Fully Dynamic Maximal Independent Set in Expected Poly-Log Update Time

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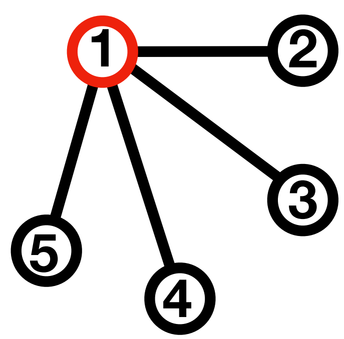
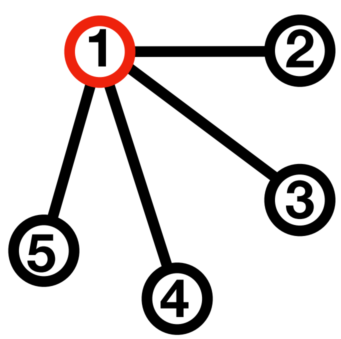
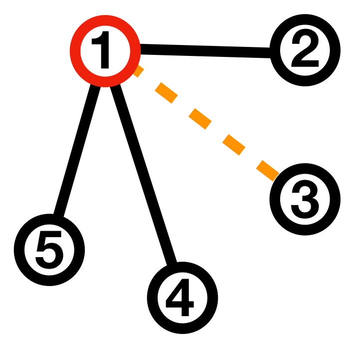
Definition: dynamic MIS

- Given an undirected simple graph G = (V, E)
- A sequence of edge ins / del by an oblivious adversary
- Explicitly maintain a maximal independent set

<u>Example:</u> dynamic MIS

Input: Updates to G				
Picture				
Output: Changes to the MIS				

Example: dynamic MIS

Input: Updates to G		Delete (1, 3)		
Picture				
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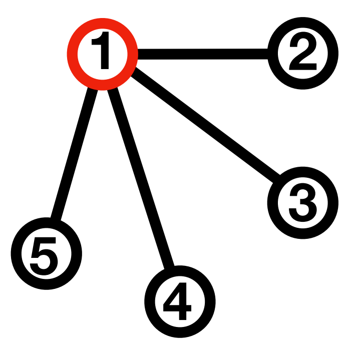
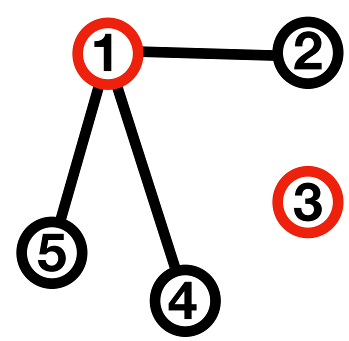
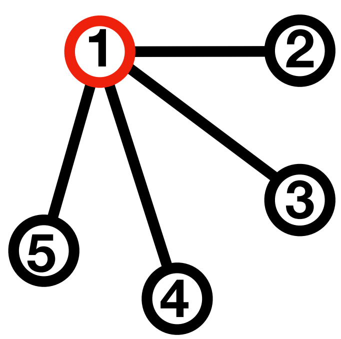
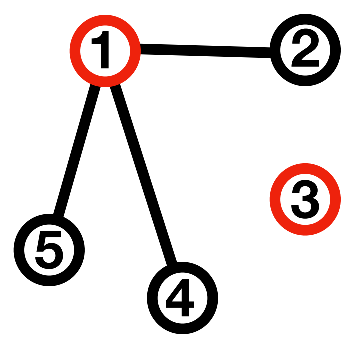
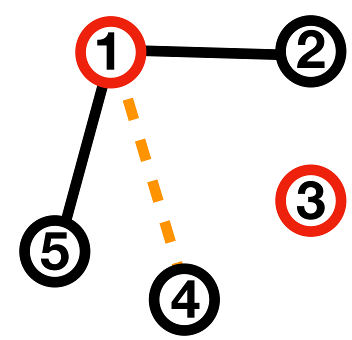
Picture				
Output: Changes to the MIS		3 joins MIS		

<u>Example:</u> dynamic MIS

Input: Updates to G		Delete (1, 3)	Delete (1, 4)	
Picture				
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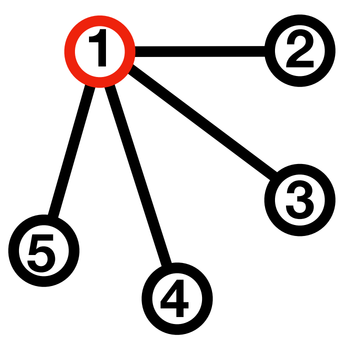
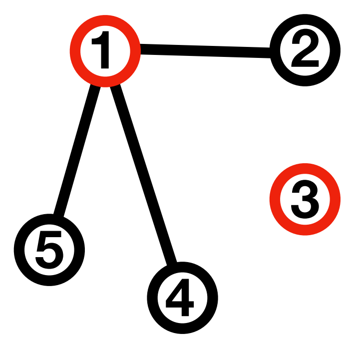
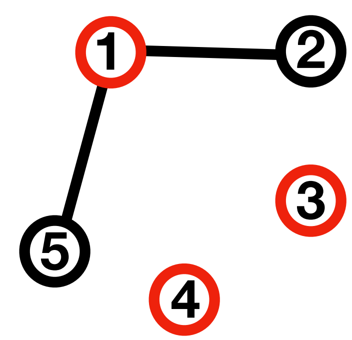
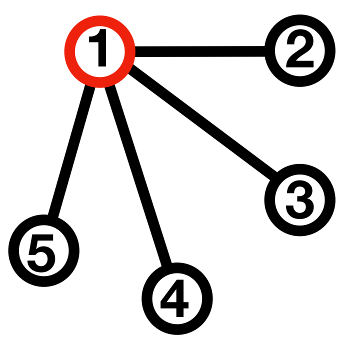
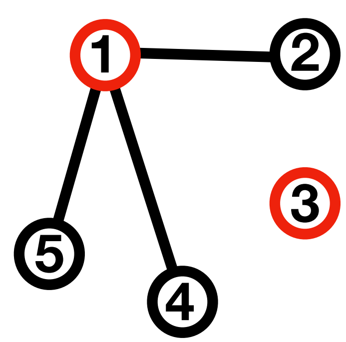
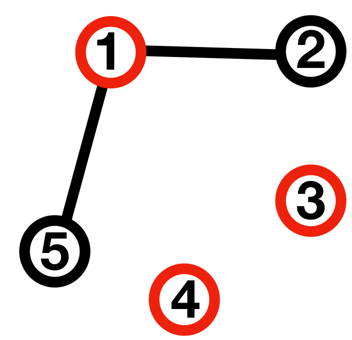
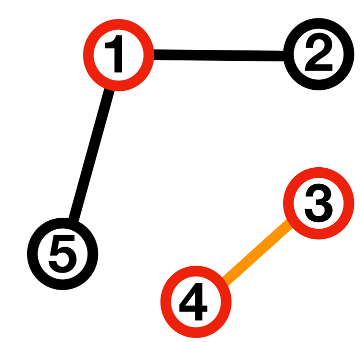
Input: Updates to G		Delete (1, 3)	Delete (1, 4)	
Picture				
Output: Changes to the MIS		3 joins MIS	4 joins MIS	

<u>Example:</u> dynamic MIS

Input: Updates to G		Delete (1, 3)	Delete (1, 4)	Insert (3, 4)
Picture				
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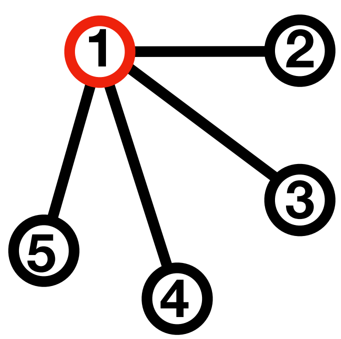
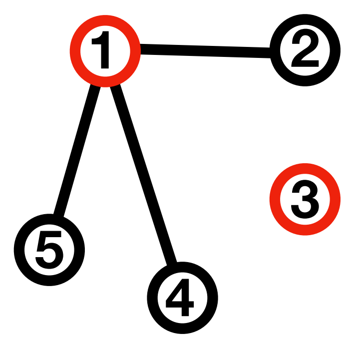
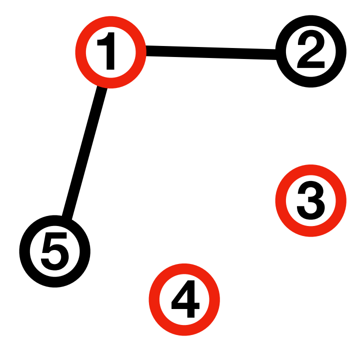
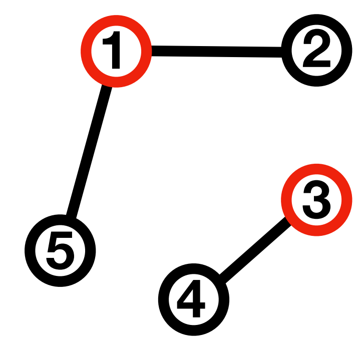
Picture				
Output: Changes to the MIS		3 joins MIS	4 joins MIS	4 leaves MIS

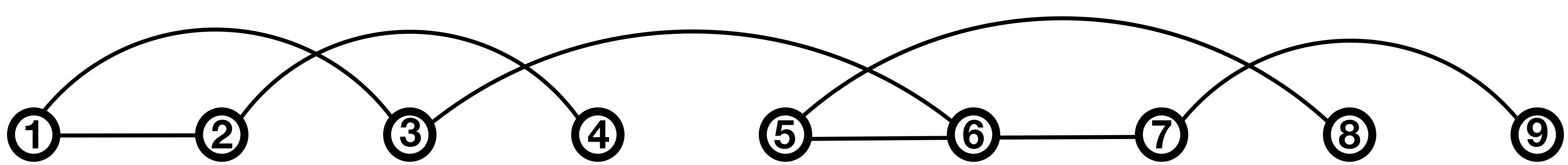
A short history

- n = # of vertices, m = # of edges, Δ = max-deg of G

Ref	[AOSS'18]	[GK'18] [DZ'18]	[DZ'18]	[AOSS'19]	Ours	[BDH+'19]
Update time	$O(m^{3/4})$	$O(m^{2/3})$	$O(m^{1/2})$	$\tilde{O}(n^{1/2})$	$O(\log^4 n)$	$O(\log^2 n \log^2 \Delta)$
Det?	Yes	Yes	No	No	No	No

<u>Definition:</u> random-greedy MIS

- Given a random order π of all vertices
- Go over each vertex in order π, add it to MIS if not yet dominated



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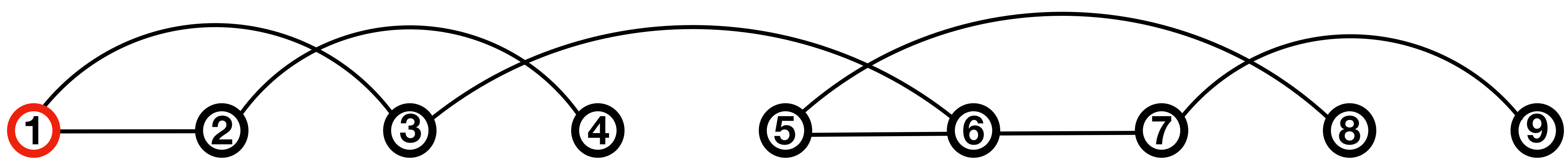

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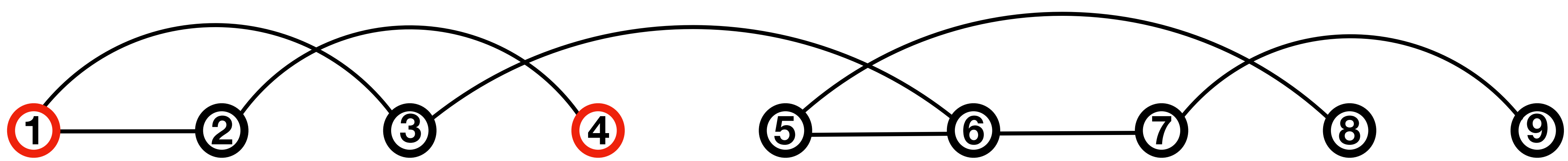

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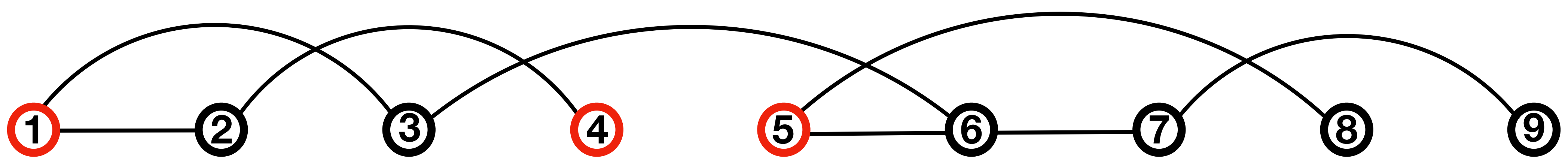

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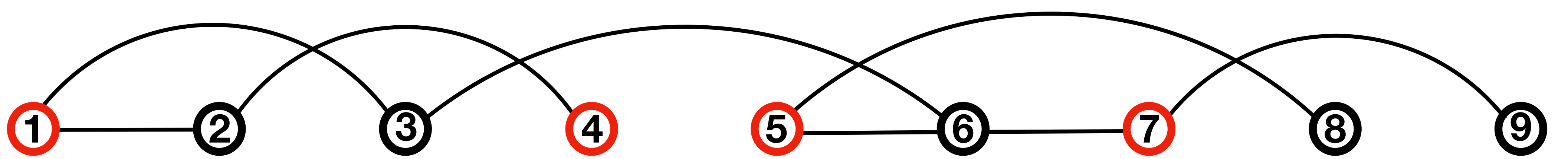

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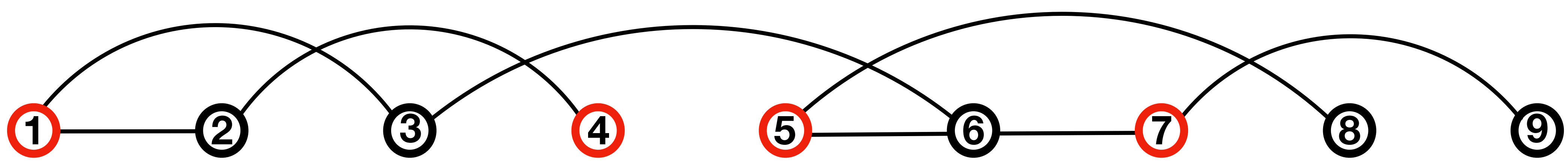

Main Theorem:

There is an algorithm that maintains a greedy MIS with respect to π in expected time $O(\log^4 n)$, expectation taken over the uniformly random choice of π

A degree-reduction lemma [AOSS'19]

Lemma:

Let $U \subseteq V$ be the set of vertices not dominated by vertices whose order $\leq k$, then $\Delta(G[U]) \leq \tilde{O}(n/k)$ w.h.p. over π

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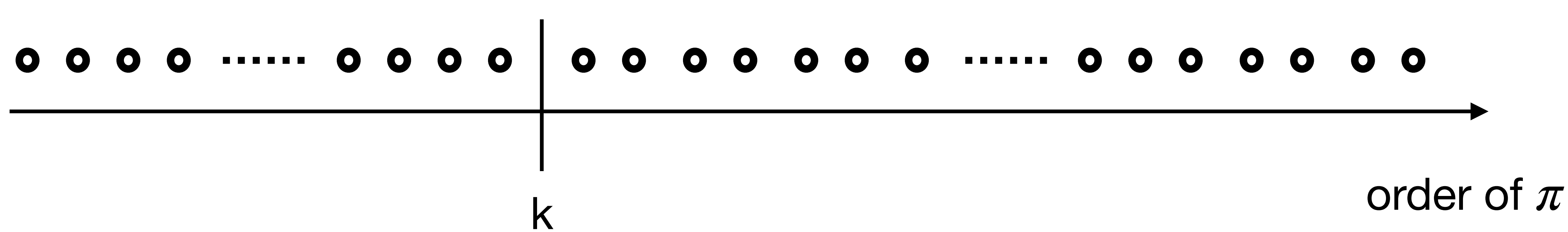

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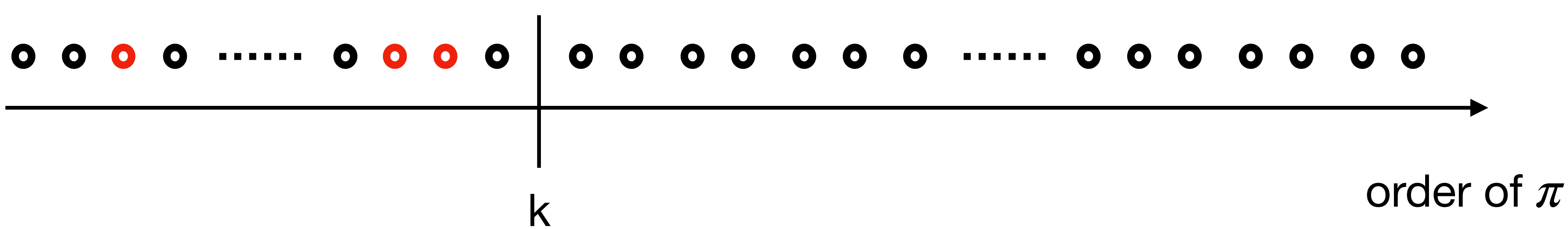

