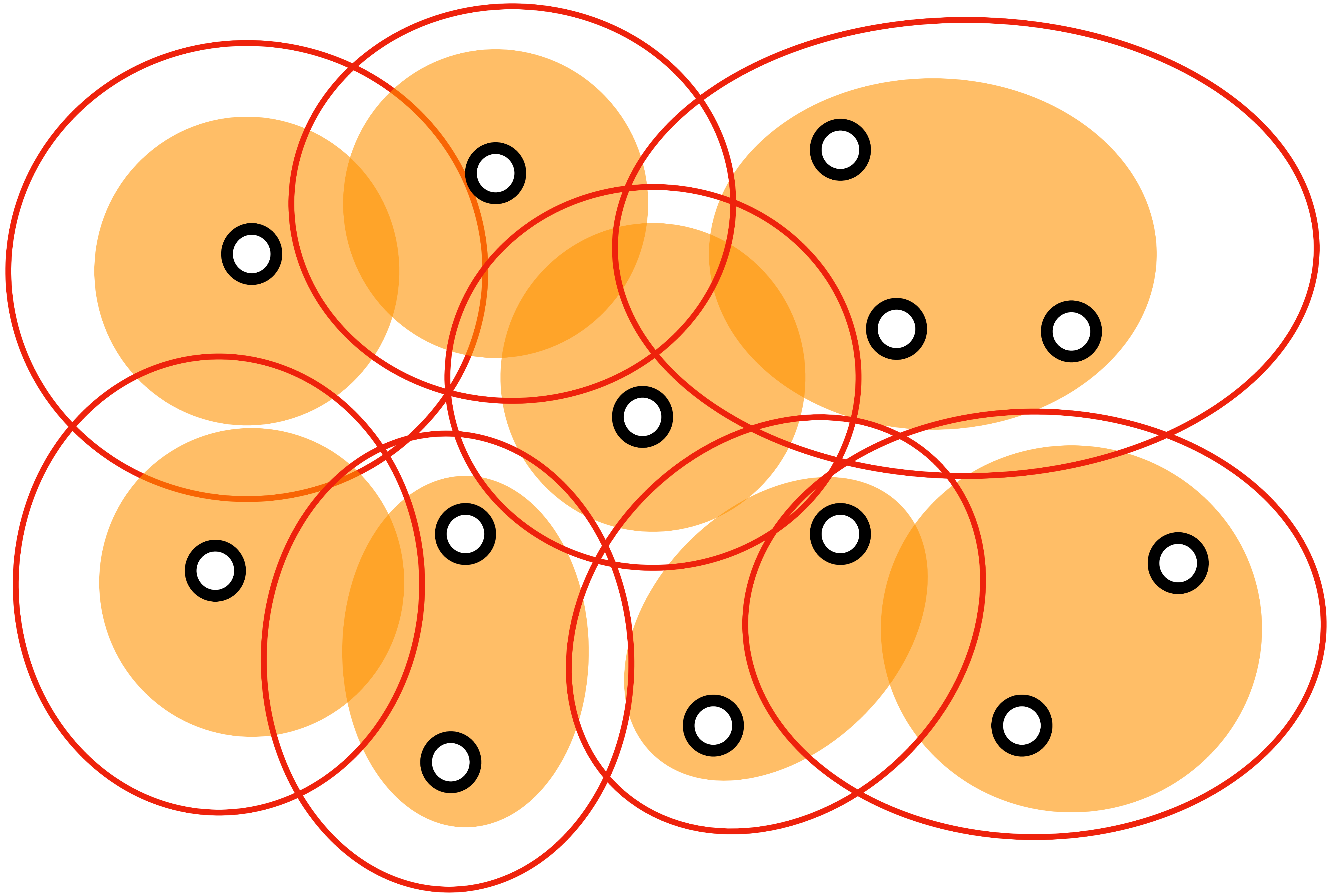

Total size of $\{B(c,2r)\}$ is almost-linear

Covering shortest paths

- For any distance scale $D = 2,4,8,\ldots\ldots$
- Apply the ball-covering lemma with **radius** $R = \Theta(D^{1/2})$

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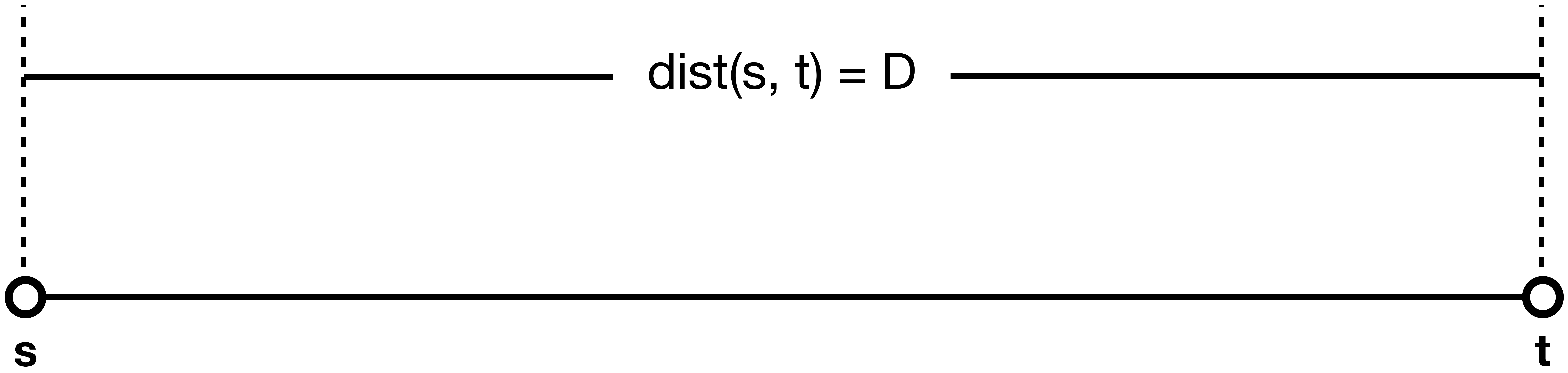

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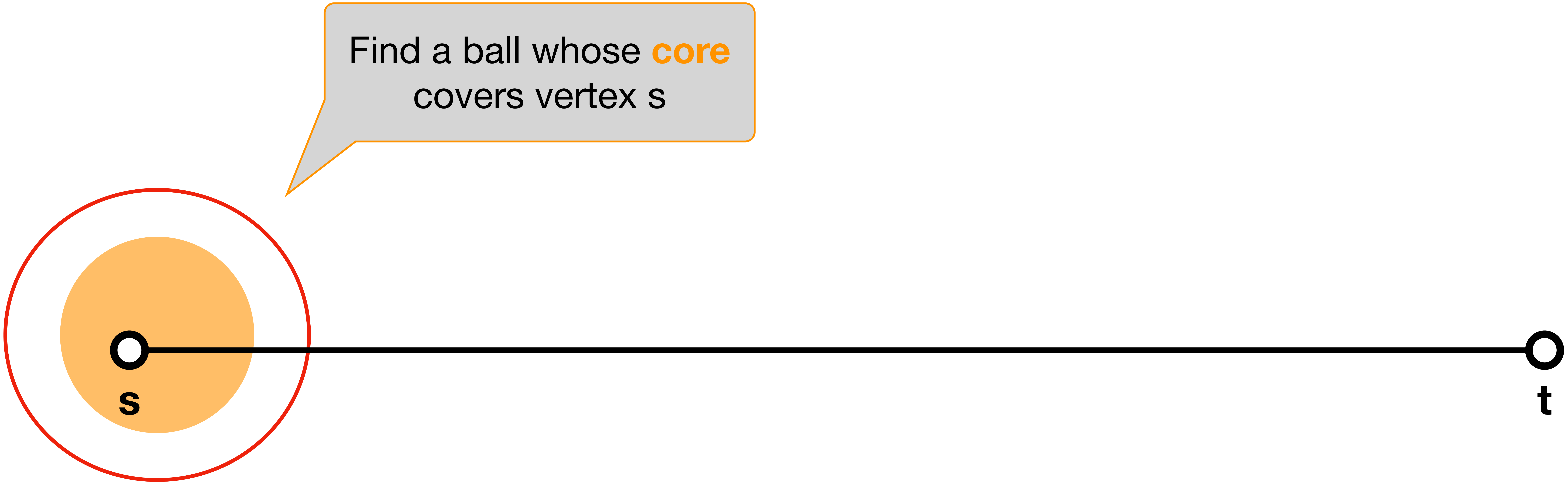

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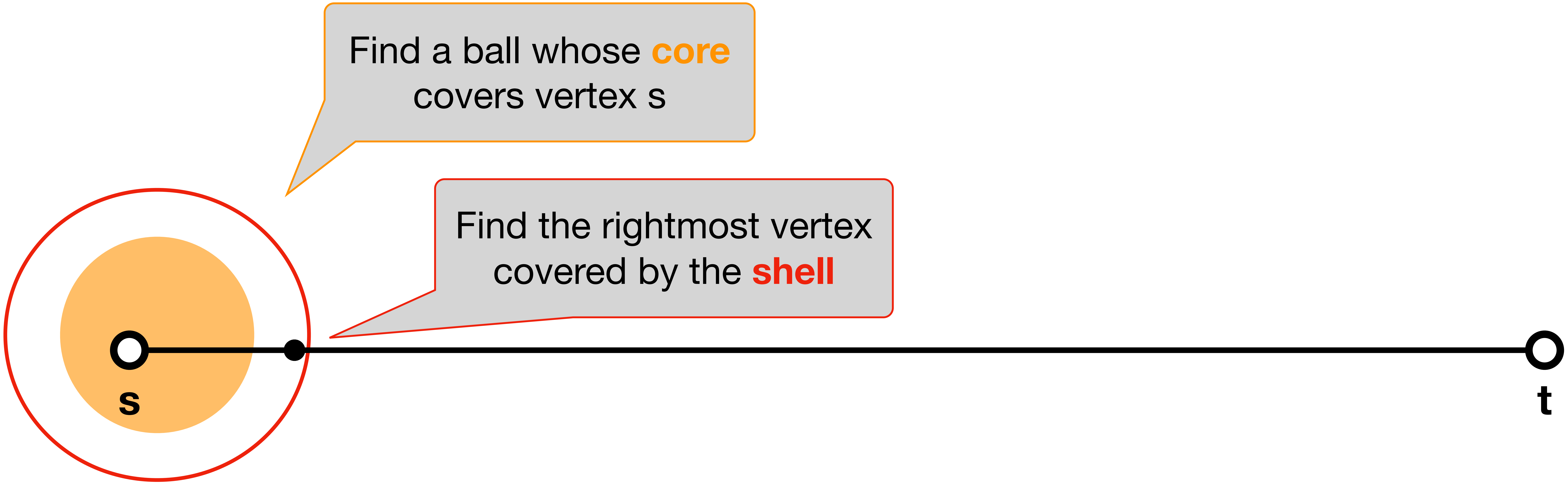