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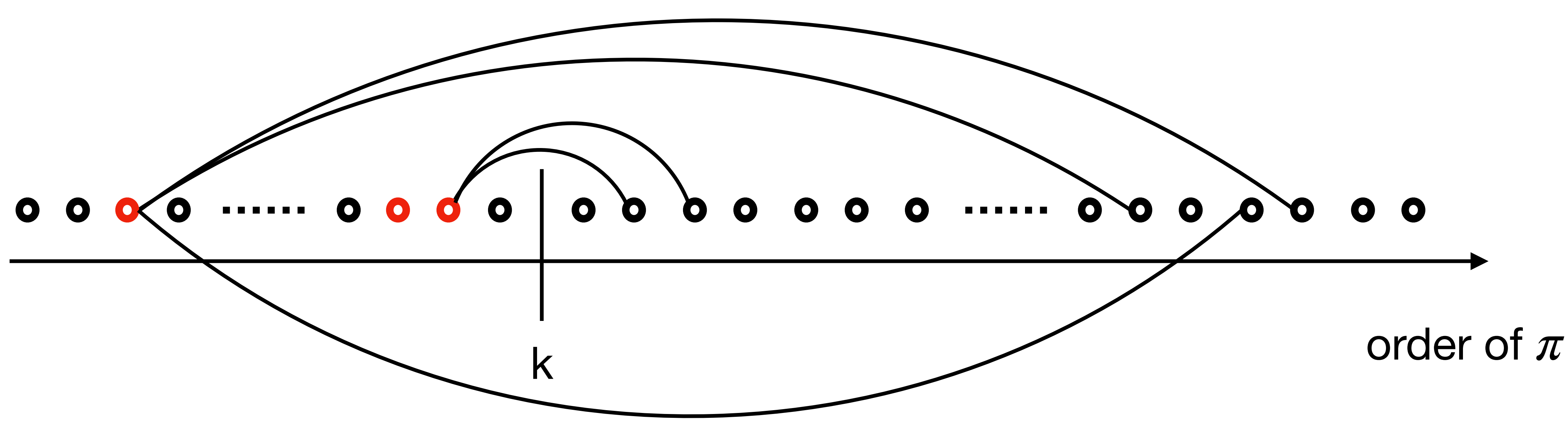

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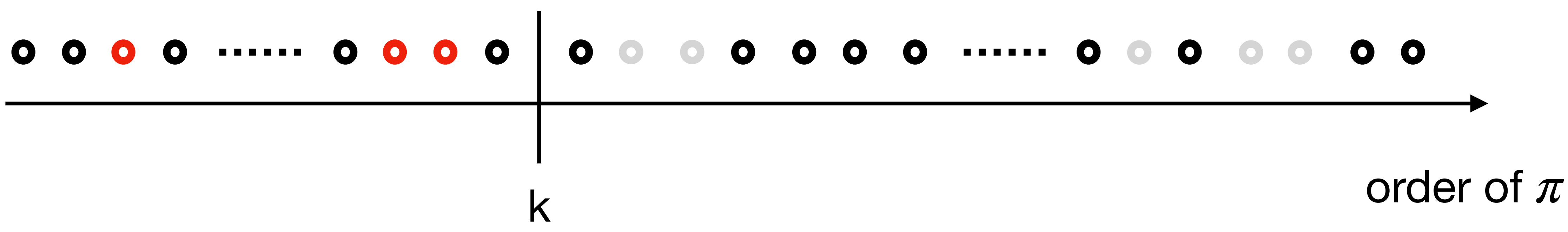

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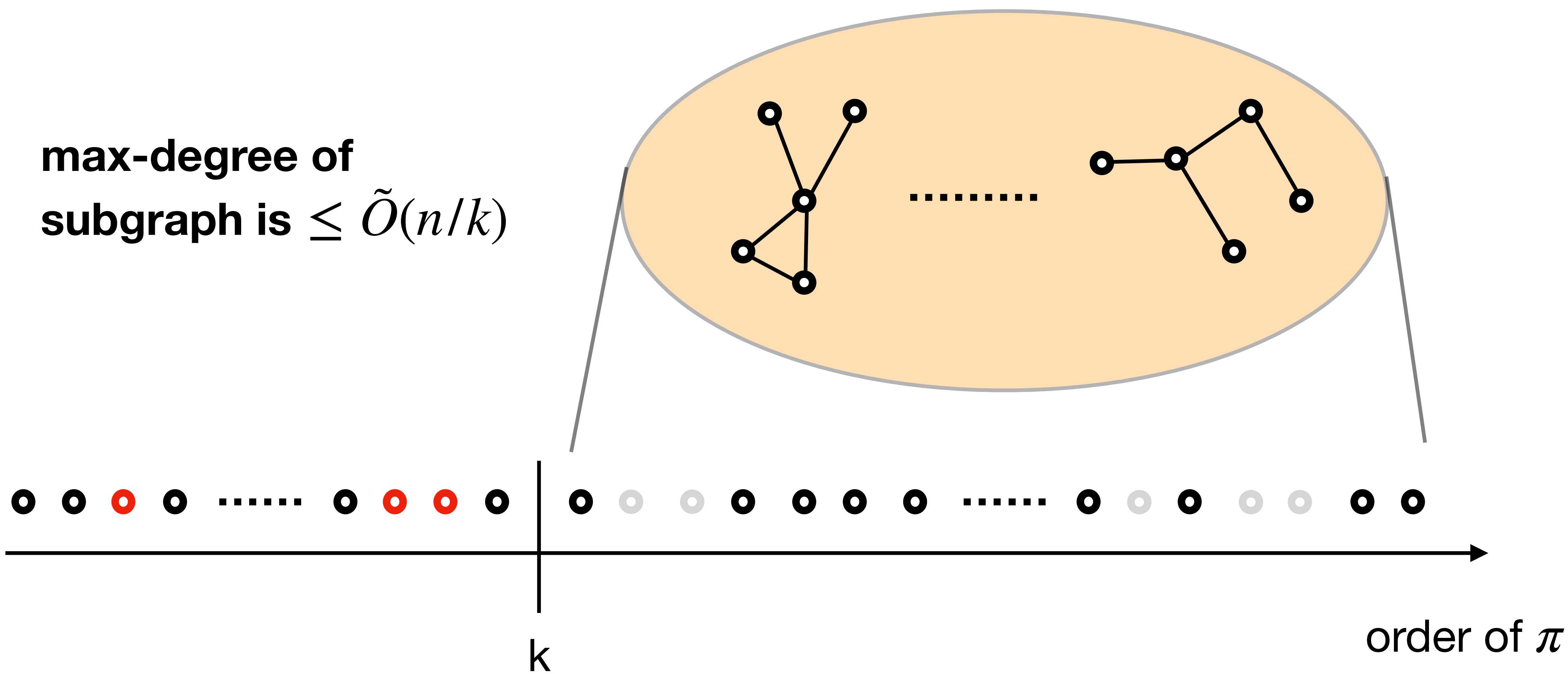

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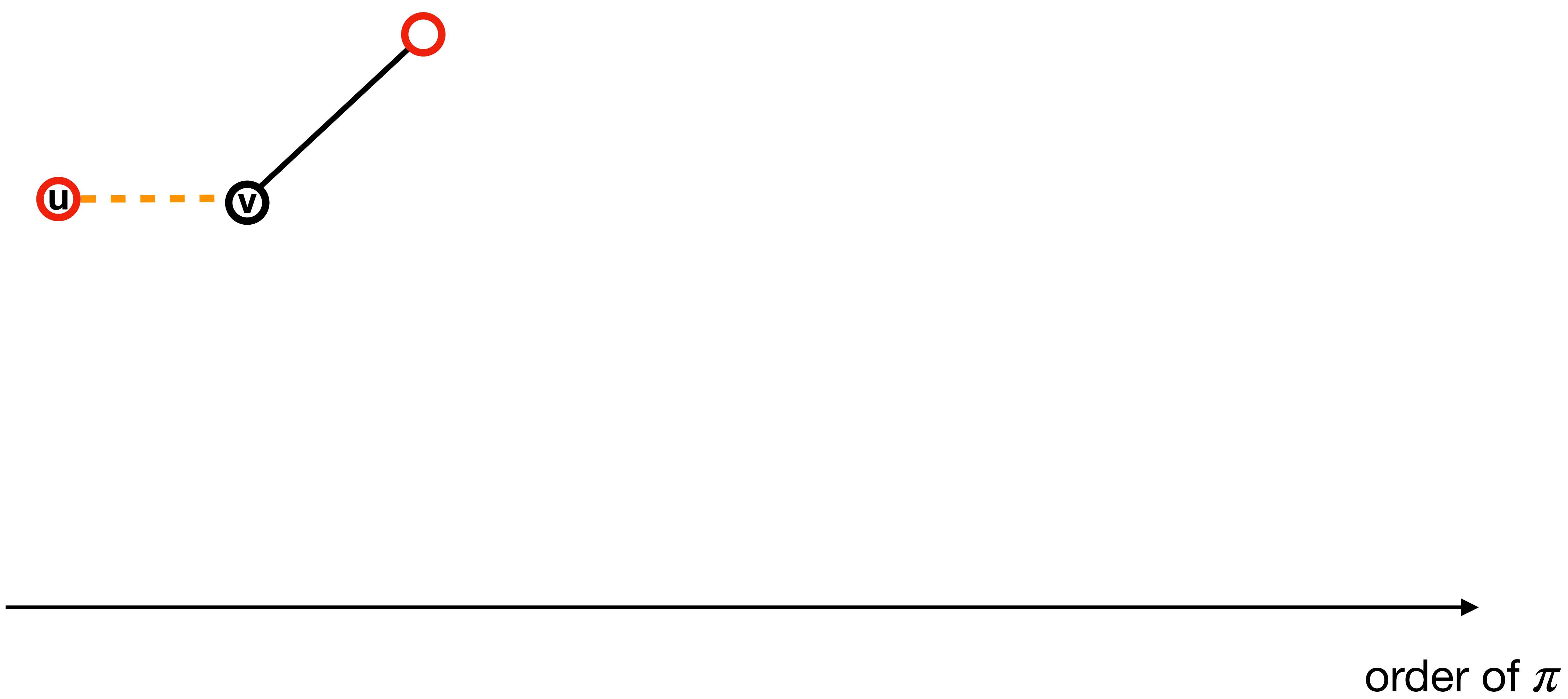

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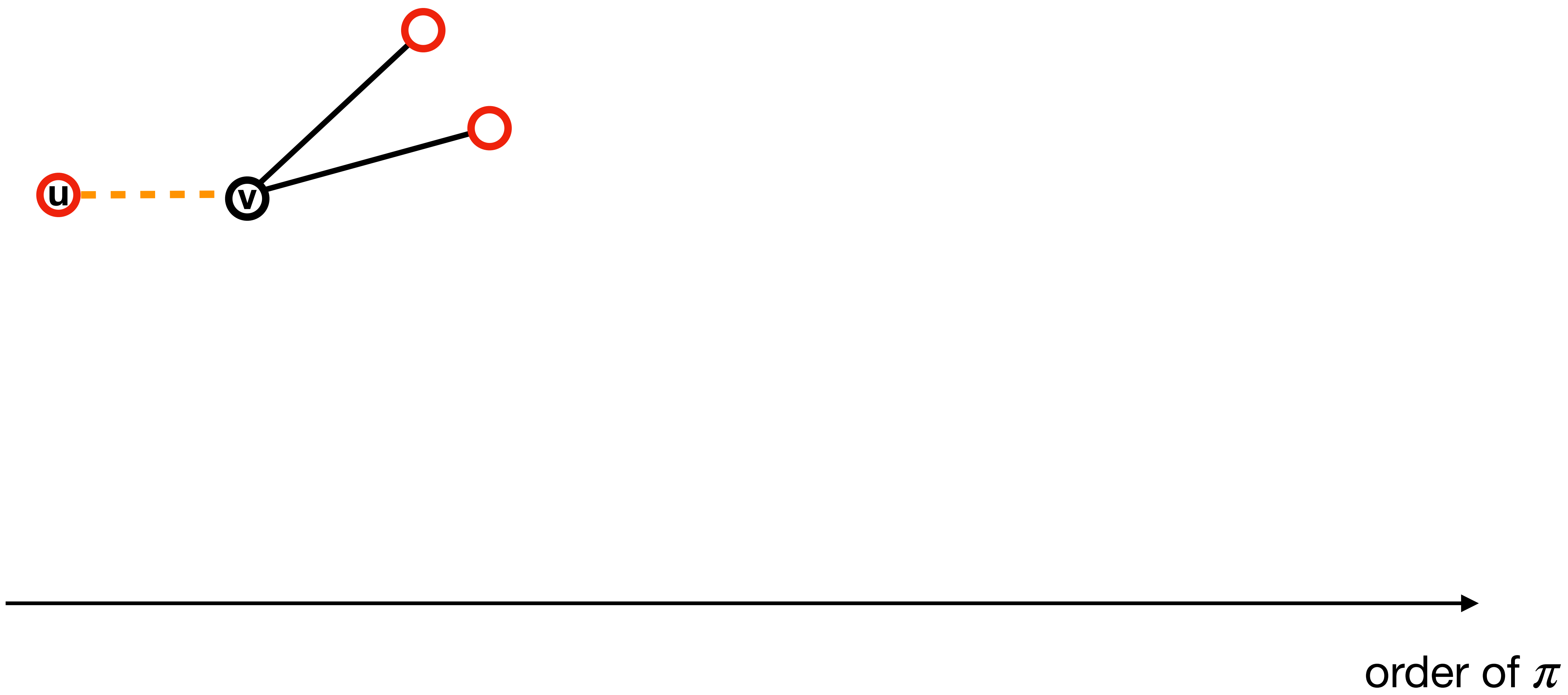

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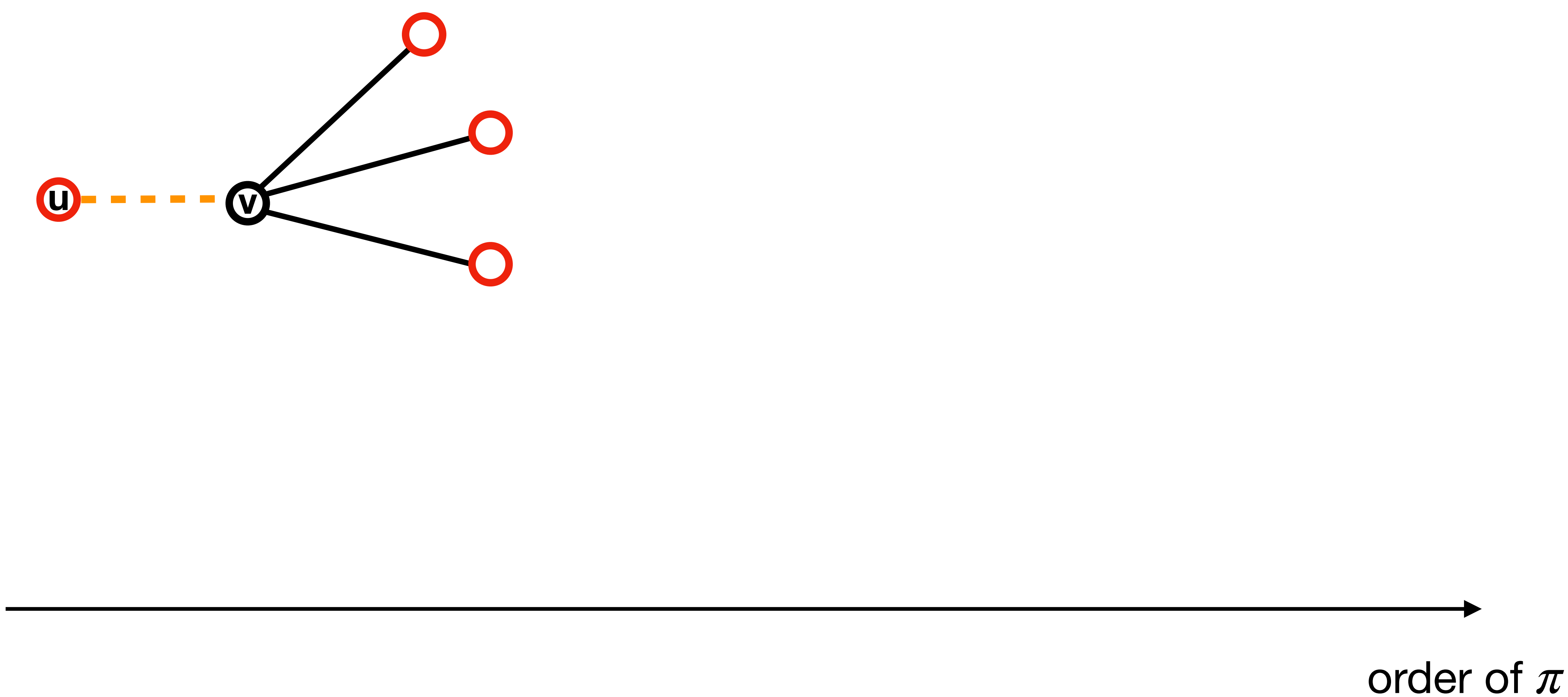