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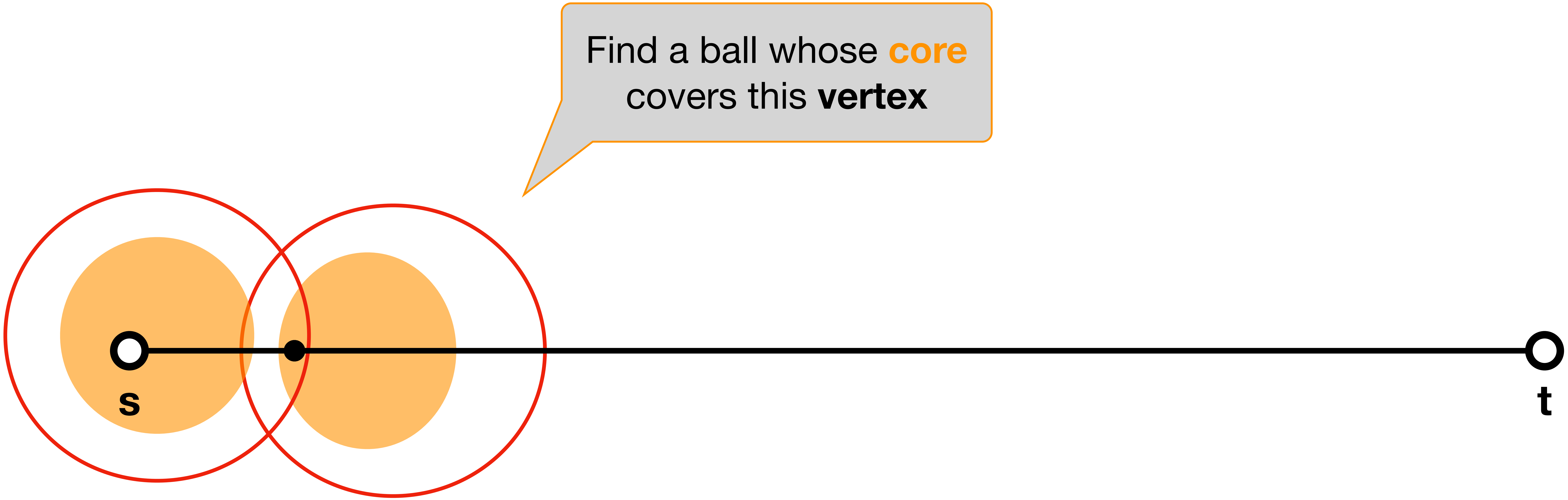

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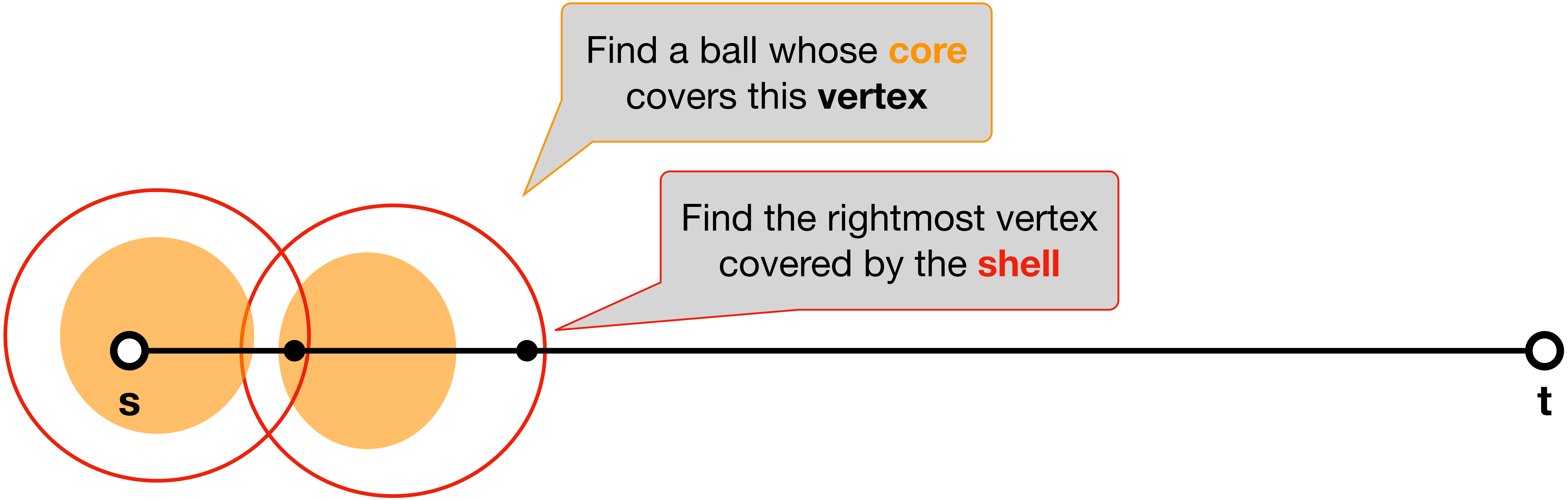

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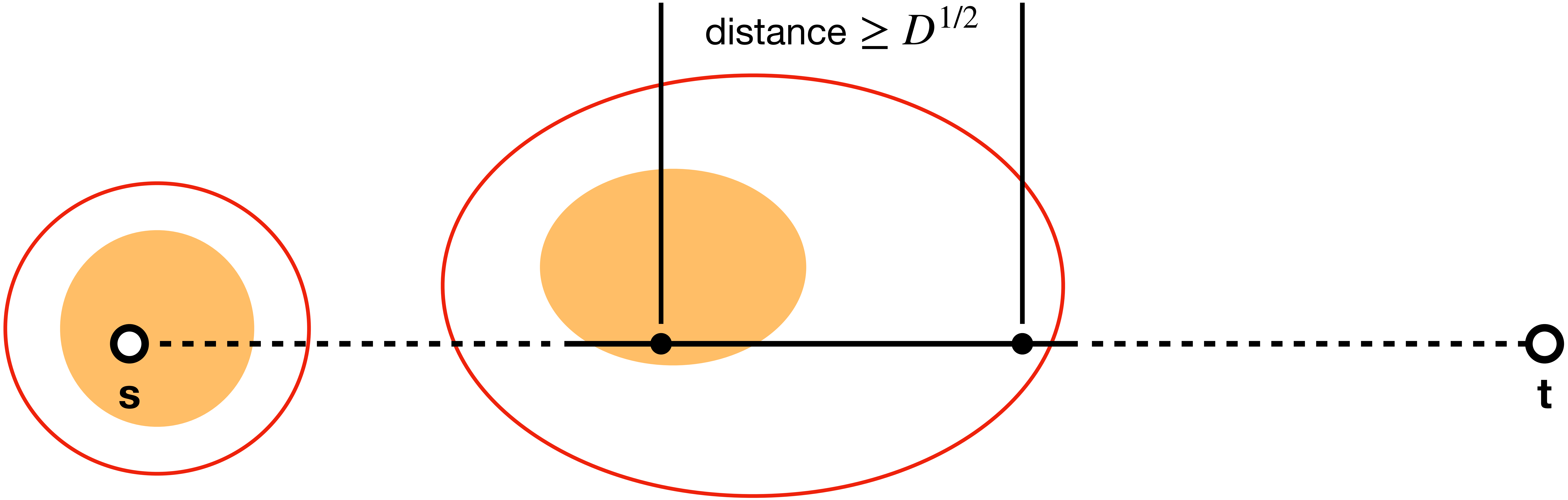

Covering shortest paths

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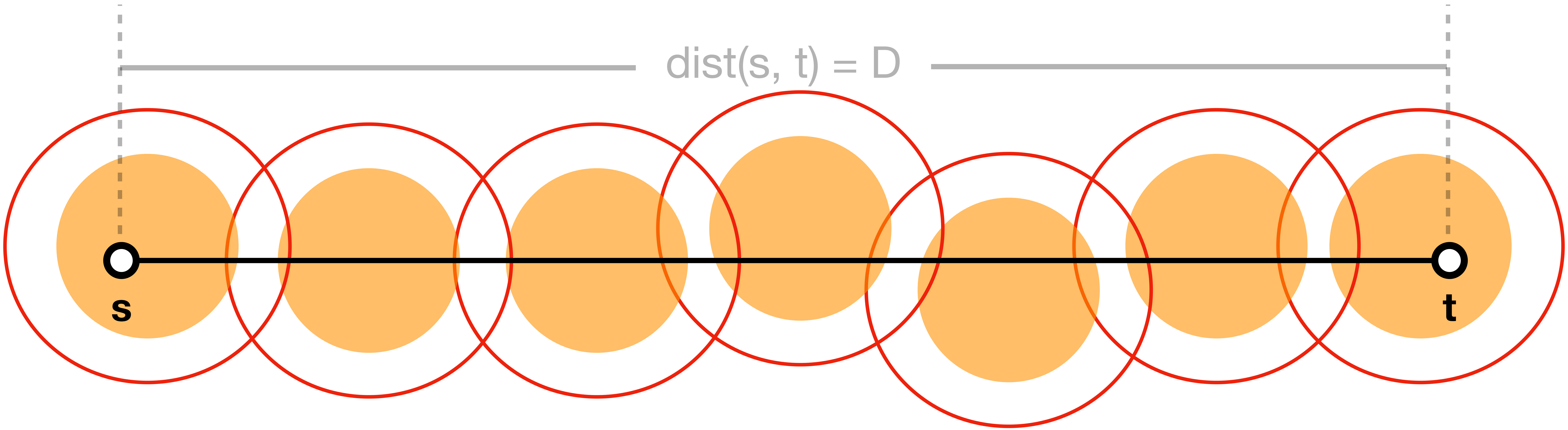

Spanner construction

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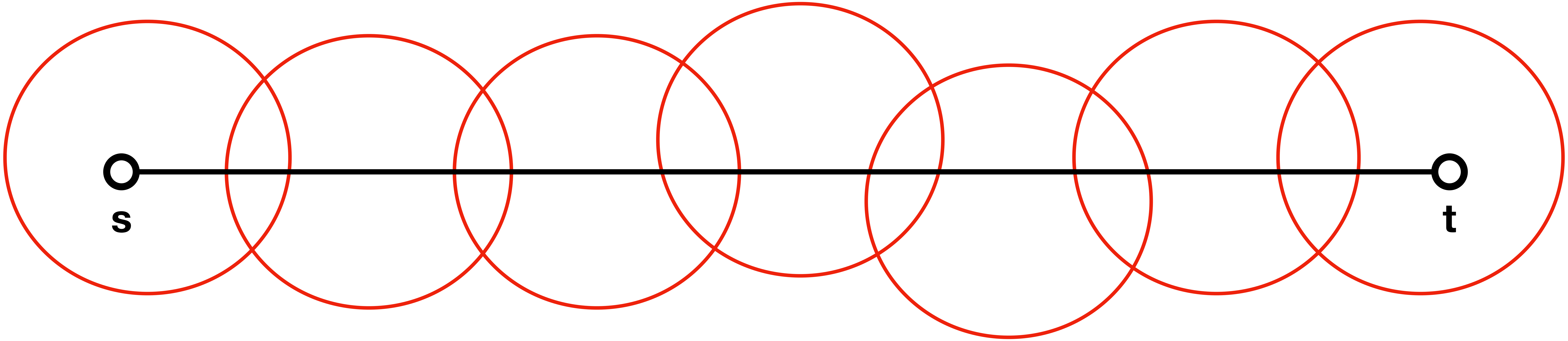
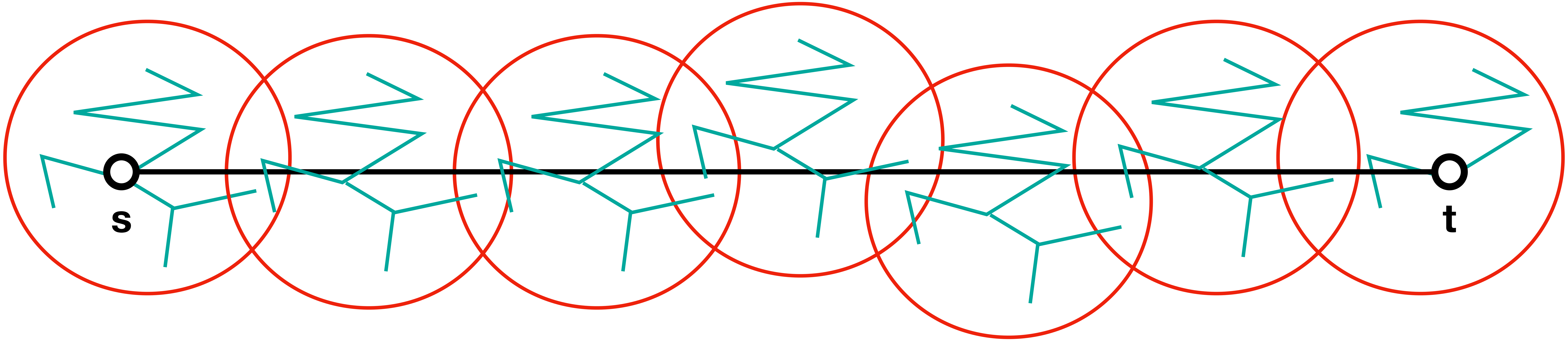


Spanner construction

- **Observation:** The **#balls** below is bounded by $D^{1/2}$
- Build a +6 additive spanner within each ball $B(c, 2r)$ of size $|B(c,2r)|^{4/3}$



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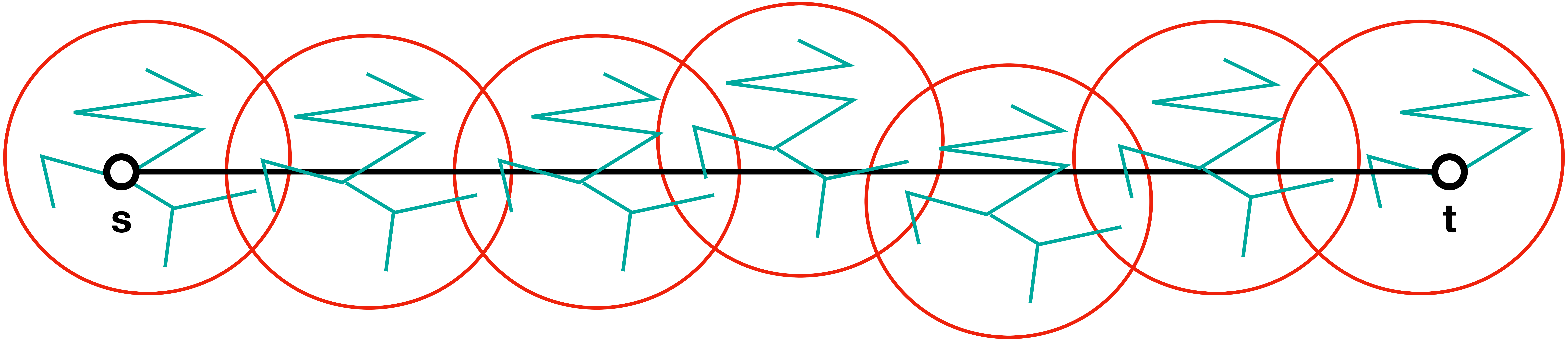

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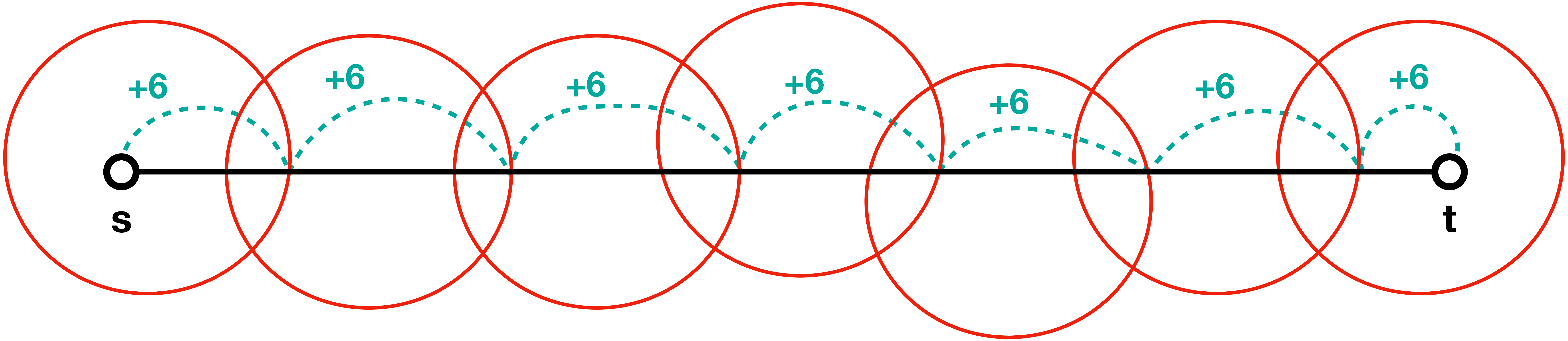
- Total stretch is $\leq 6 \cdot \#\text{balls} = O(D^{1/2})$



Handling large balls

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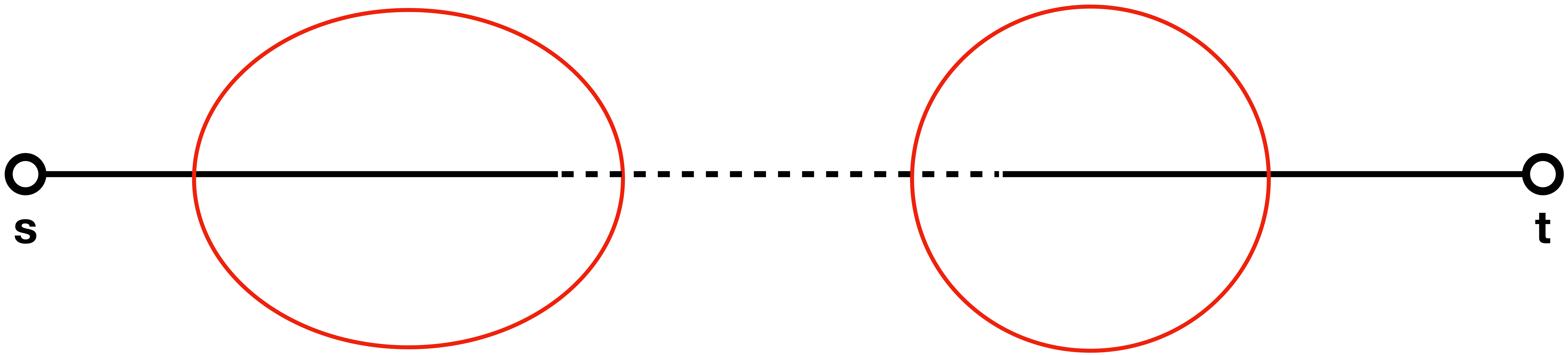

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- ~~**Assumption:**~~ If each ball $|B(c,2r)| < n^{3/7}$, then spanner size $< n^{8/7}$
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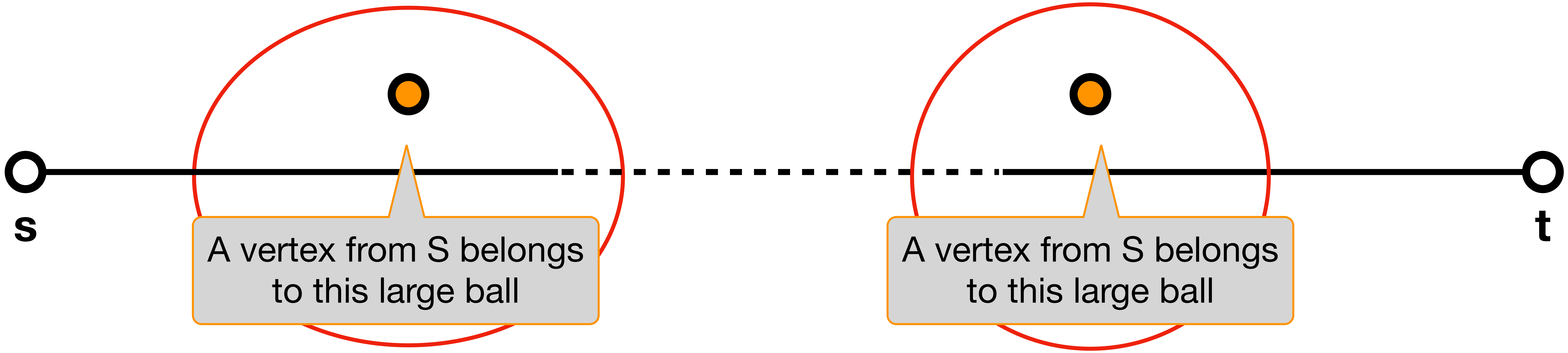