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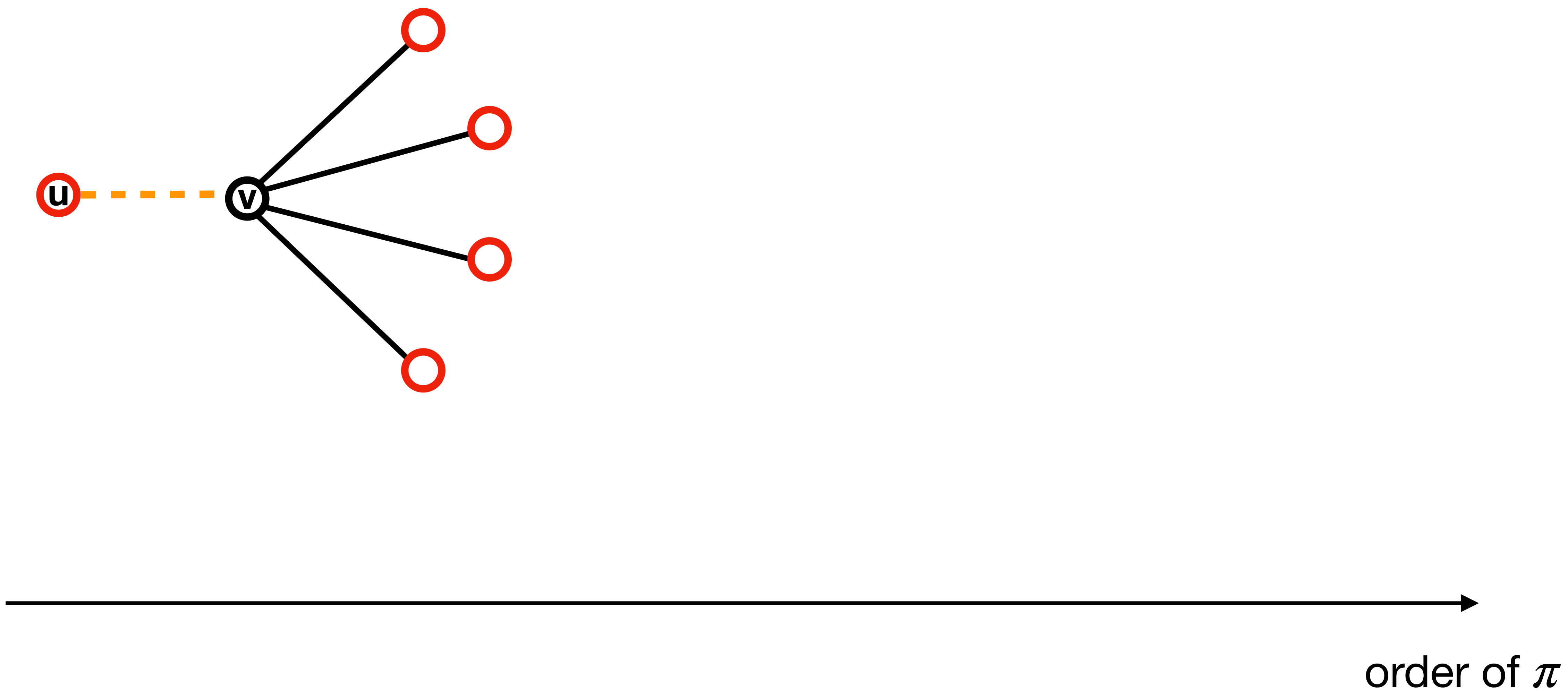

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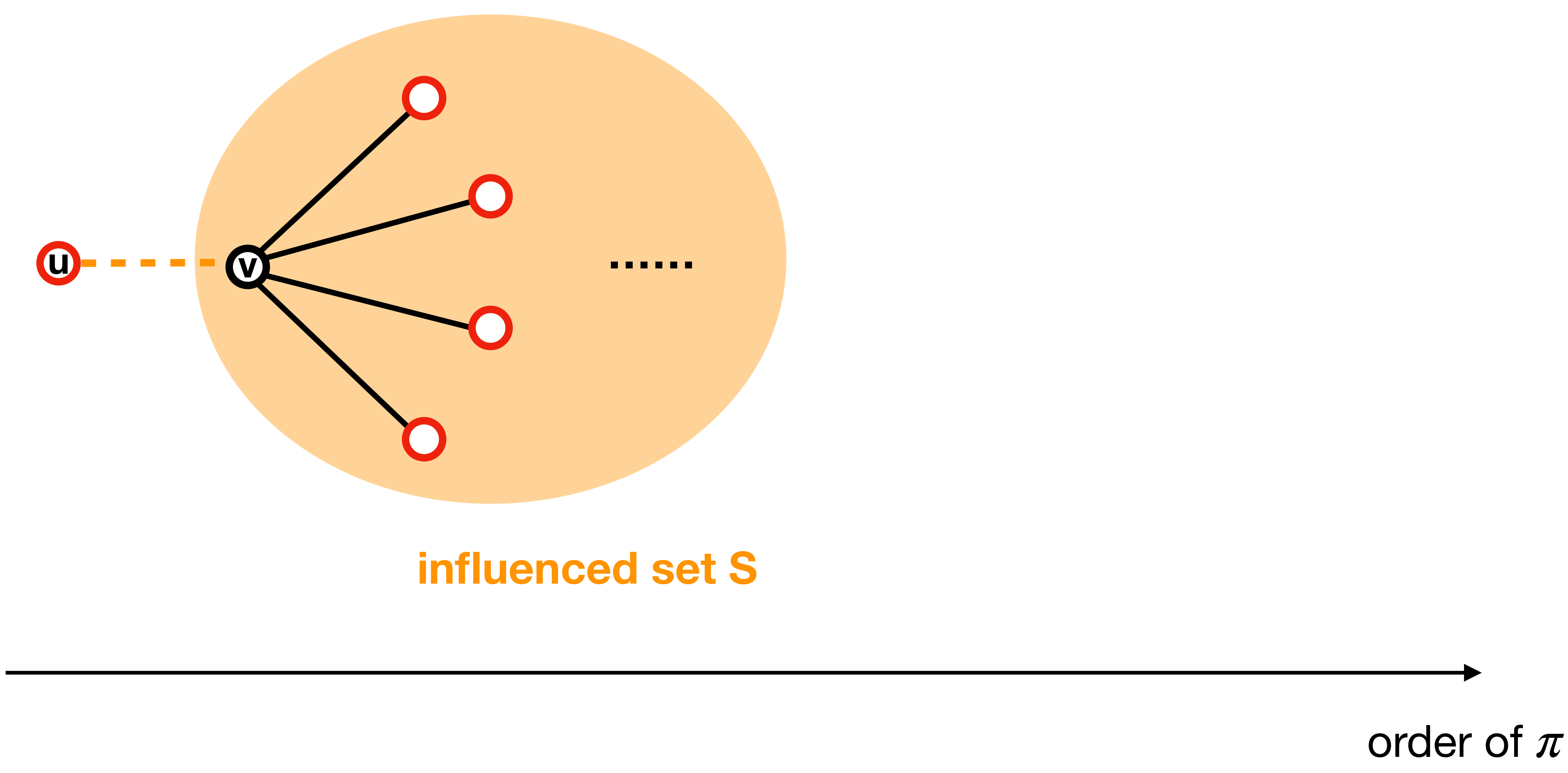

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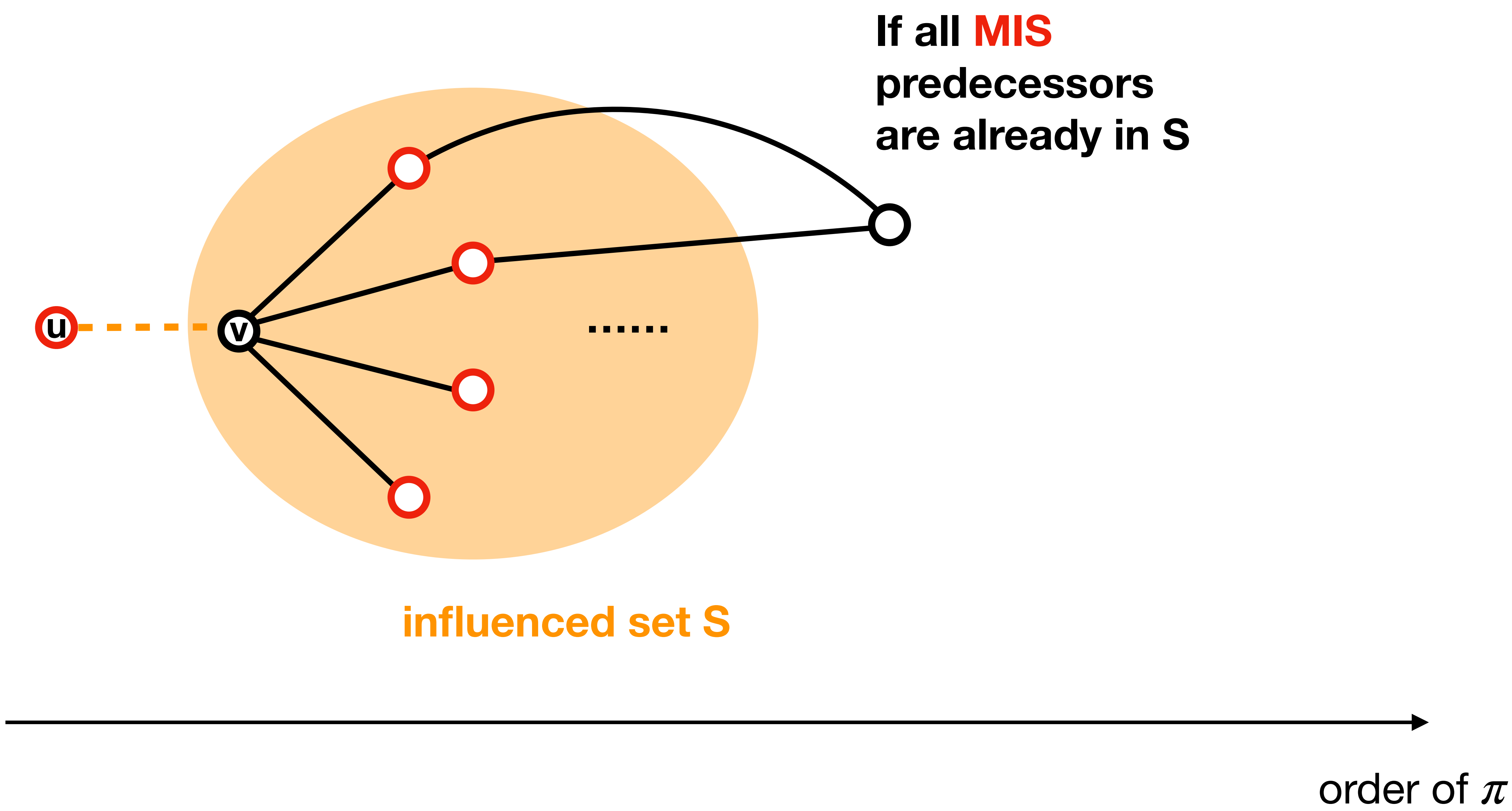

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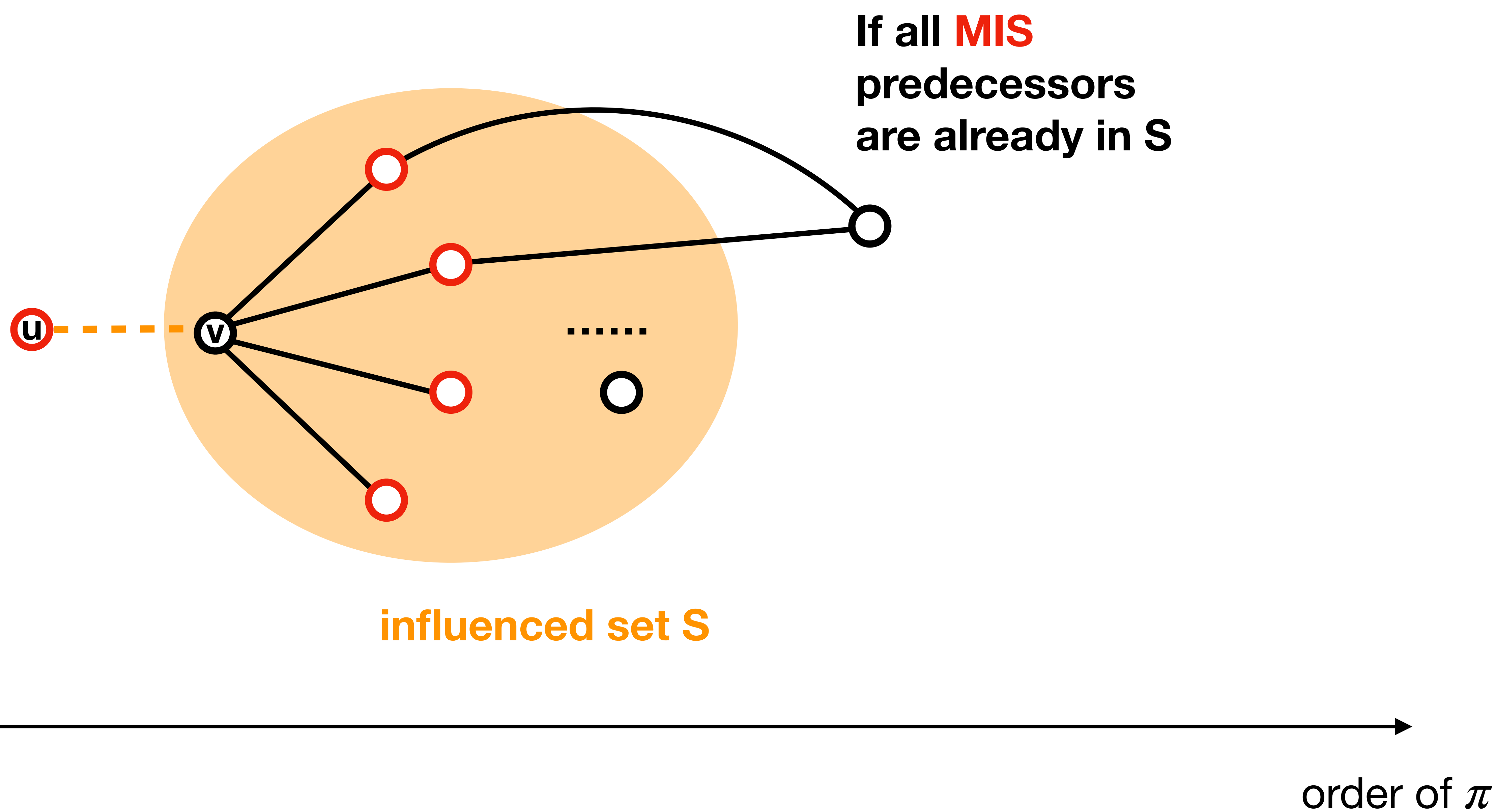

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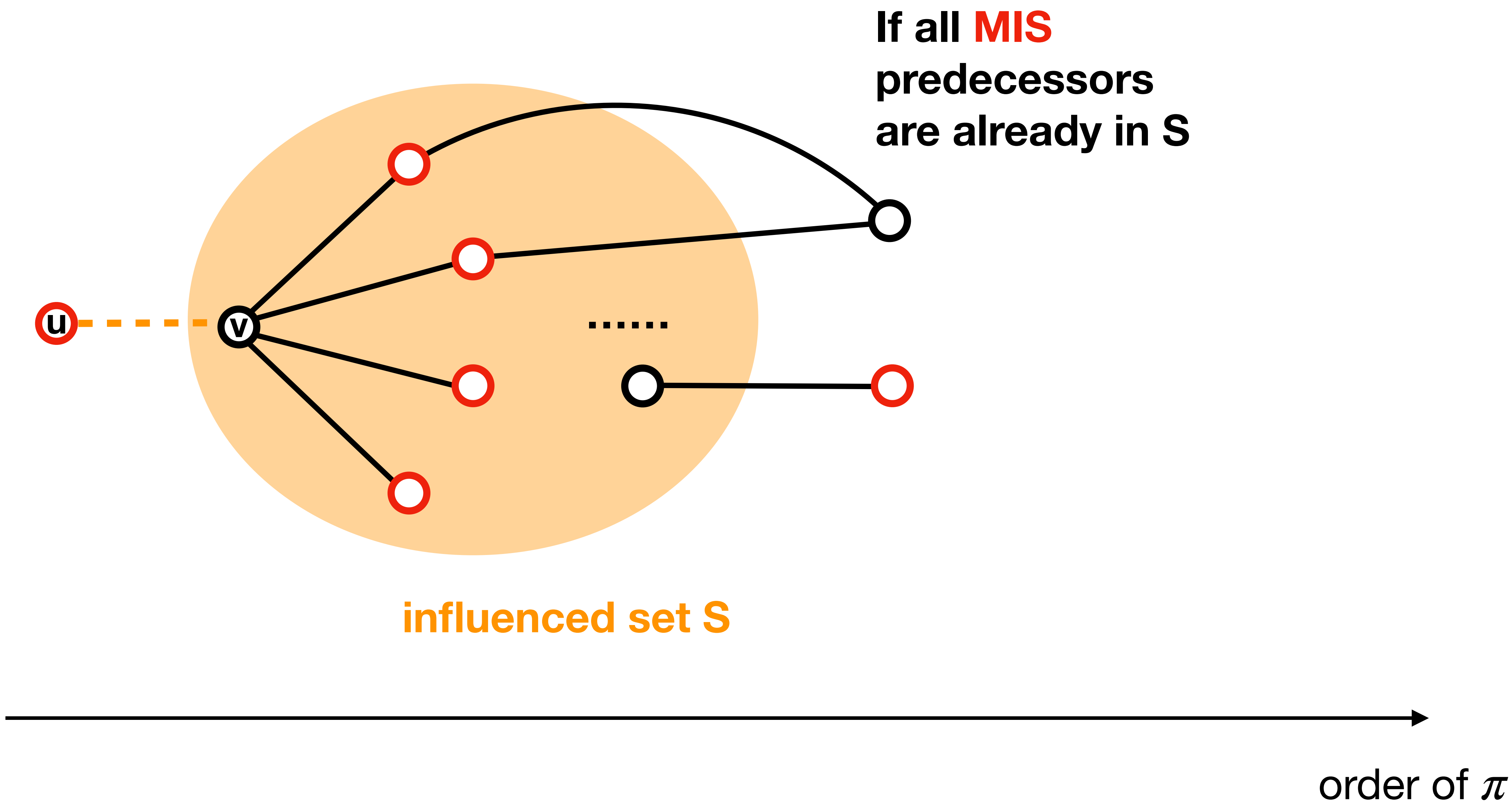

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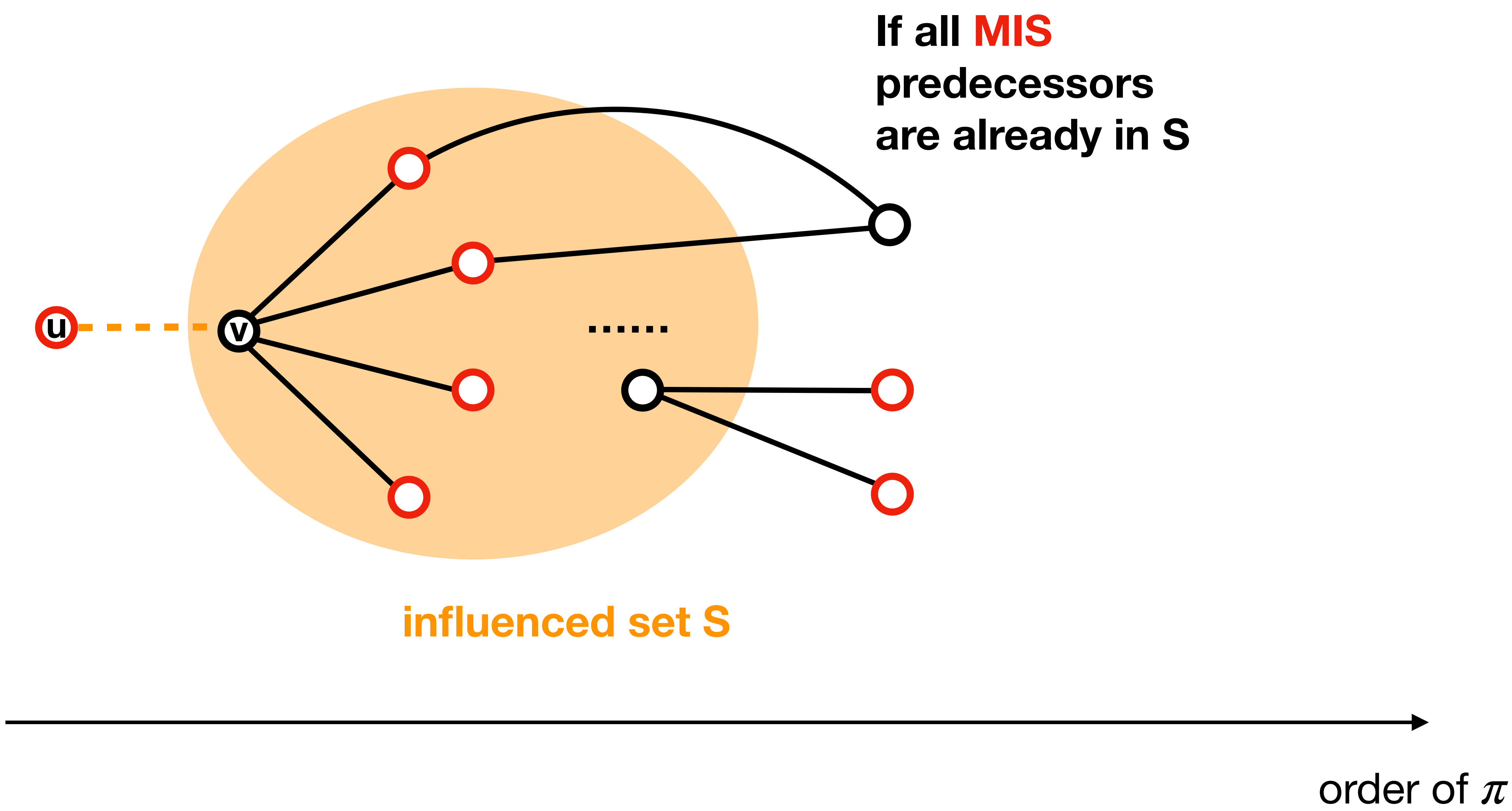

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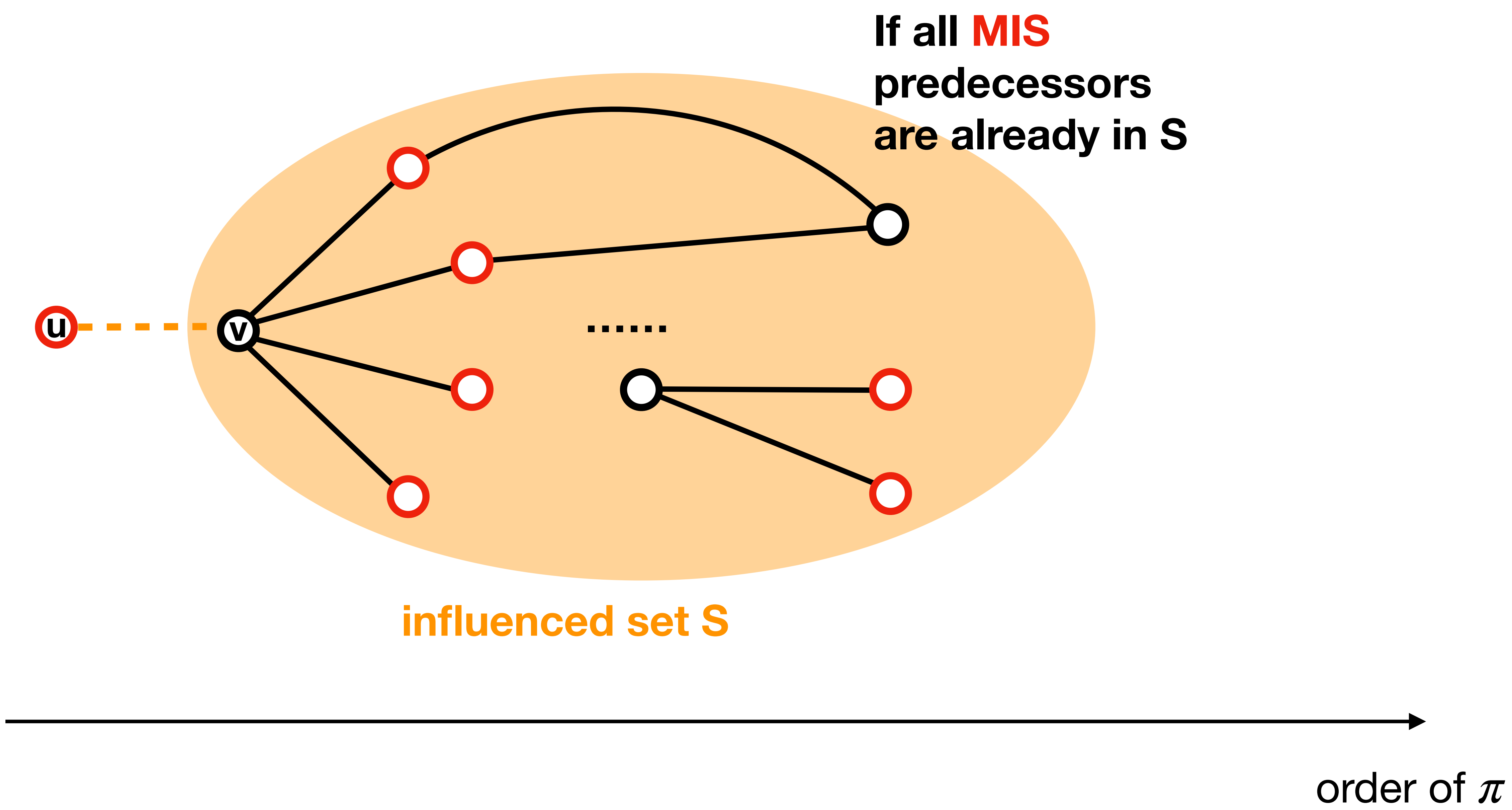

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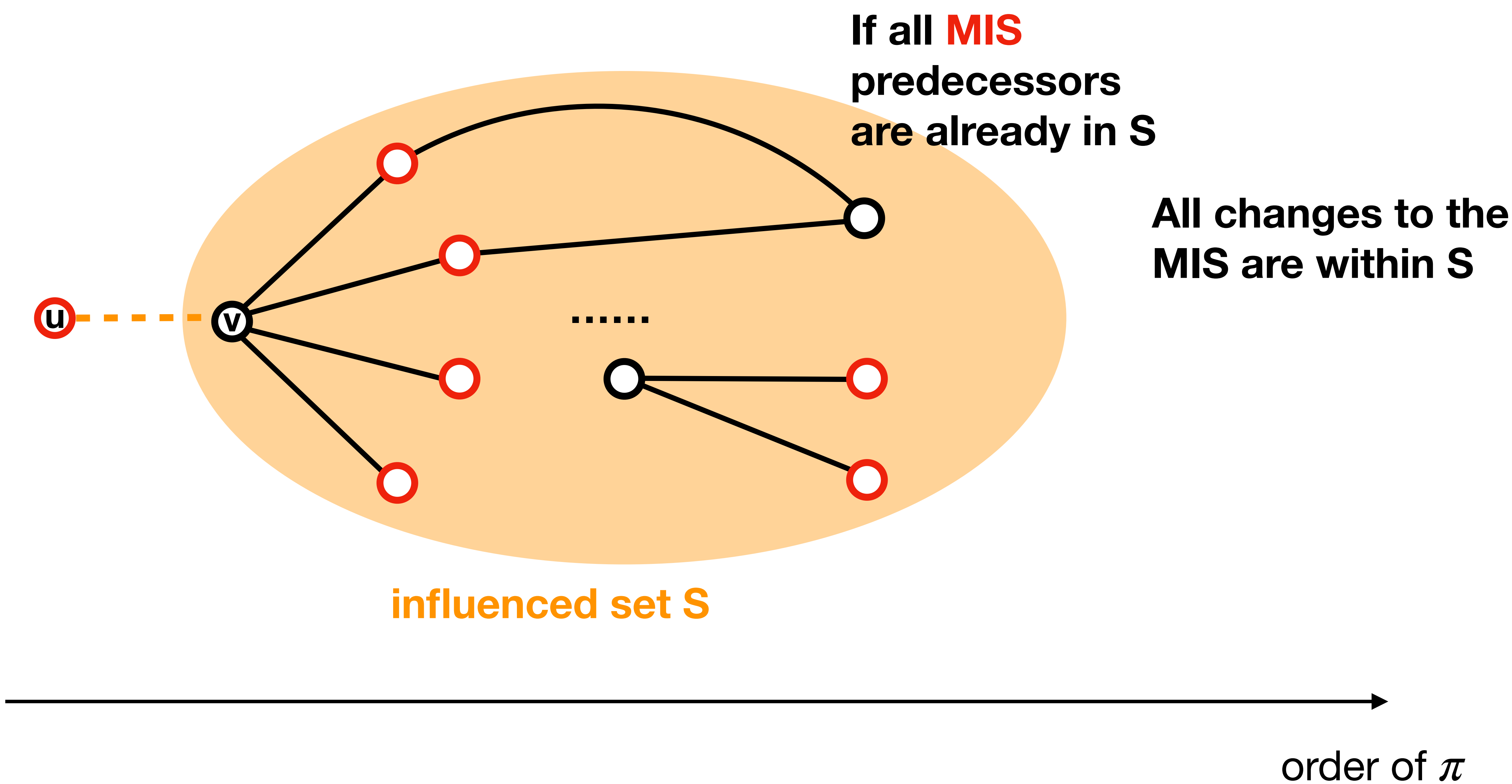

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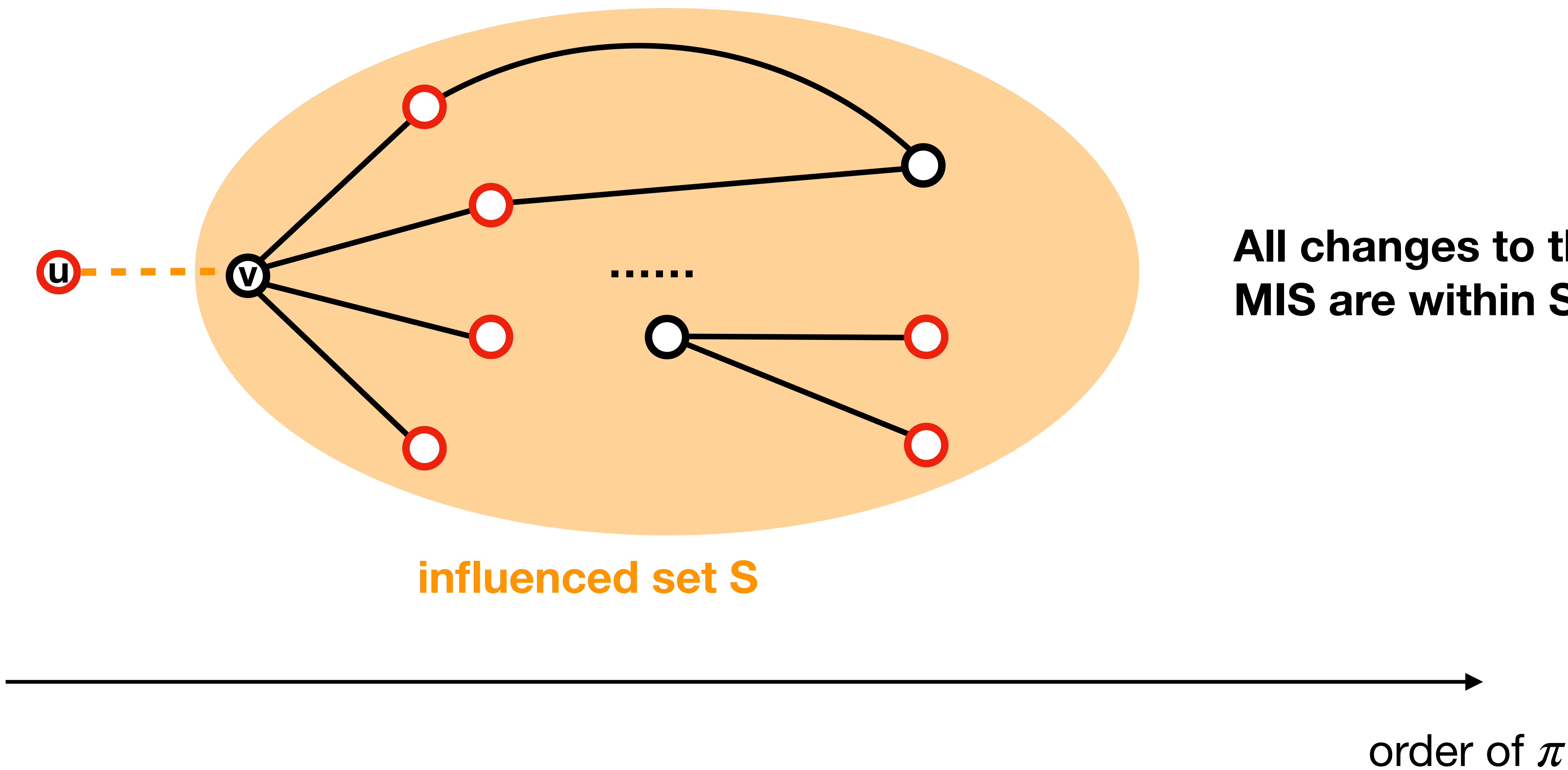
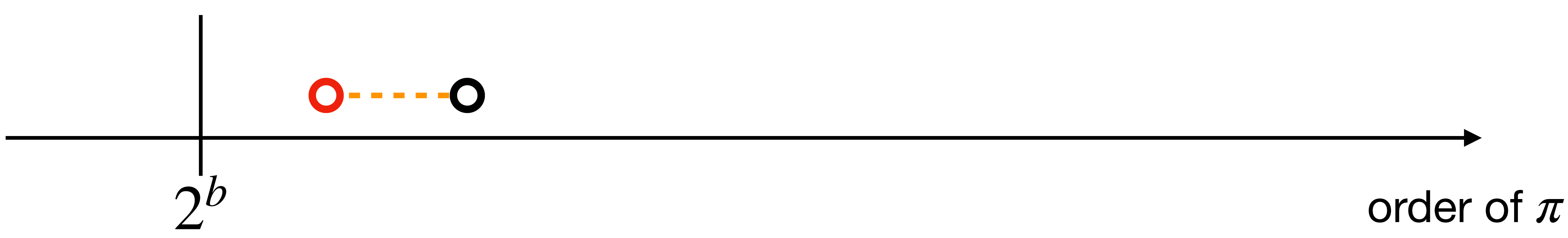


All changes to the MIS are within S

A sketch of algorithm

Data structure:

For each 2^b, explicitly maintain the induced subgraph G_b of vertices not dominated by MIS vertices of order $\leq 2^b$