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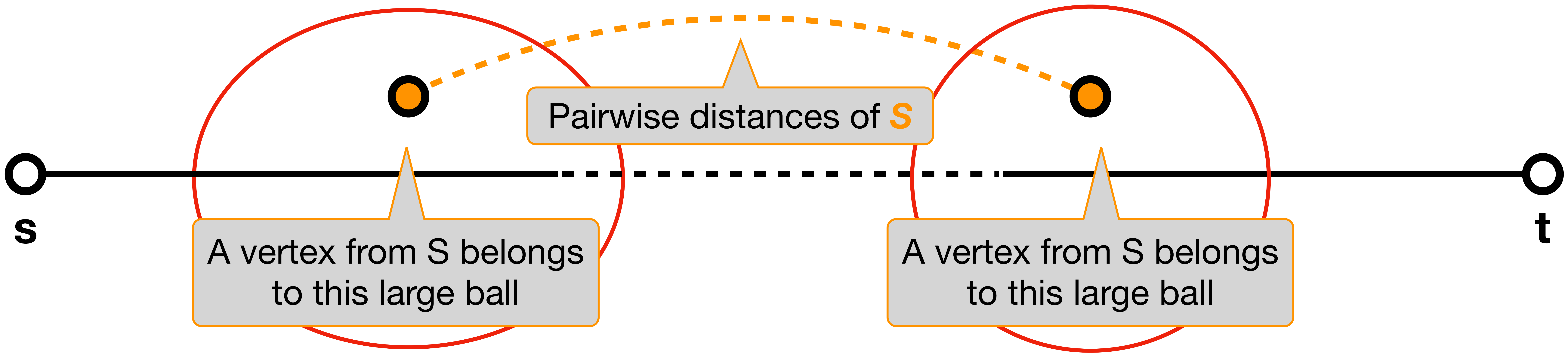

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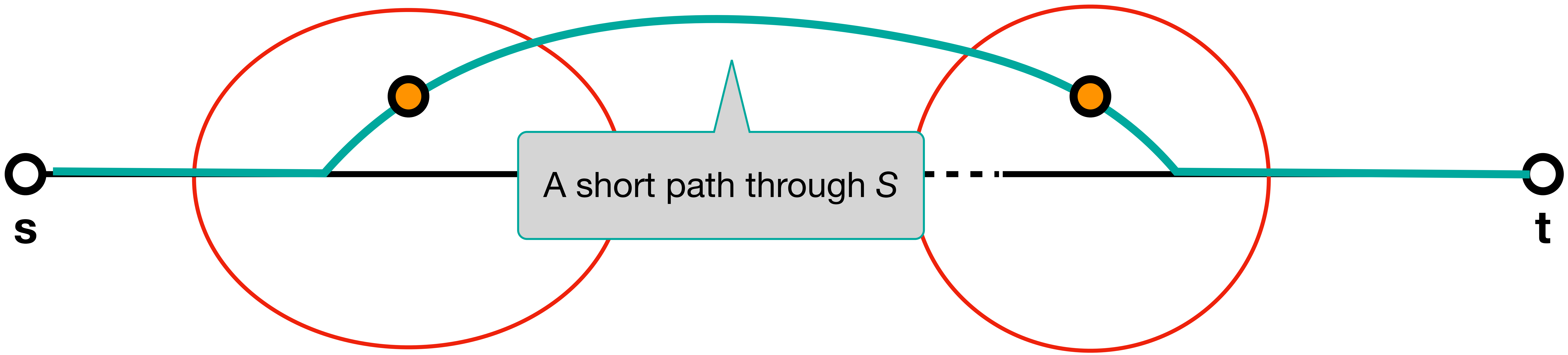
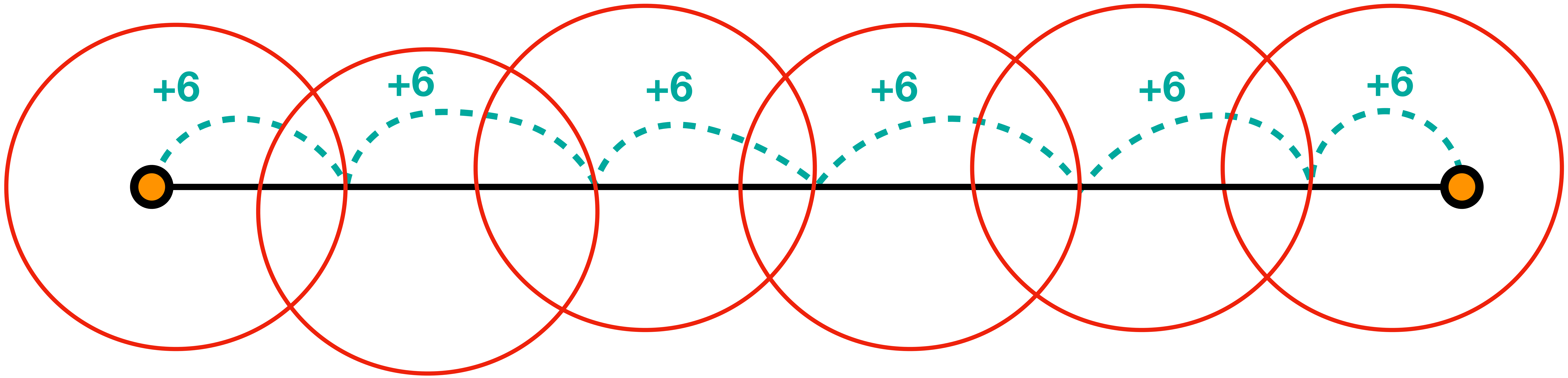
- A random set S of size $10n^{4/7} \log n$ **hits all large balls**
- Only need to preserve pairwise distances among vertices in S
- Route *s* to *t* **through the two hit**ting vertices in S



Reduction to pairwise spanners

Goal: Approximately preserve pairwise distances in $S \subset V$



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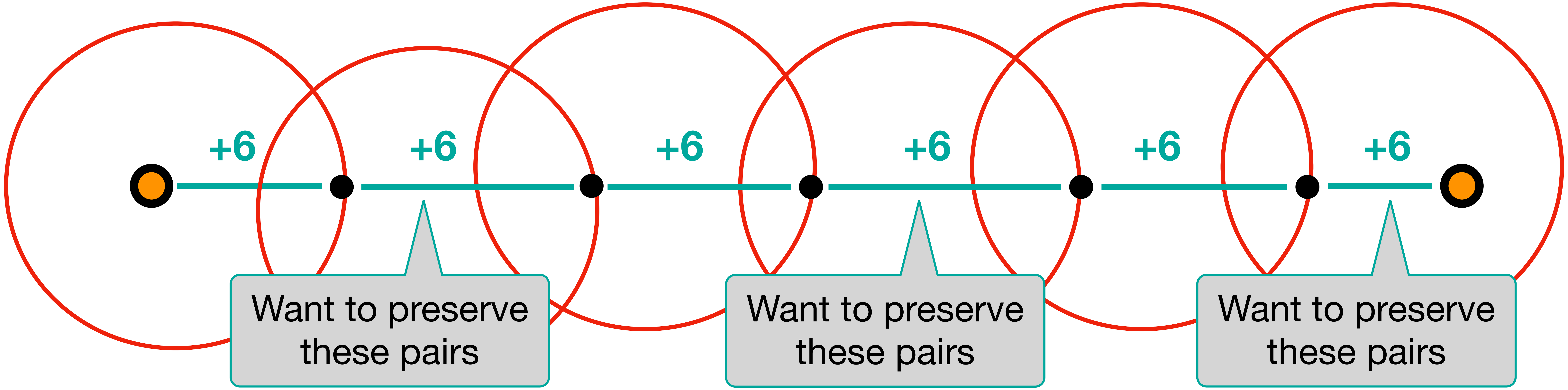

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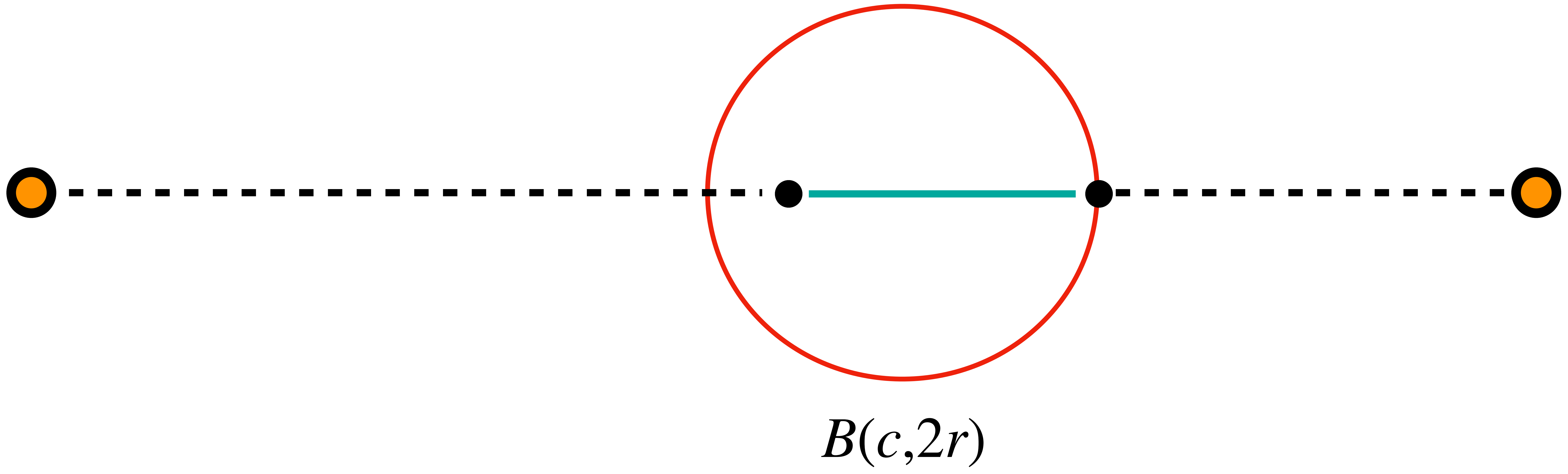

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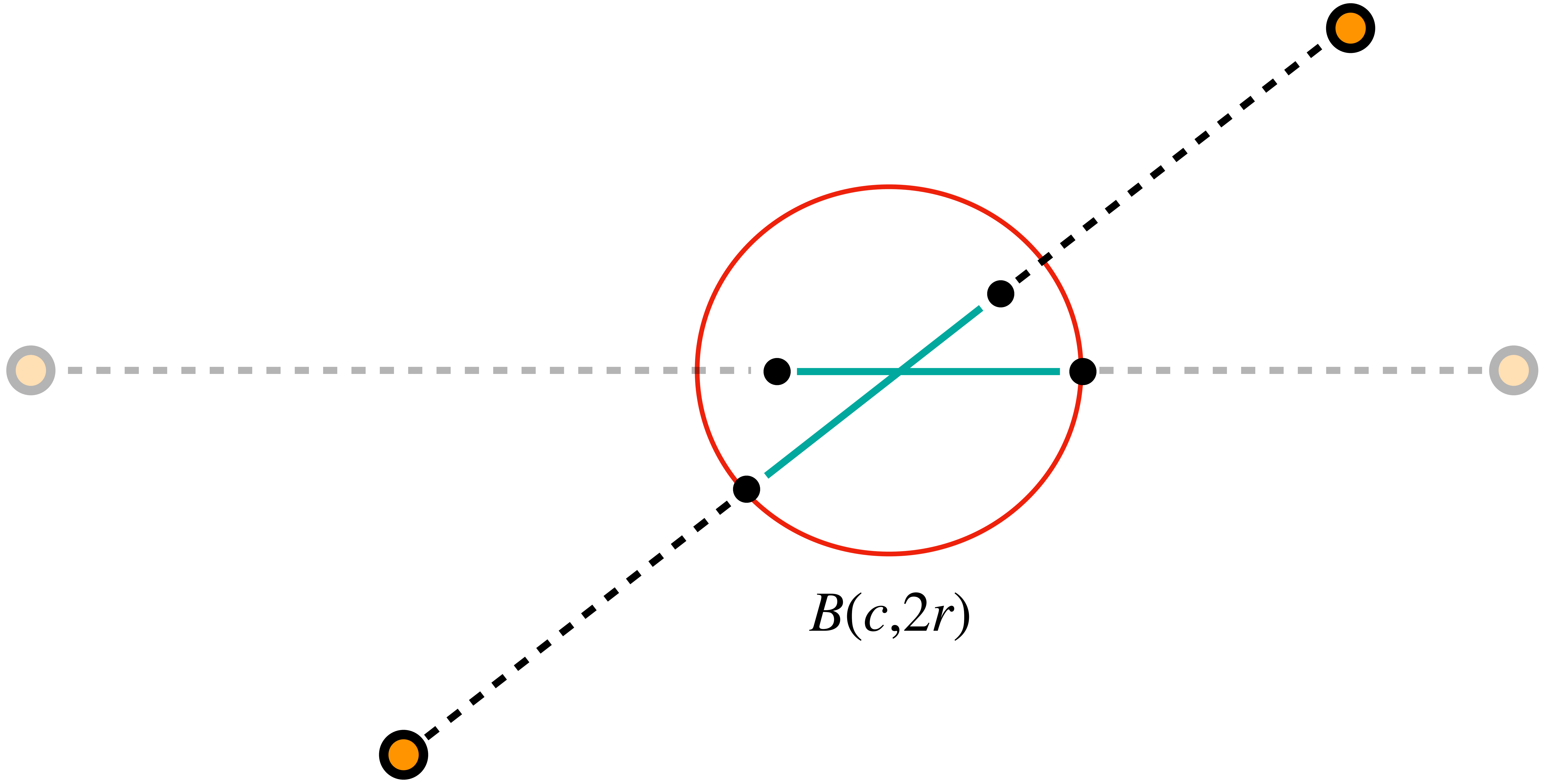

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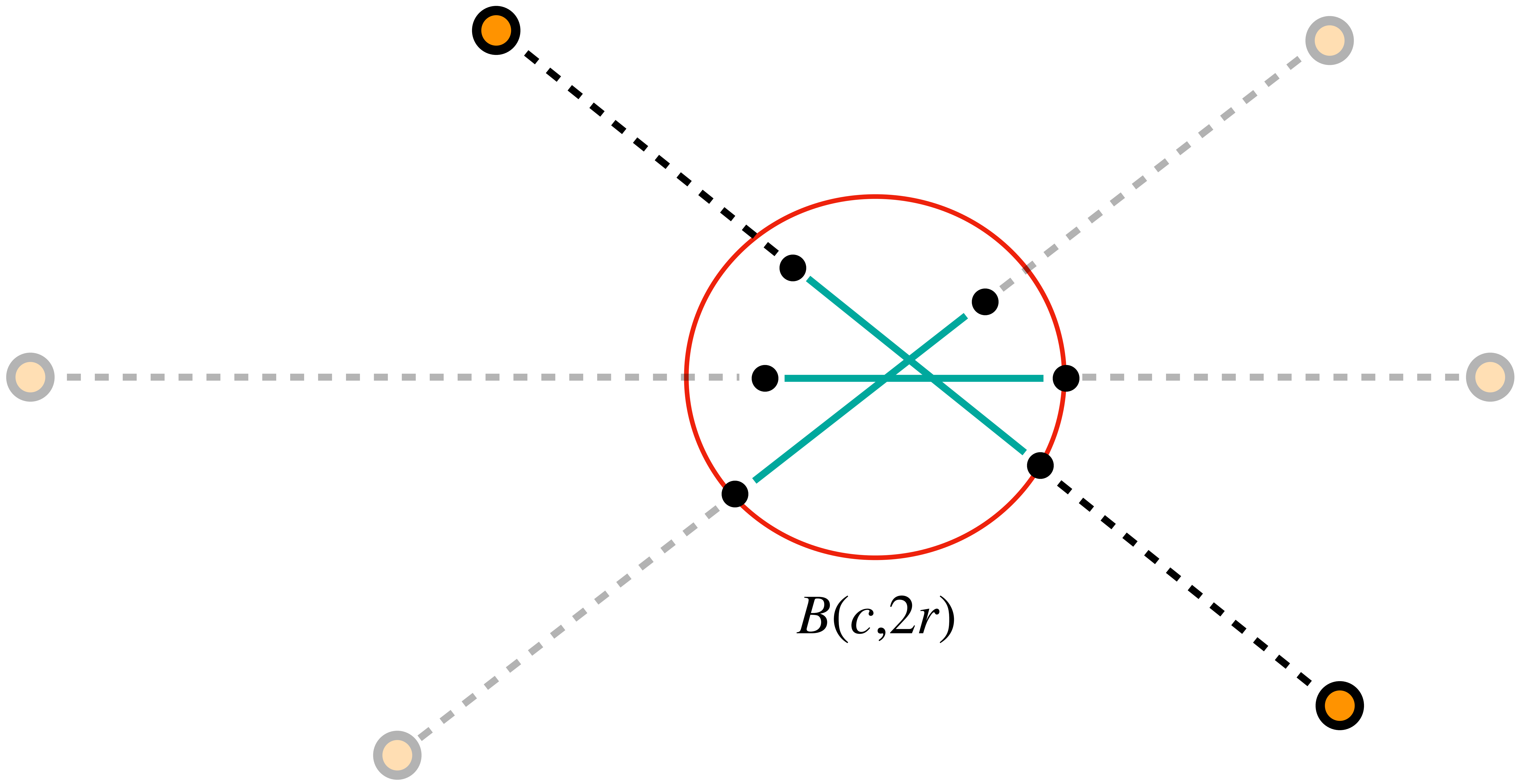

Reduction to pairwise spanners

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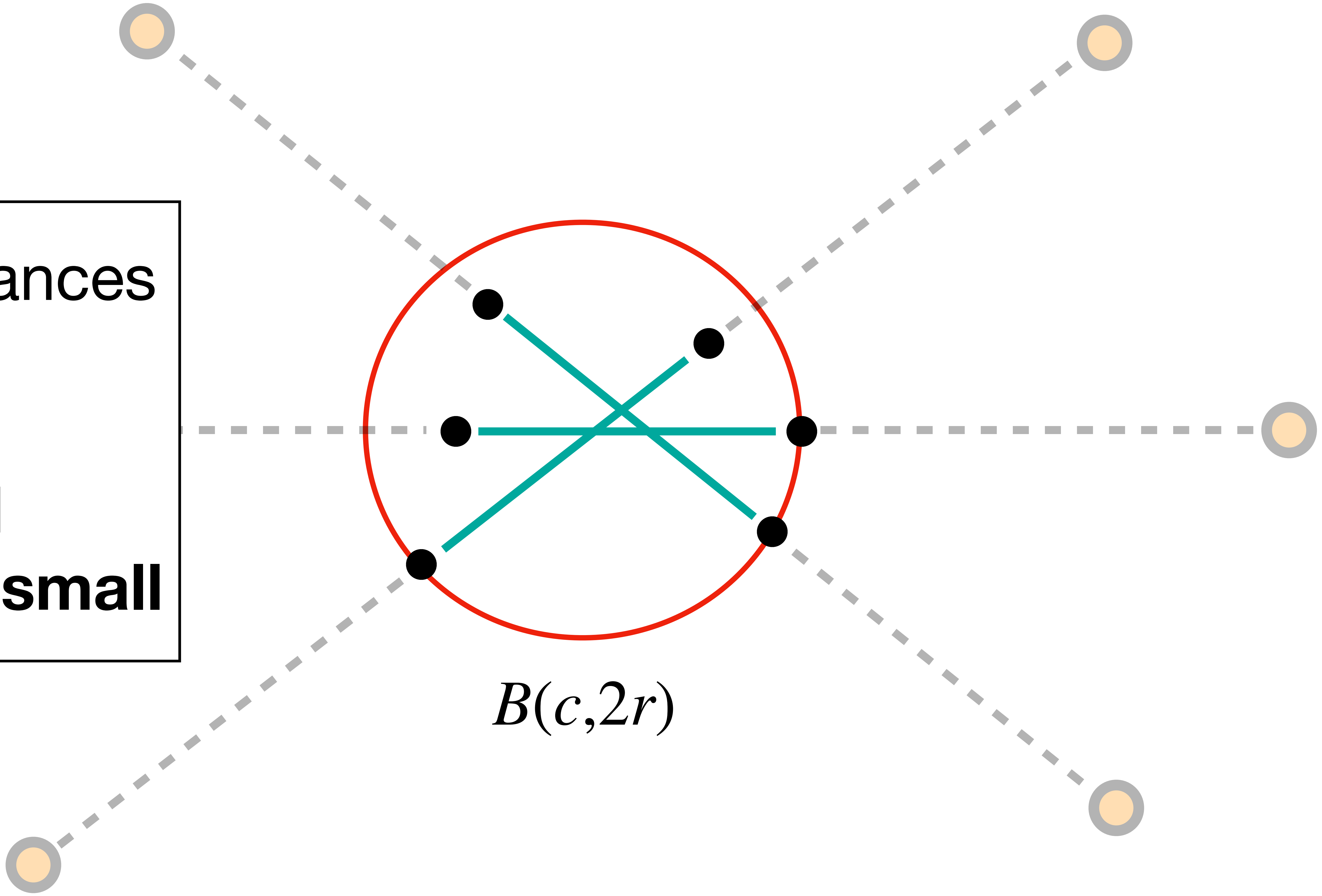
Want to preserve distances between **these pairs**