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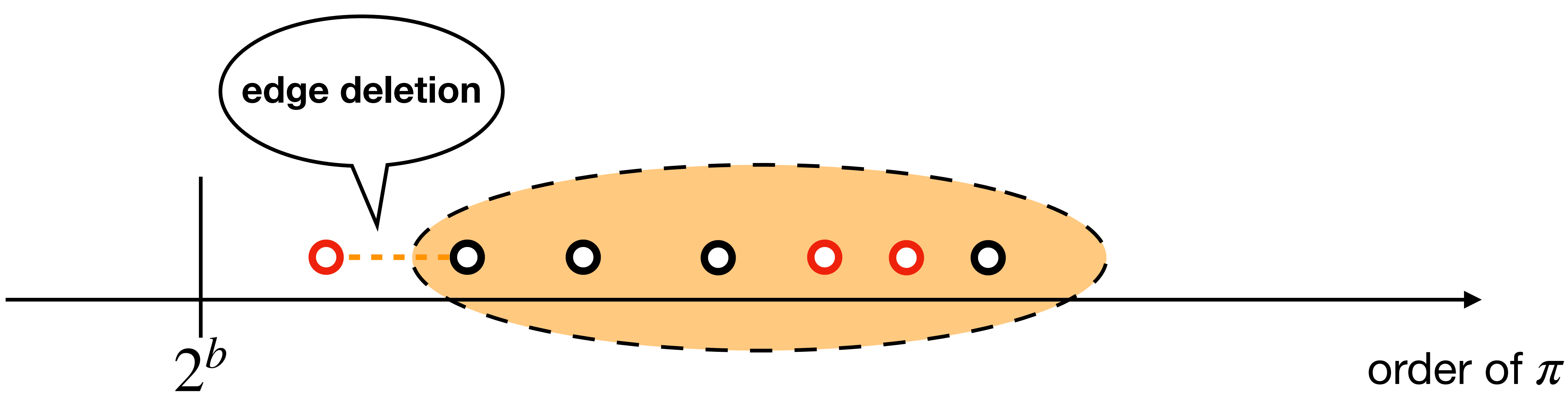

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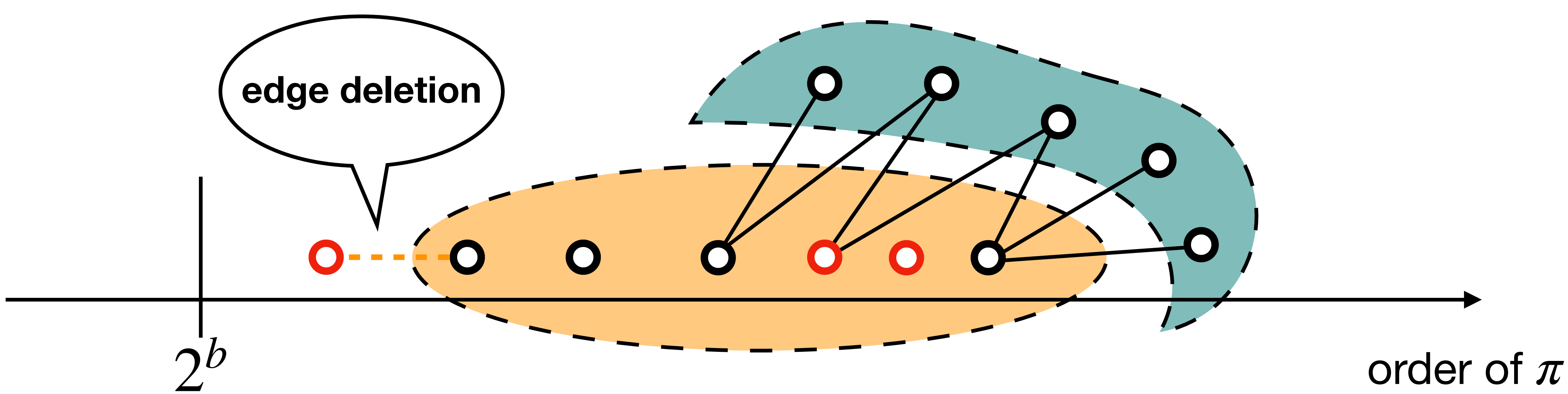
(2) Compute the **vertex updates to subgraphs**



The main algorithm

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Update algorithm:

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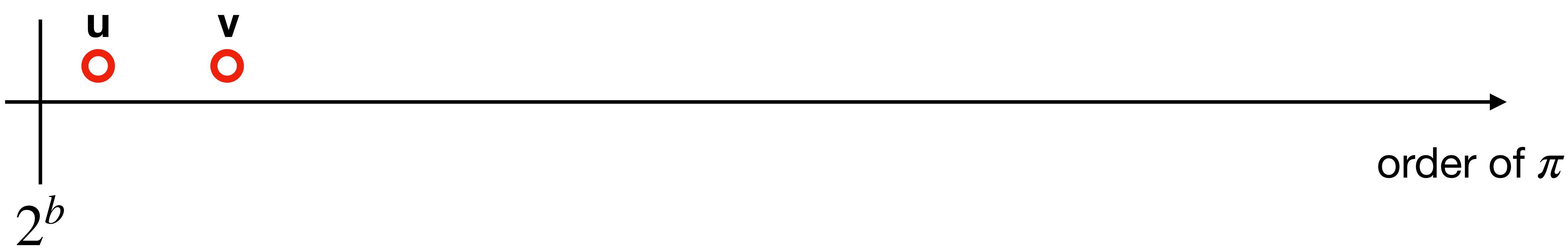

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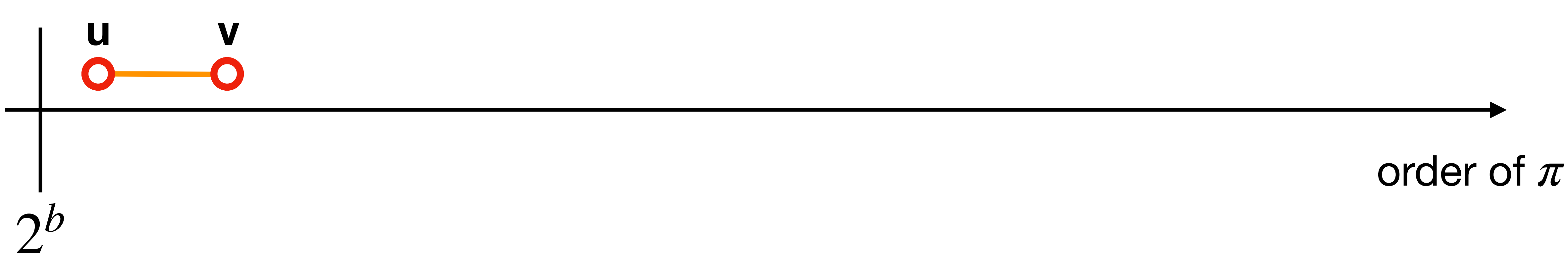

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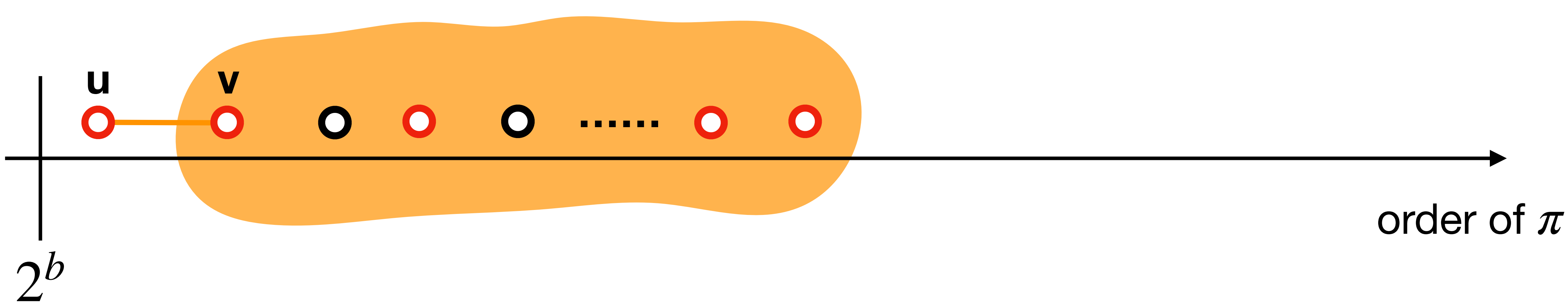

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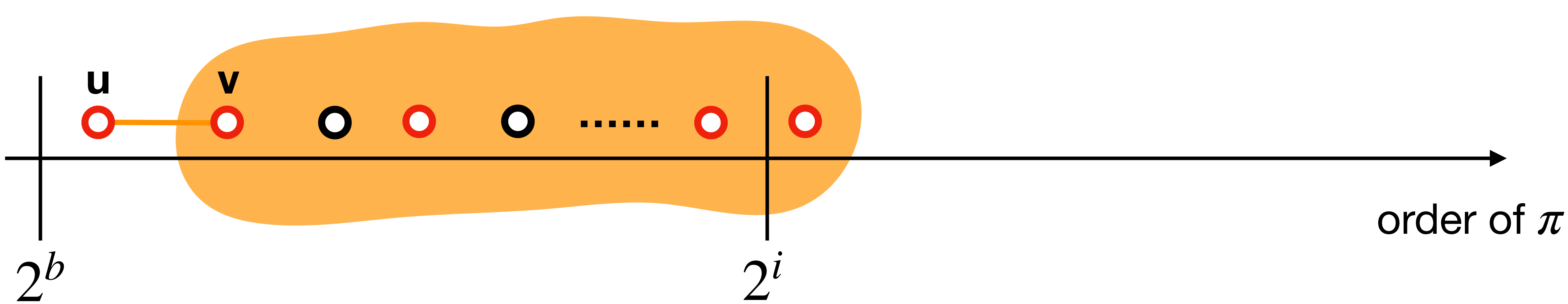

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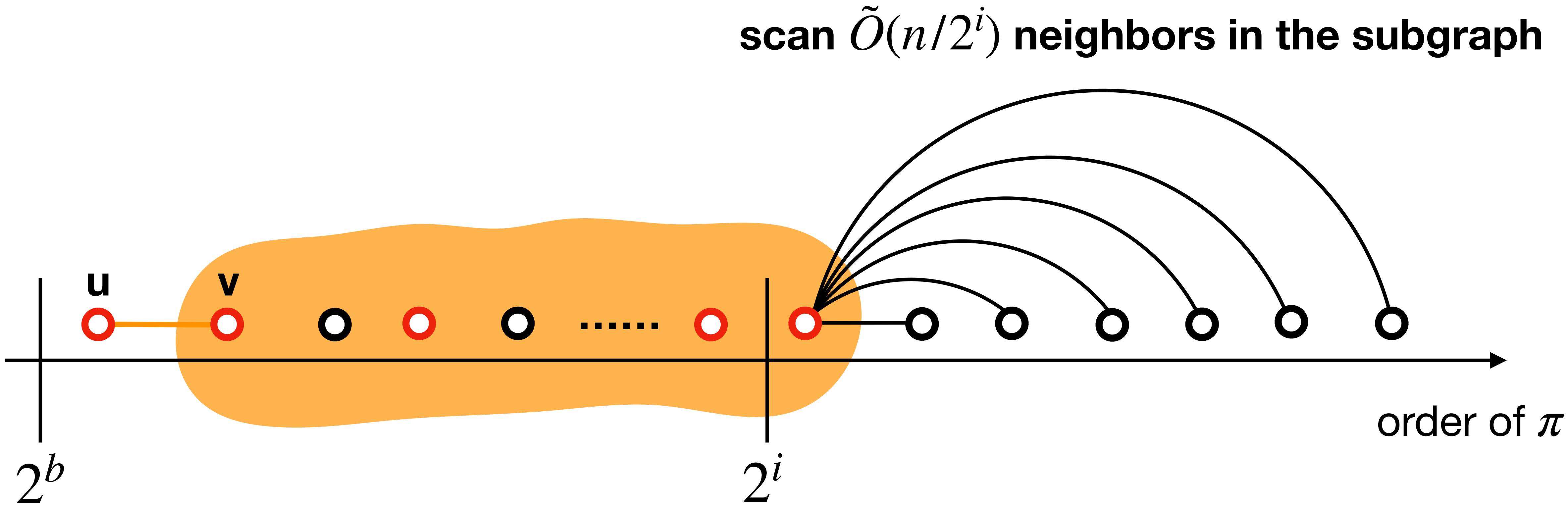

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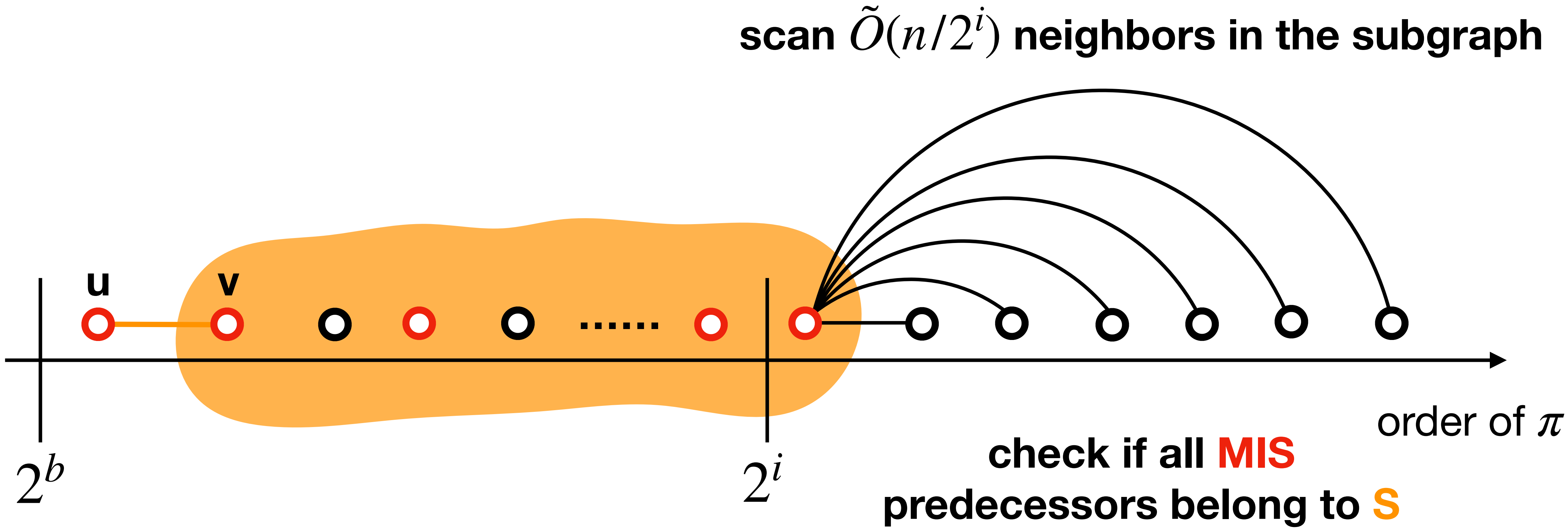

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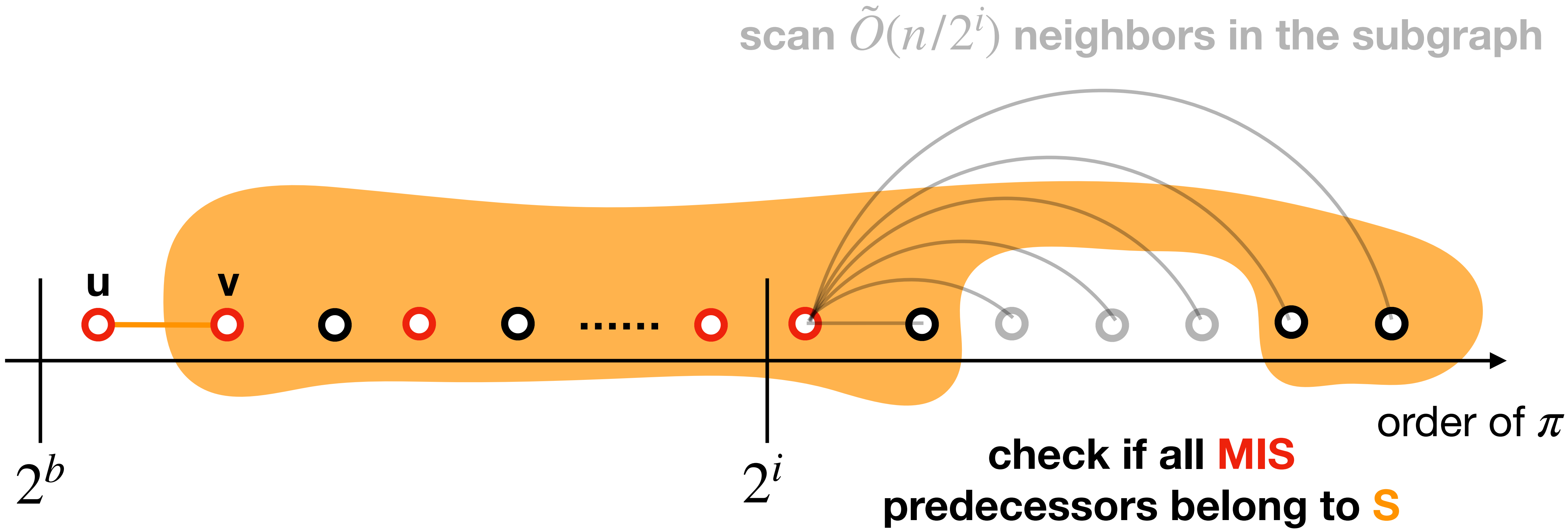