Weaker than standard spanners, if **#pairs is small**



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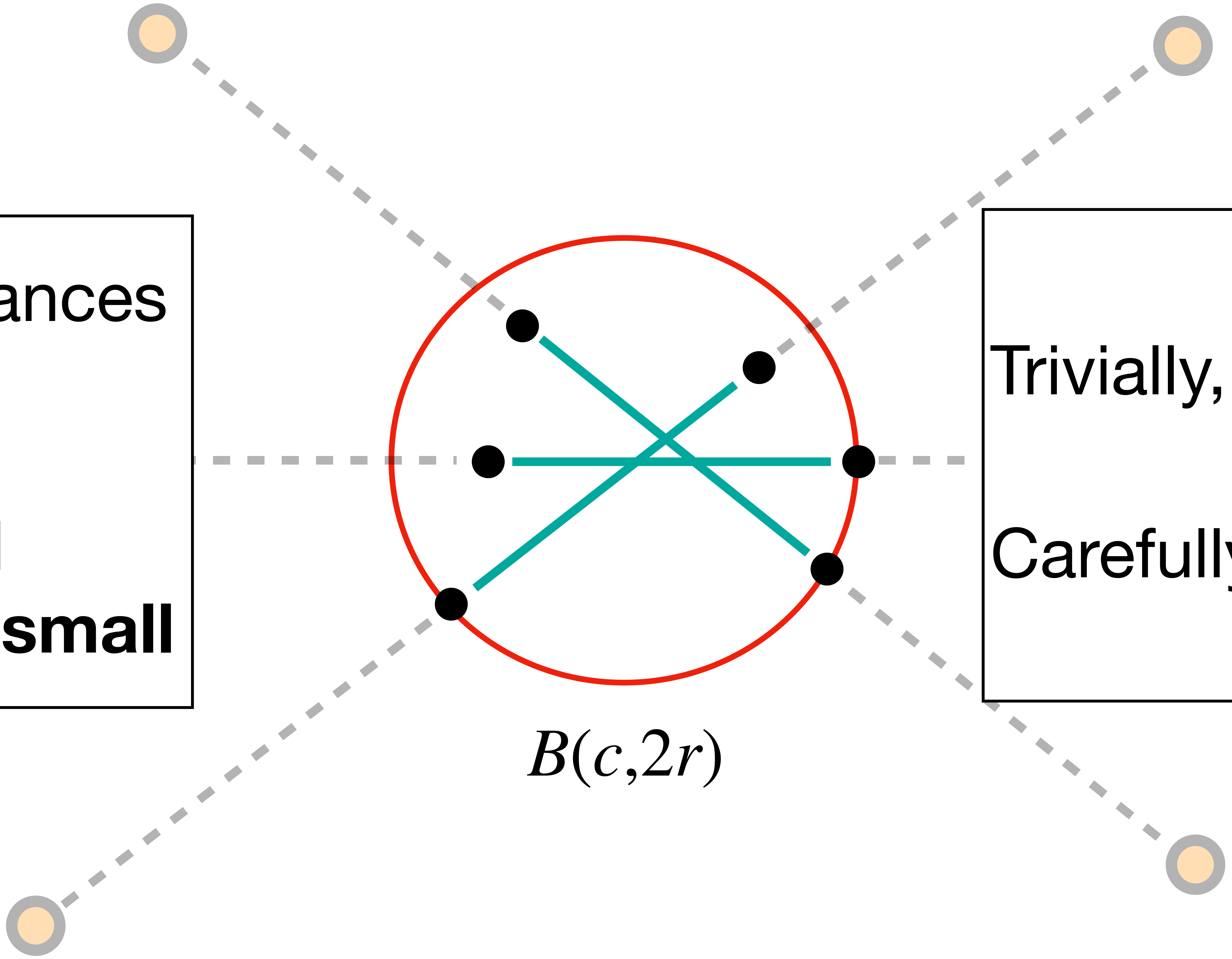


Trivially, #pairs $\leq |S|^2$

Carefully, #pairs $\leq |S|$

Pairwise Spanners

Definition:

Given a graph $G = (V, E)$ and a set of pairs $P \subseteq V^2$, a pairwise spanner $H \subseteq G$ has stretch function f, if $\text{dist}_H(s, t) \leq f(\text{dist}_G(s, t))$ for any $(s, t) \in P$

Theorem: [Kavitha, 2015]

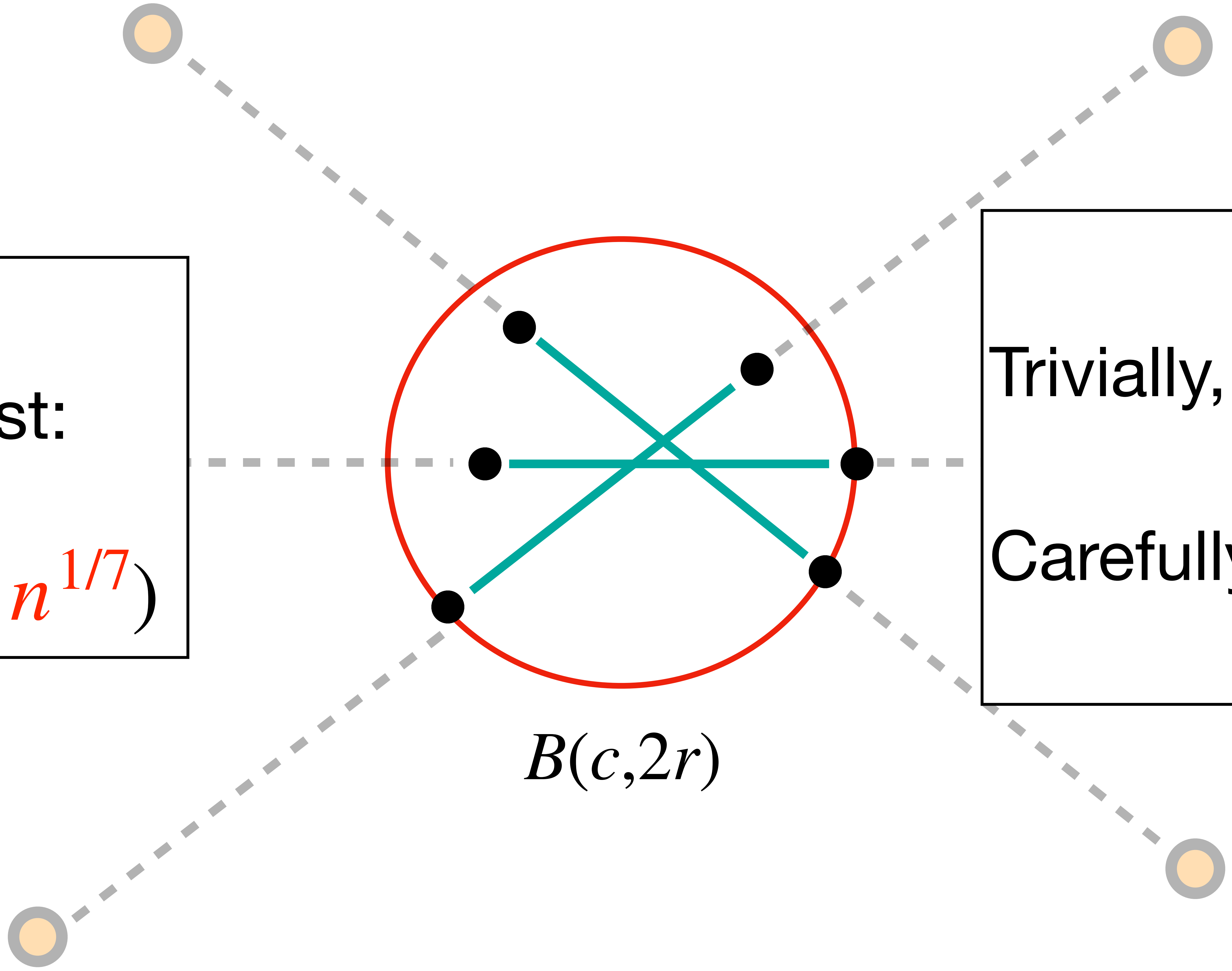
There exists a pairwise spanner for $f(d) = d+6$ with $\tilde{O}(n\,|\,P\,|^{1/4})$ edges

Reduction to pairwise spanners

<u>Goal:</u> Approximately preserve pairwise distances in $S \subset V$

Pairwise spanner within $B(c,2r)$ has edges at most:

$\tilde{O}(|B||S|^{1/4}) \leq \tilde{O}(|B|n^{1/7})$



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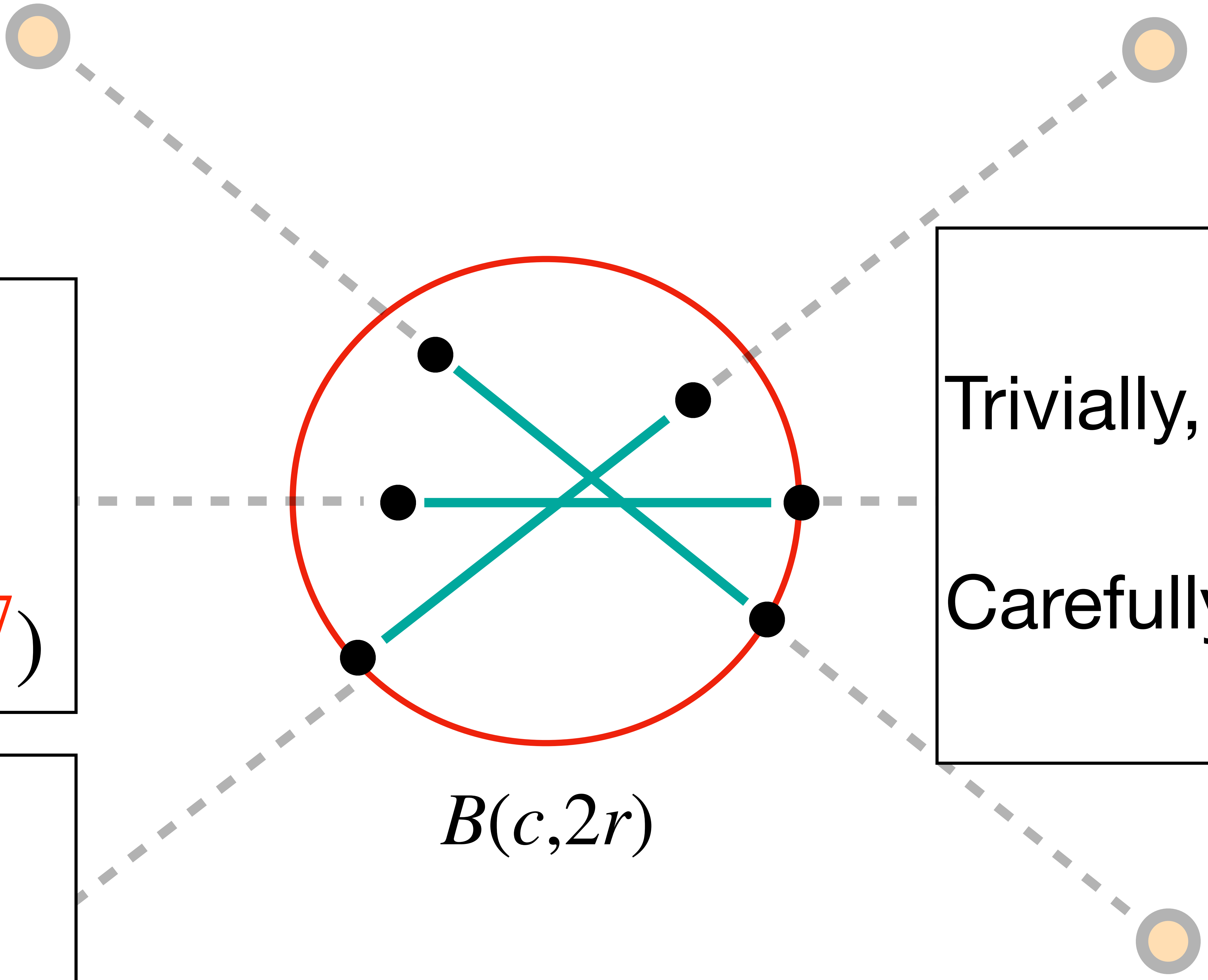


$B(c,2r)$

Trivially, #pairs $\leq |S|^2$

Carefully, #pairs $\leq |S|$

Total spanner size:

$$\sum_{B(c,2r)} |B|n^{1/7} = n^{8/7+\epsilon}$$

Today's plan

Sublinear additive spanners:

- Example: $f(d) = d + O(d^{1/2}), |E(H)| = n^{8/7+\epsilon}$
- Sketch: $f(d) = d + O_{k,\epsilon}(d^{1-1/k}), |E(H)| = n^{1+\frac{1+\epsilon}{2^{k+1}-1}}$

Pairwise Sublinear Additive Spanners

Theorem: [Kavitha, 2015]
There exists a pairwise spanner for $f(d) = d + 6$ with $\tilde{O}(n|P|^{1/4})$ edges

- This leads to $f(d) = d + O(d^{1/2})$ spanners with $n^{8/7+\epsilon}$ edges

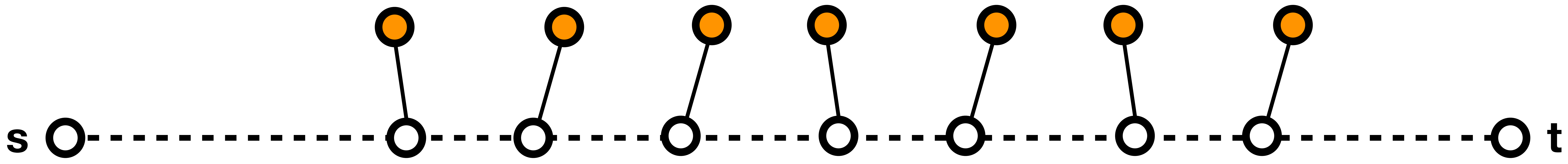
- In general, we want $f(d) = d + O(d^{1-1/k})$ spanners with $n^{1+\frac{1+\epsilon}{2^{k+1}-1}}$ edges

Missing component:
A pairwise spanner for $f(d) = d + O(d^{1-\frac{1}{k-1}})$ with $\tilde{O}(n|P|^{1/2^k})$ edges

A path-buying scheme [Kavitha, 2015]

Algorithm: [Kavitha, 2015]
Construct a pairwise spanner **H** for $f(d) = d + 6$ with $\tilde{O}(n|P|^{1/4})$ edges

1. Add to **H** all edges incident on low-degree vertices (degree $\leq |P|^{1/4}$)
2. Take a random set of size $10n \log n / |P|^{1/4}$ that dominates all high-degree vertices



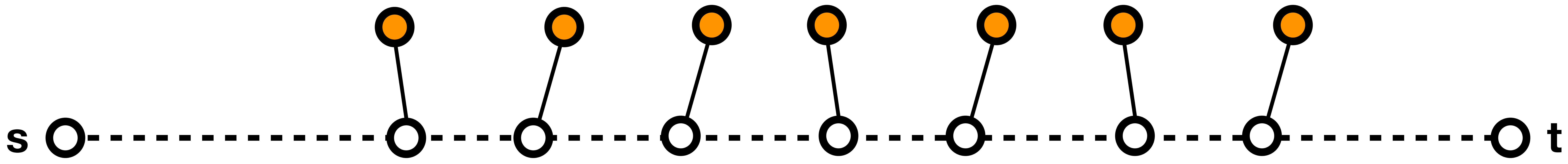
A path-buying scheme [Kavitha, 2015]

Algorithm: [Kavitha, 2015]

Construct a pairwise spanner **H** for $f(d) = d + 6$ with $\tilde{O}(n|P|^{1/4})$ edges

3. Pivot vertices **u, v** are called "settled", if the their distance is preserved

$$\text{dist}_H(u, v) \leq \text{dist}_G(u, v) + 2$$



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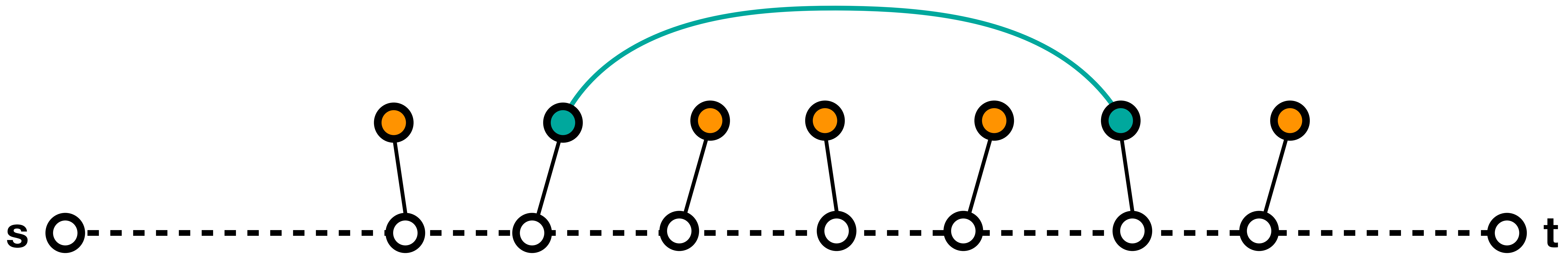
A good path in H



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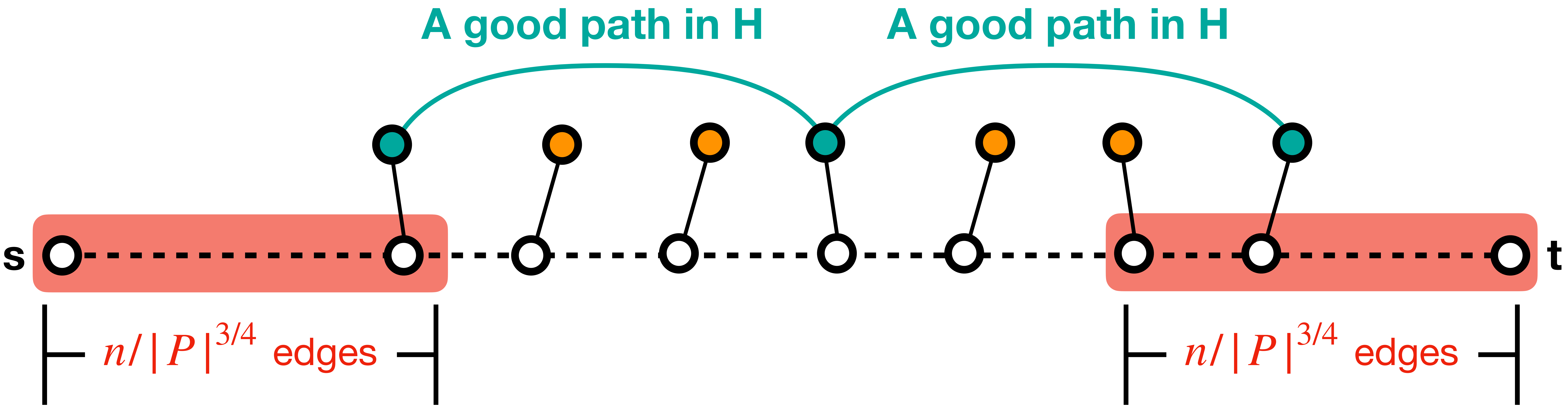
Construct a pairwise spanner **H** for $f(d) = d + 6$ with $\tilde{O}(n|P|^{1/4})$ edges