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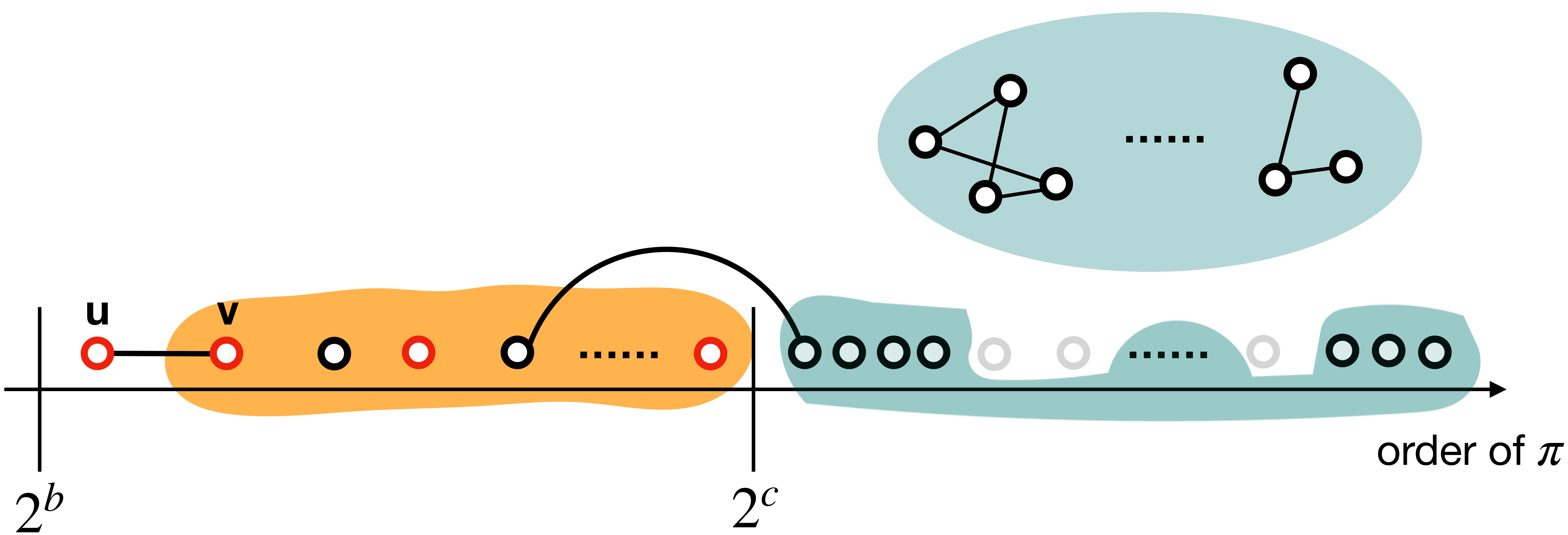
Update algorithm:

(2) Update all induced subgraphs $G_{c(>b)}$



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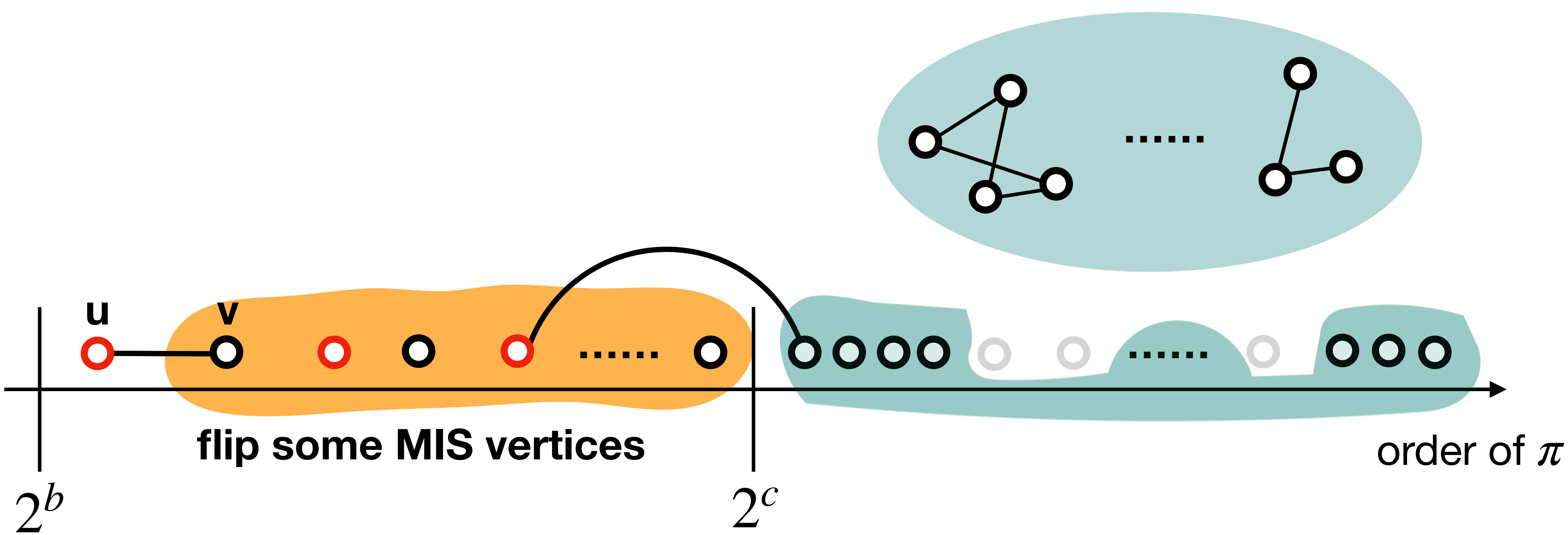

The main algorithm

Data structure:

For each 2^b, explicitly maintain the induced subgraph G_b of vertices not dominated by MIS vertices of order $\leq 2^b$

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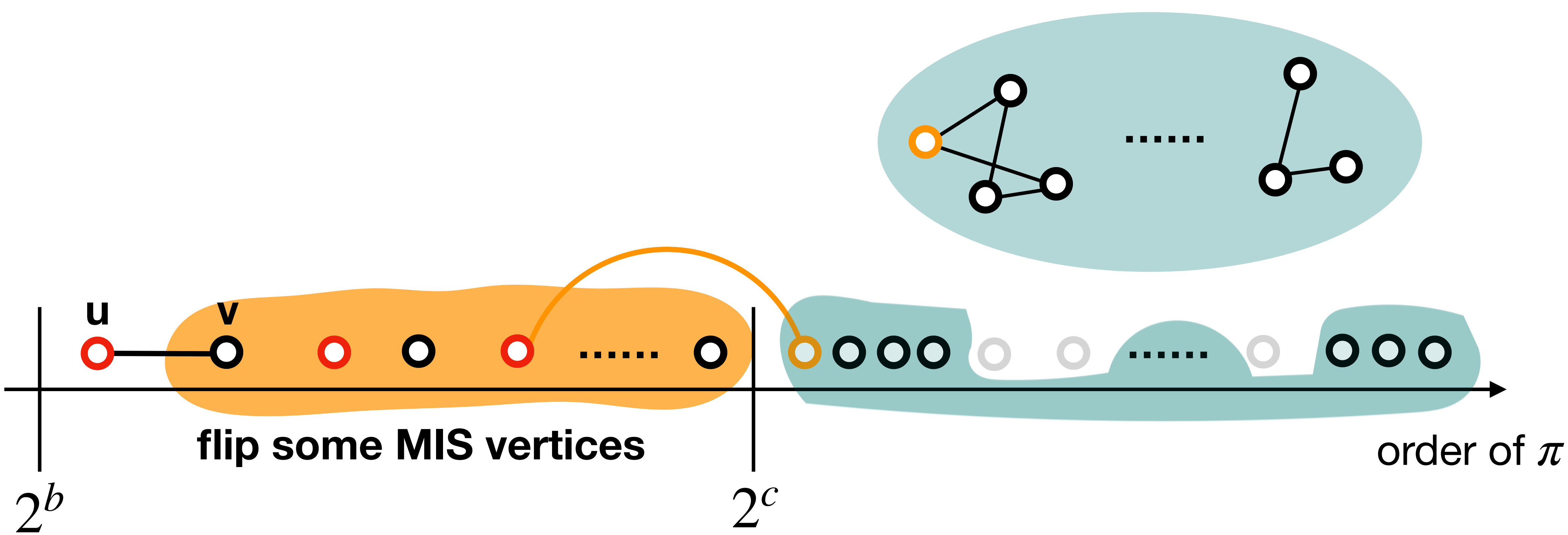

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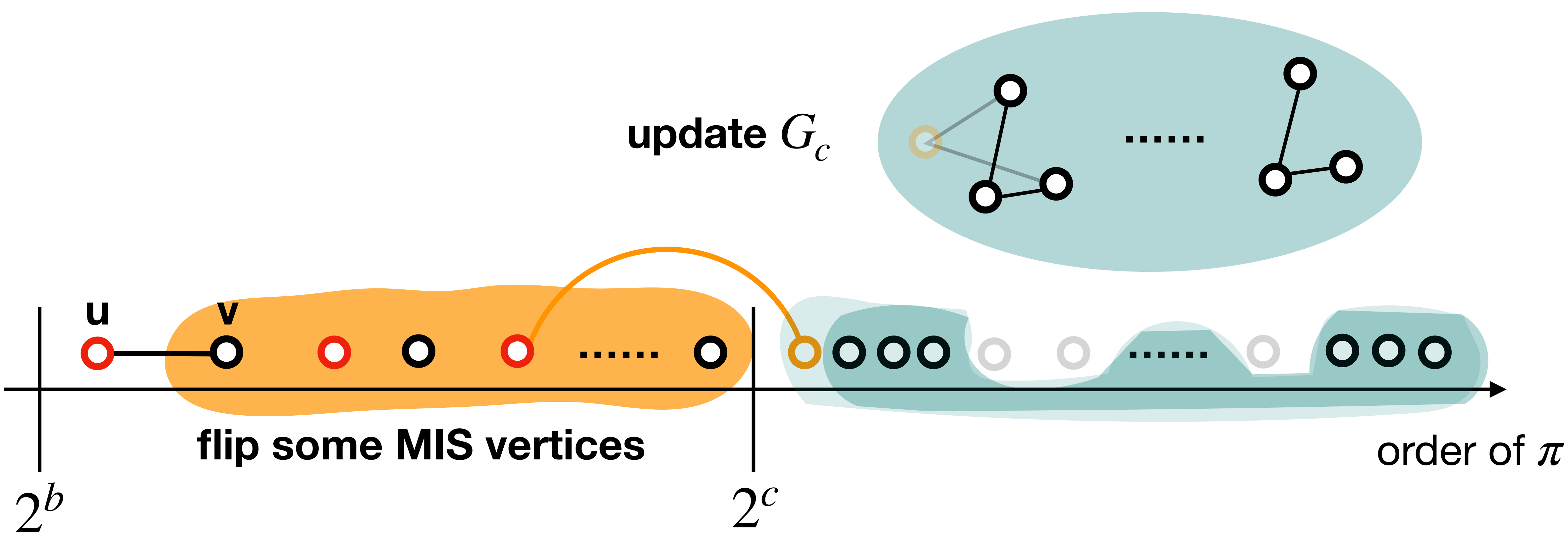
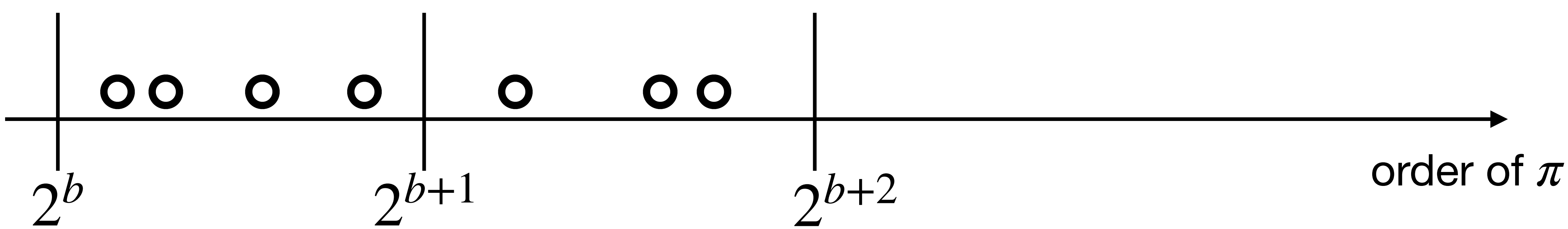


A sketch of analysis

Say $2^b < \pi(u) < \pi(v) \leq 2^{b+1}$, want update time $\leq \tilde{O}(n^2/2^{2b})$



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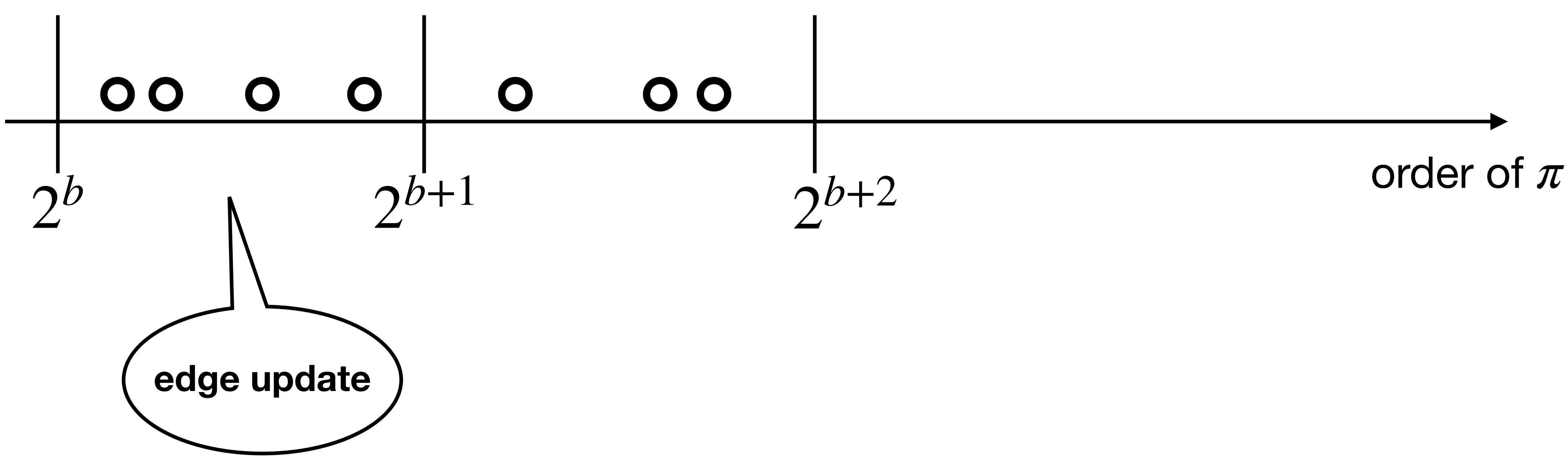

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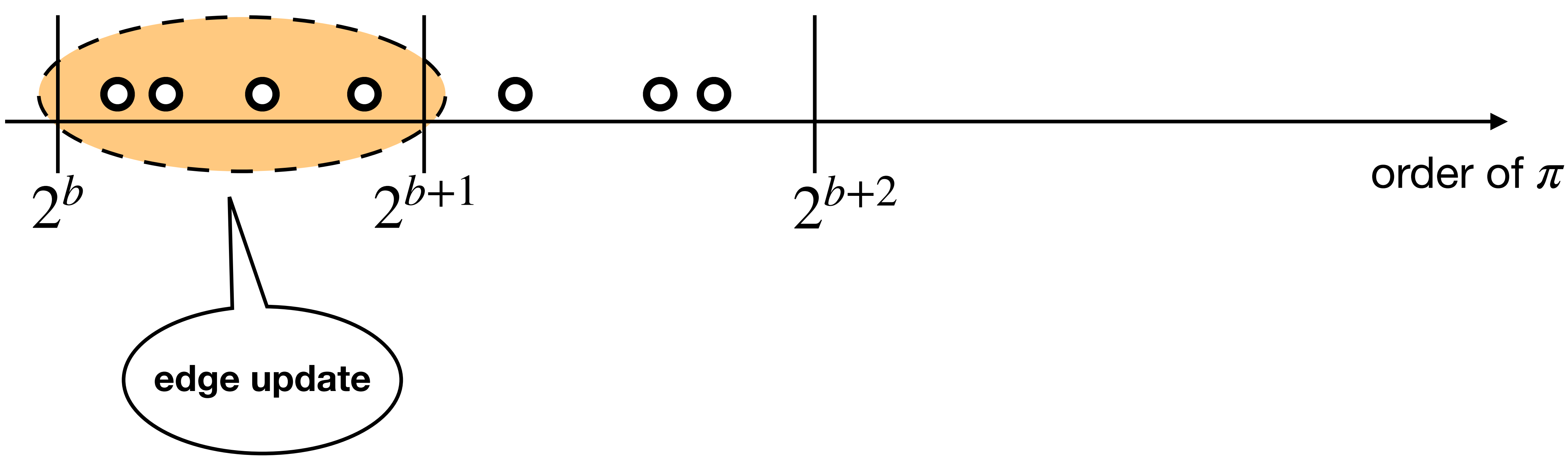
 # MIS-flips [CHK'16]
$\leq \mathrm{E}_\pi[\,|S|\,] \leq 1$



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$\leq \mathrm{E}_\pi[\,|S|\mid$ **first half**$] \leq 2 \cdot \mathrm{E}_\pi[\,|S|\,] \leq 2$

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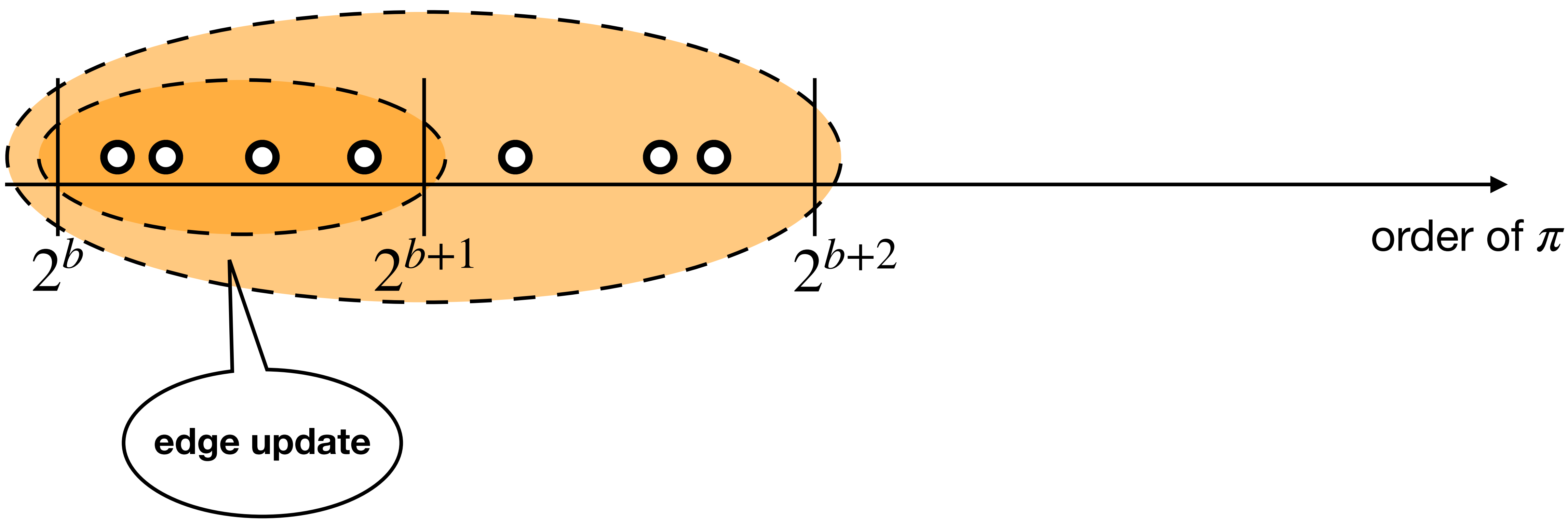

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$\leq \mathrm{E}_\pi[\,|S|\,|$ **first** $2^b] \leq 2^{i-b} \cdot \mathrm{E}_\pi[\,|S|\,] \leq 2^{i-b}$

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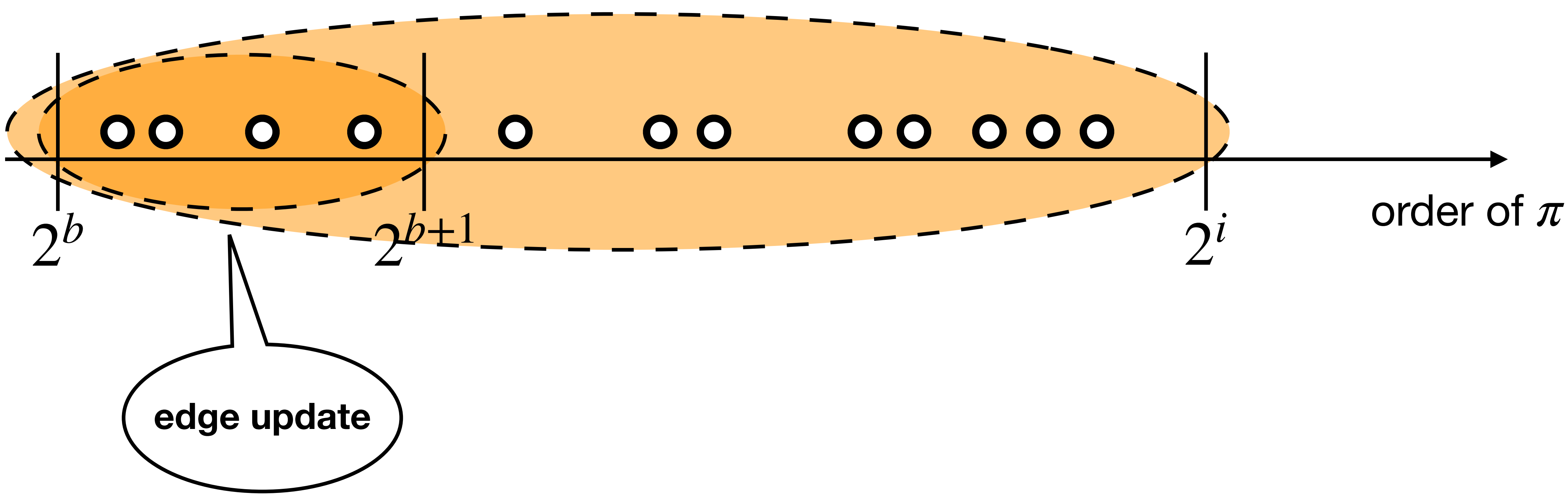

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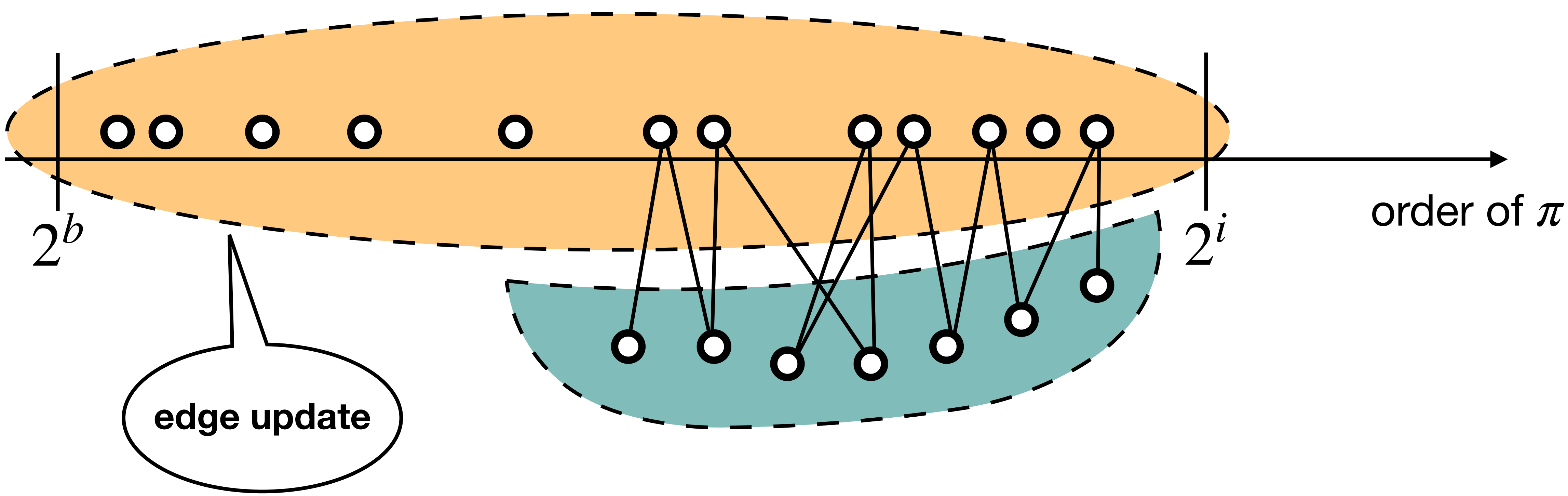
$\leq$ **max-deg of** $G_i \cdot \mathrm{E}_\pi[\,|S|\mid$ **first** $2^b] \leq n/2^i \cdot 2^{i-b} = n/2^b$



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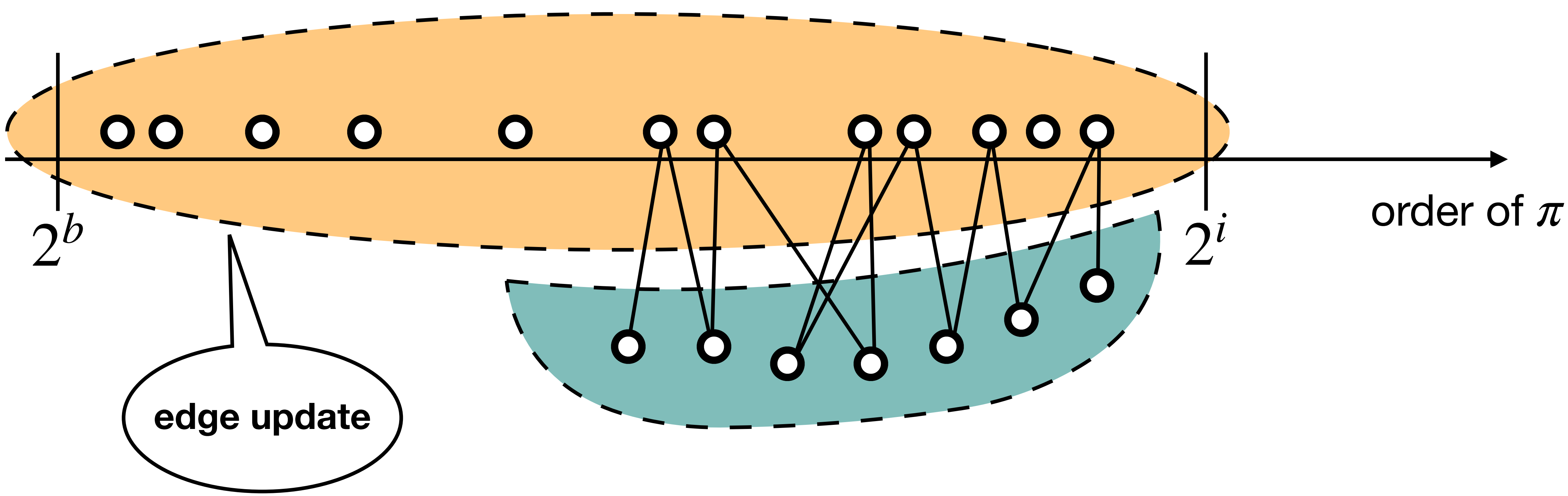
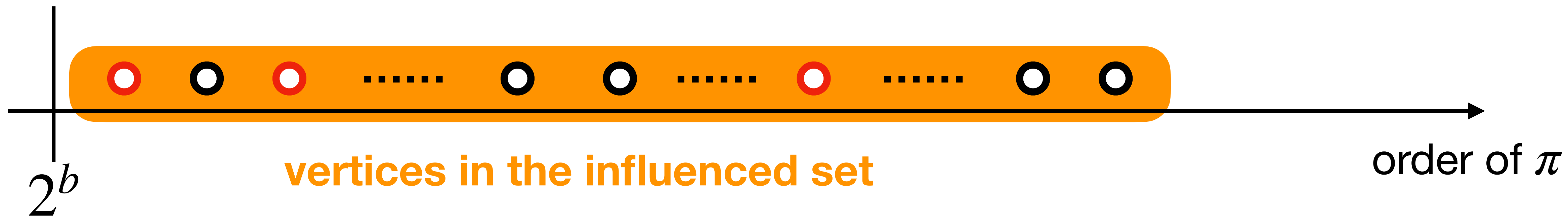
total work $\leq$ (**# subgraph changes**) $\cdot$ **max-deg of** $G_b \leq n^2/2^{2b}$



Running time analysis

Say $2^b < \pi(u) < \pi(v) \leq 2^{b+1}$, want update time $\leq \tilde{O}(n^2/2^{2b})$

Extending [CHK'16], we could prove that $\mathrm{E}_\pi[\,|S|\,] \leq O(n/2^b)$



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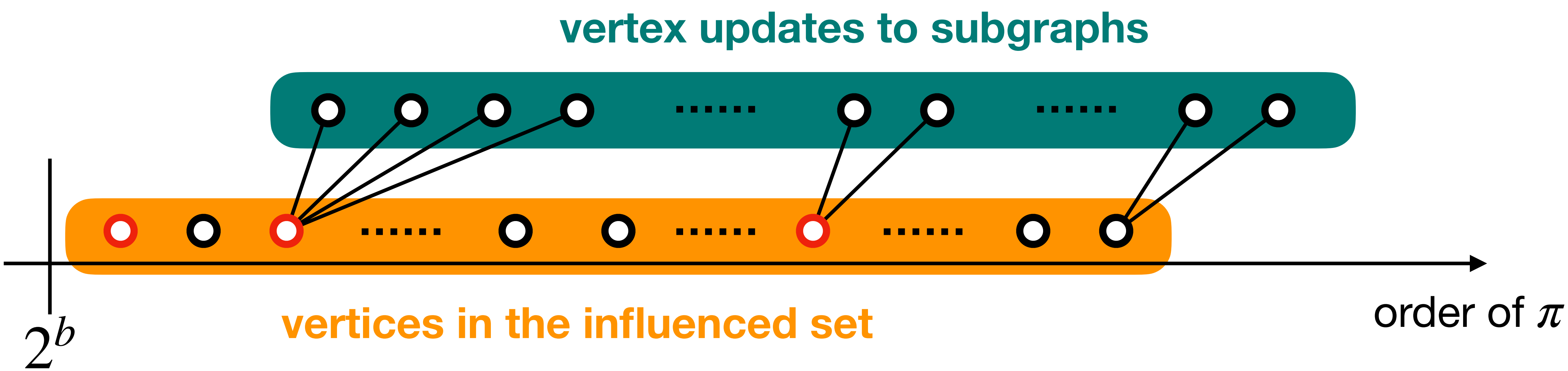

size of [teal box] $\leq$ **max-deg of** G_b *** |S|**

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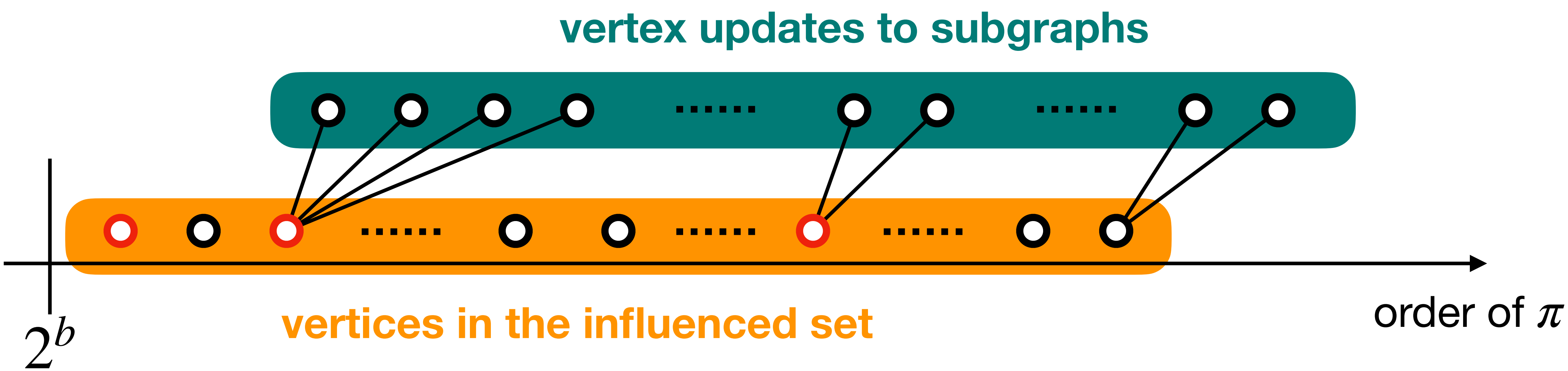

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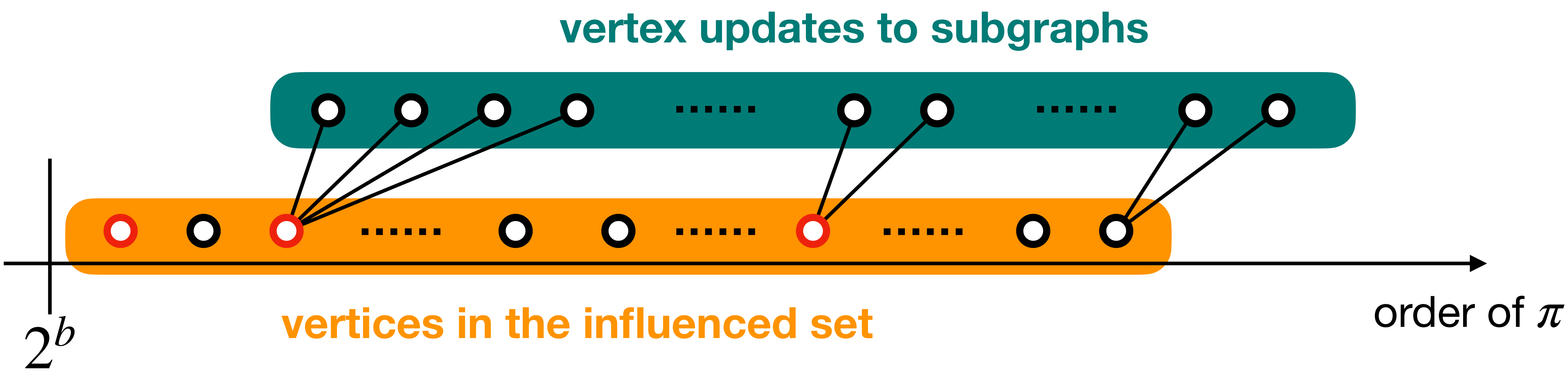