4. If many **pivot** vertices are settled, then we can add $n/|P|^{3/4}$ edges to reach a **bridge structure**



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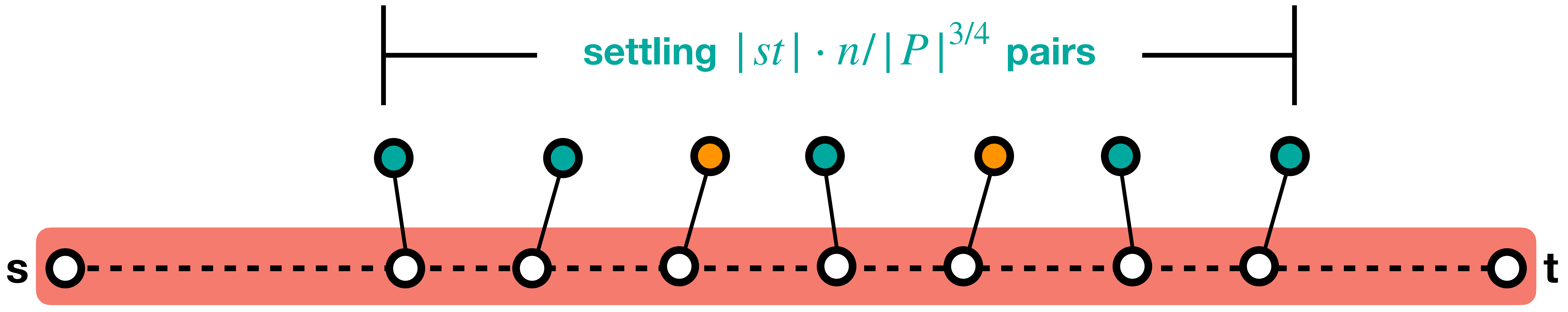
Construct a pairwise spanner **H** for $f(d) = d + 6$ with $\tilde{O}(n|P|^{1/4})$ edges

5. If many **pivot** vertices are not settled, then we can add the entire s-t path to H, settling at least $|st| \cdot n/|P|^{3/4}$ **new pairs**



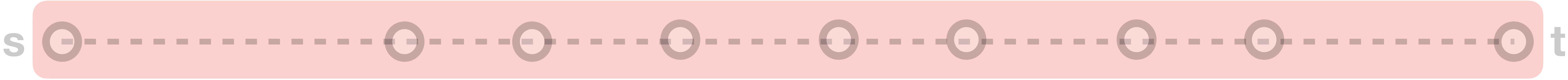
Add |st| edges to H

A path-buying scheme [Kavitha, 2015]

Algorithm: [Kavitha, 2015]
Construct a pairwise spanner **H** for $f(d) = d + 6$ with $\tilde{O}(n|P|^{1/4})$ edges

For each demand pair $(s, t) \in P$,
(1) either we add $n/|P|^{3/4}$ edges, or (2) each edge settles $n/|P|^{3/4}$ pairs on average

Total size = $|P| \cdot n/|P|^{3/4}$ + #pairs / $(n/|P|^{3/4})$ = $n|P|^{1/4}$

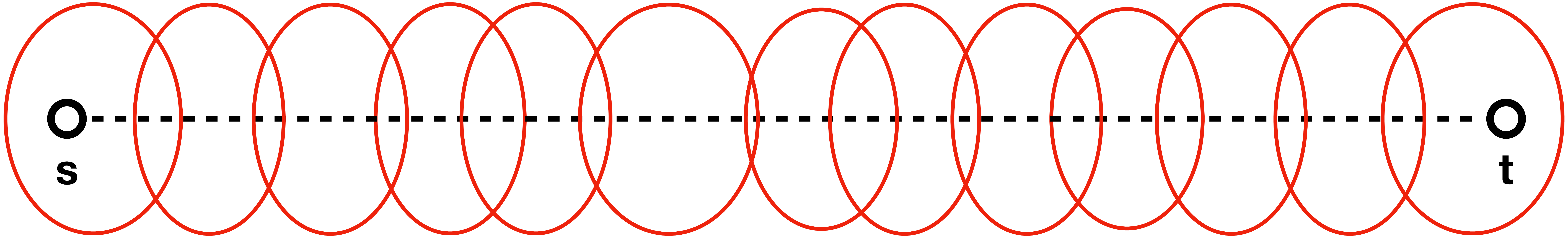


Generalizing the path-buying scheme

Algorithm:

A pairwise spanner **H** for $f(d) = d + O(d^{1-1/(k-1)})$ with $\tilde{O}(n|P|^{1/2^k})$ edges

1. Apply the ball-covering lemma with radius $R = \Theta(D^{1-1/(k-1)})$



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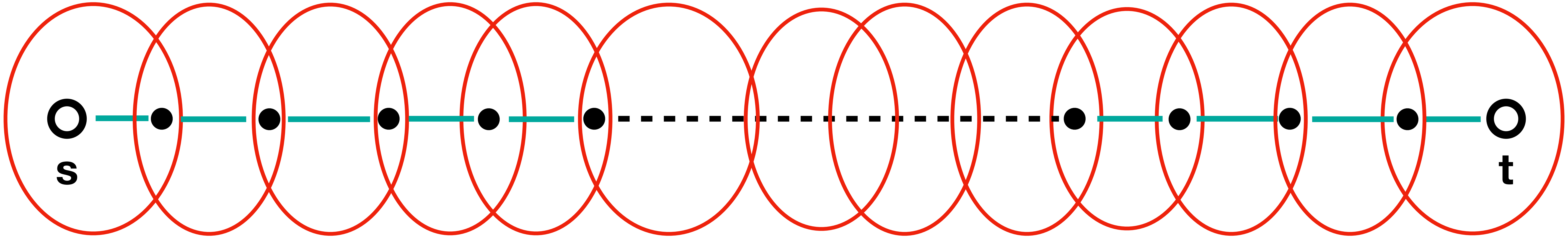
1. Apply the ball-covering lemma with radius $R = \Theta(D^{1-1/(k-1)})$

2. Decompose the st-path into **demand pairs**



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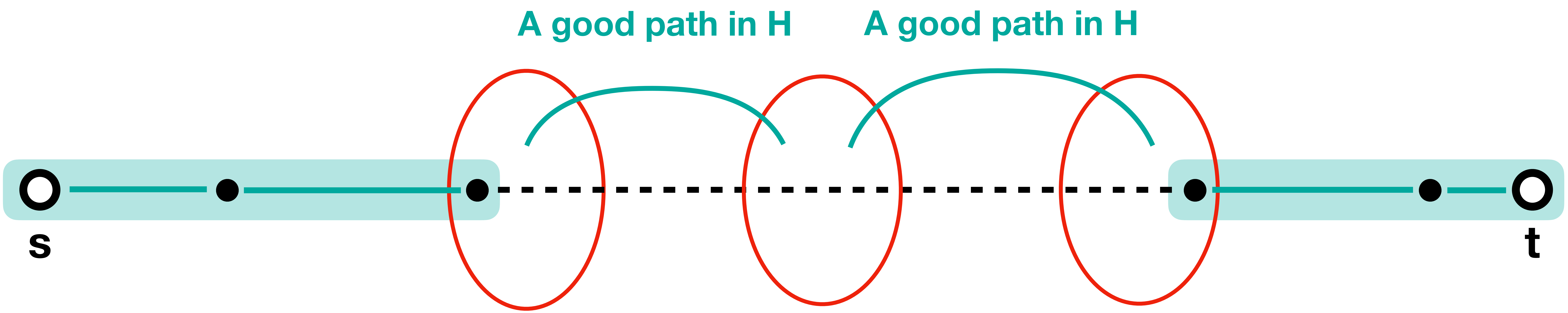
A pairwise spanner **H** for $f(d) = d + O(d^{1-1/(k-1)})$ with $\tilde{O}(n\,|\,P\,|^{1/2^k})$ edges

3. If many **ball centers** are settled, then we can **add some demand pairs** to reach a **bridge structure**



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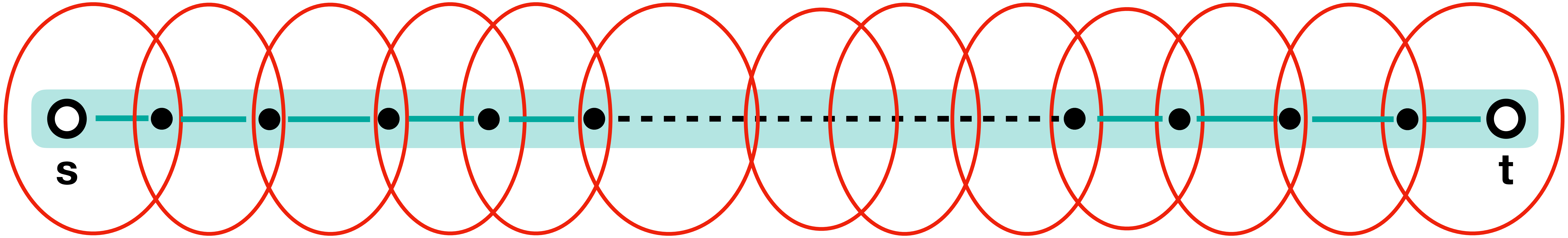
4. If many **ball centers** are not settled, then we can add all demand-pairs, settling **many new pairs** of ball centers

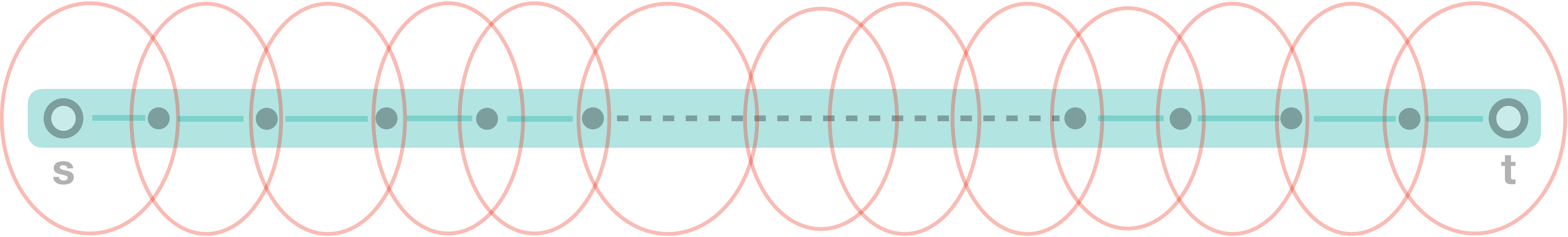
Assign all of the demand pairs to balls
Build pairwise spanners within balls **recursively**



Generalizing the path-buying scheme

Technical difficulties:

- Balls might have **different densities**.
- Divide densities into $O(1/\epsilon)$ classes, and deal with each class **separately**



Further Directions

- Our result: $f(d) = d + 2^{k^2 2^k/\epsilon} \cdot d^{1-1/k}$, size = $n^{1+\frac{1+\epsilon}{2^{k+1}-1}}$
- **<u>Question:</u>** No dependency on ϵ, better in k ?