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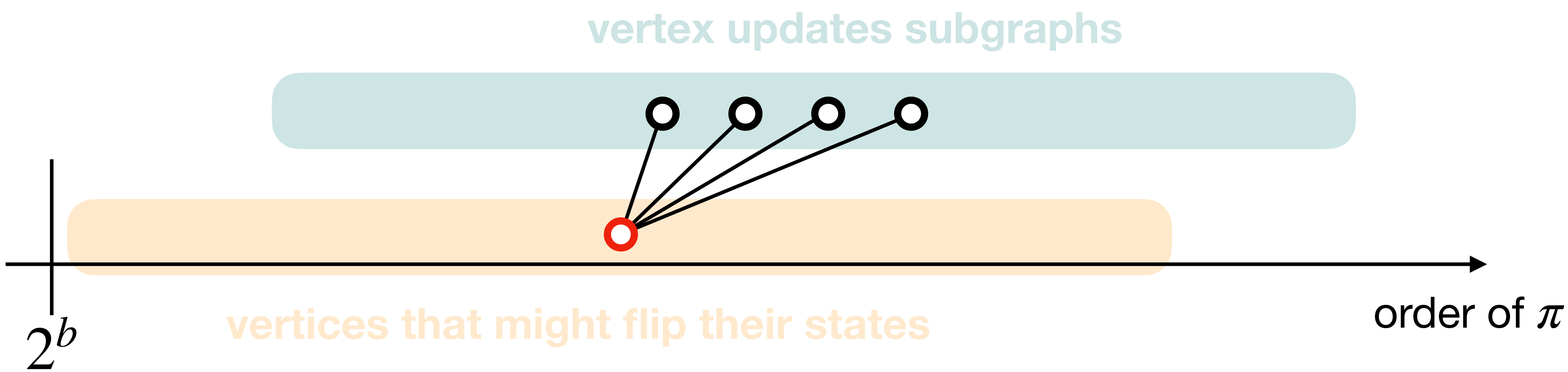
We need $\tilde{O}(n^2/2^{2b})$

Running time analysis



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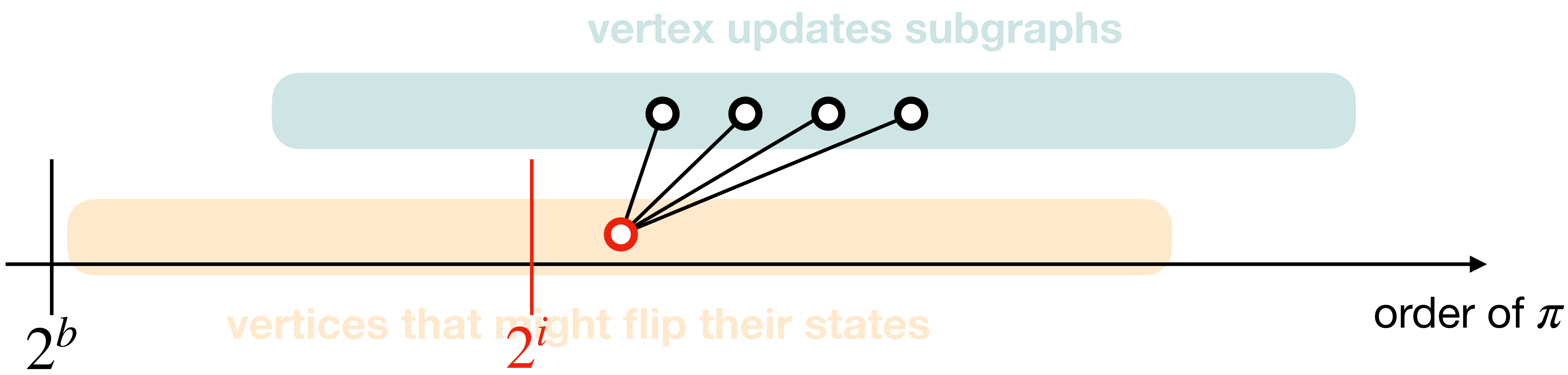

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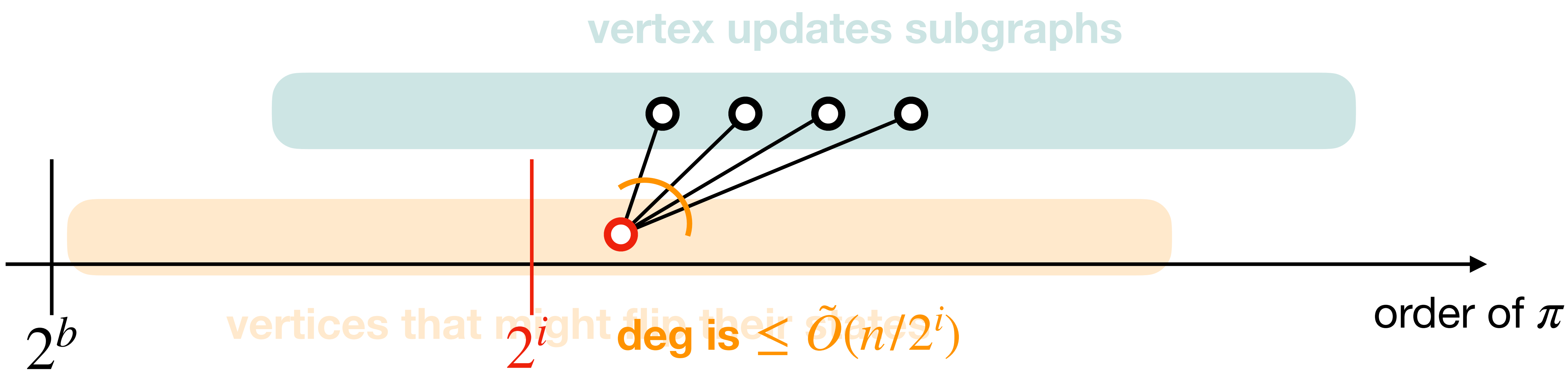

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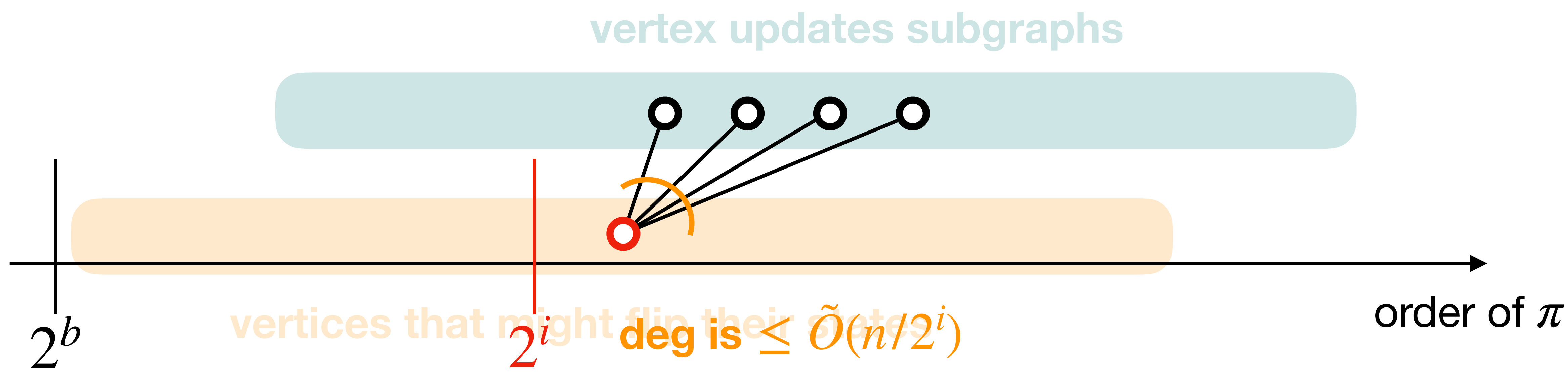

size of $\leq \sum \tilde{O}(n/2^i) \cdot |S \cap \pi[2^i, 2^{i+1})|$

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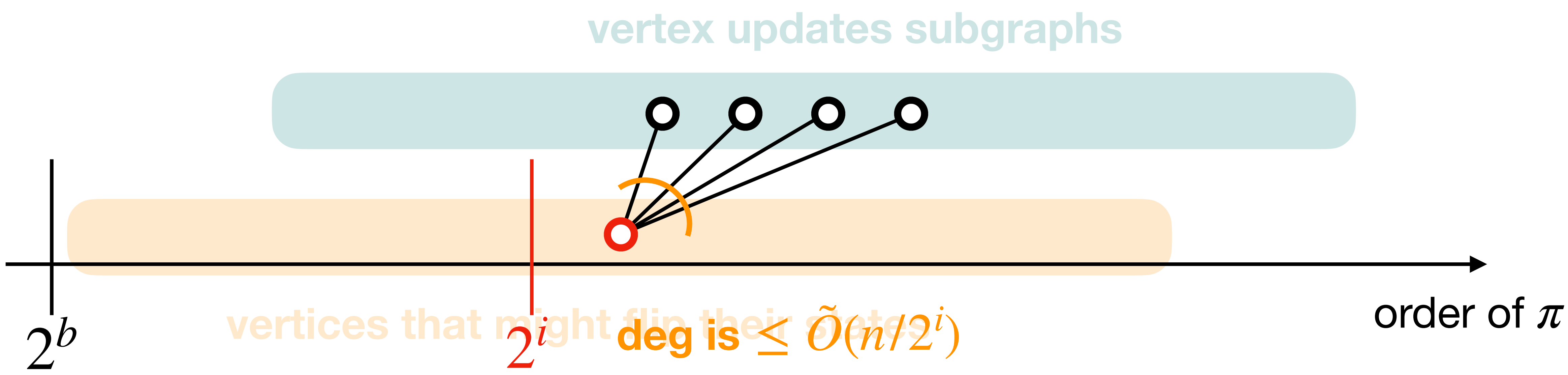

size of $\leq \sum \tilde{O}(n/2^i) \cdot |S \cap \pi[2^i, 2^{i+1})|$

If S is "evenly" distributed on $(2^b, n]$, say $|S \cap \pi[2^i, 2^{i+1})| \leq O(\frac{2^i}{n} \cdot |S|)$

Consequently, $\sum n/2^i \leq$ ~~max deg~~ $O(\log n)$ * |S|

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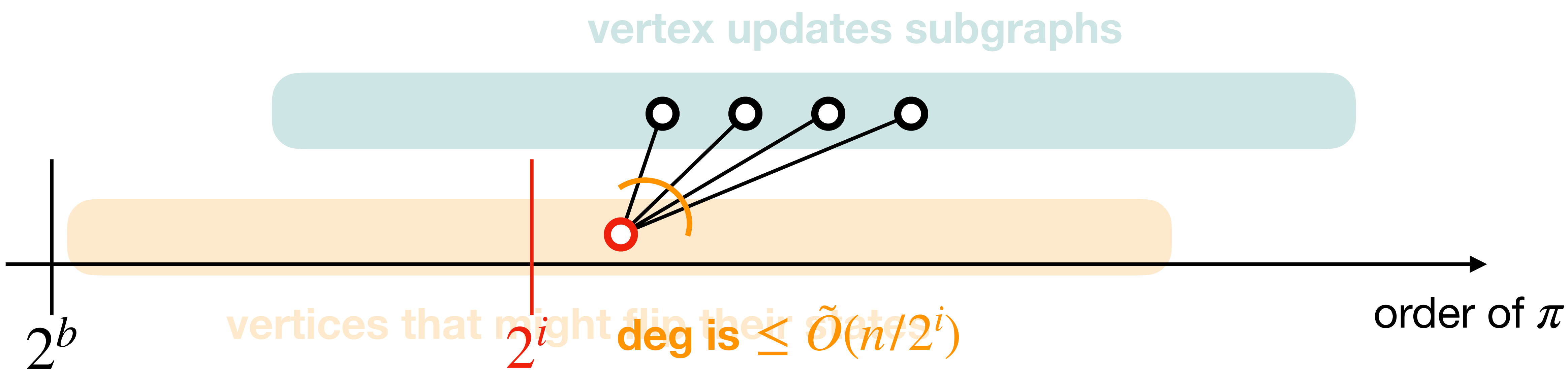

size of [teal shape] $\leq \sum \tilde{O}(n/2^i) \cdot |S \cap \pi[2^i, 2^{i+1})|$

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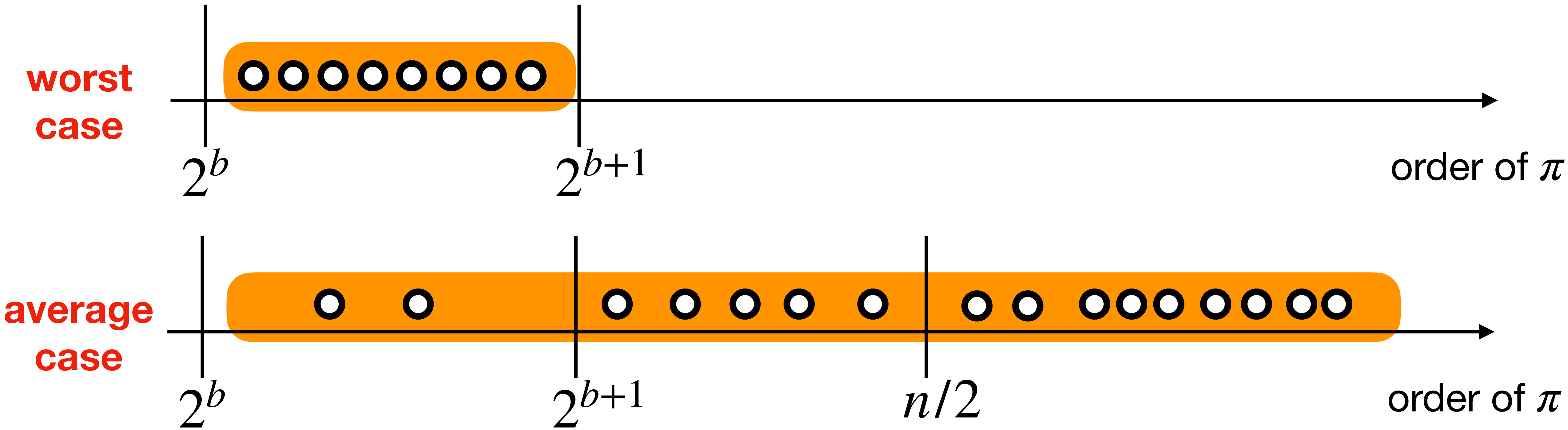
Consequently, $\sum n/2^i \leq$ ~~max-deg~~ $O(\log n)$ * |S|

Total update time becomes $\tilde{O}(n^2/2^{2b})$

Running time analysis

Analyze the how S is distributed on $(2^b, n]$



<u>Lemma:</u>

Conditioning on $2^b < \pi(u) < \pi(v) \leq 2^{b+1}$, the set S is "evenly" distributed on interval $(\pi(v), n]$

Running time analysis

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Strategy:

Gradually transform the bad case to a good case.

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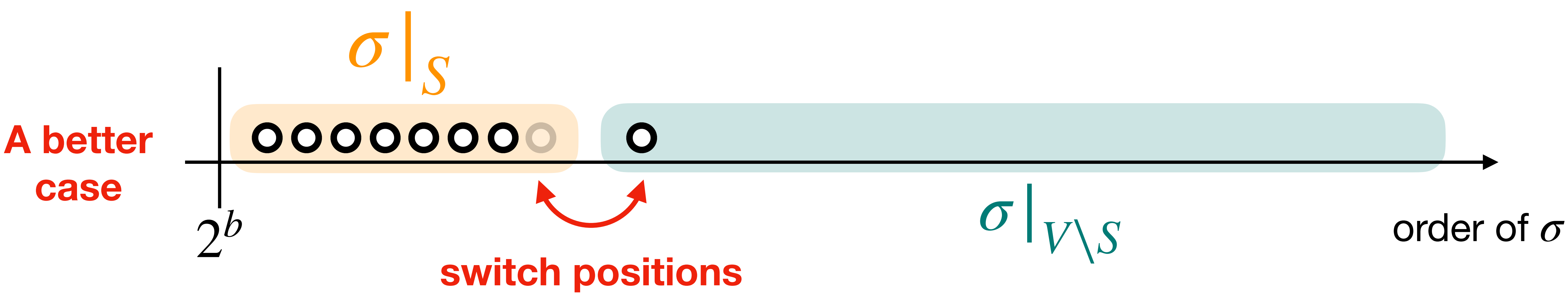

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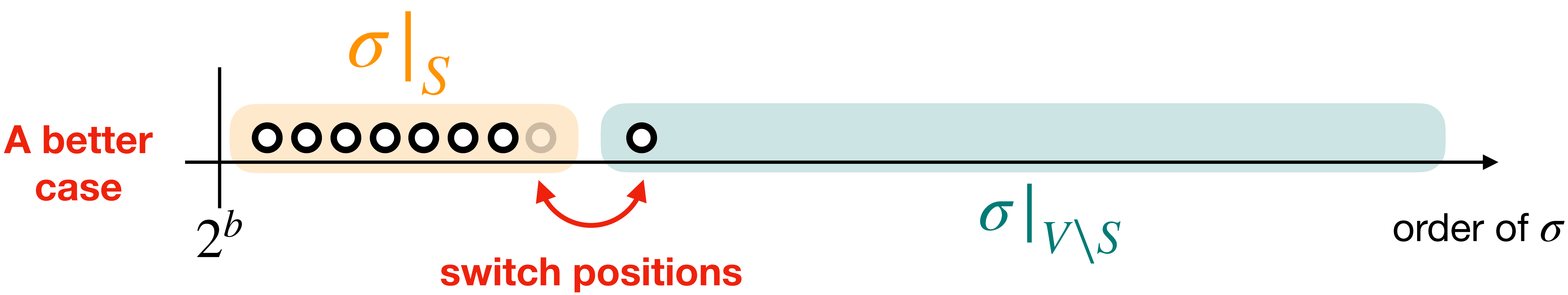
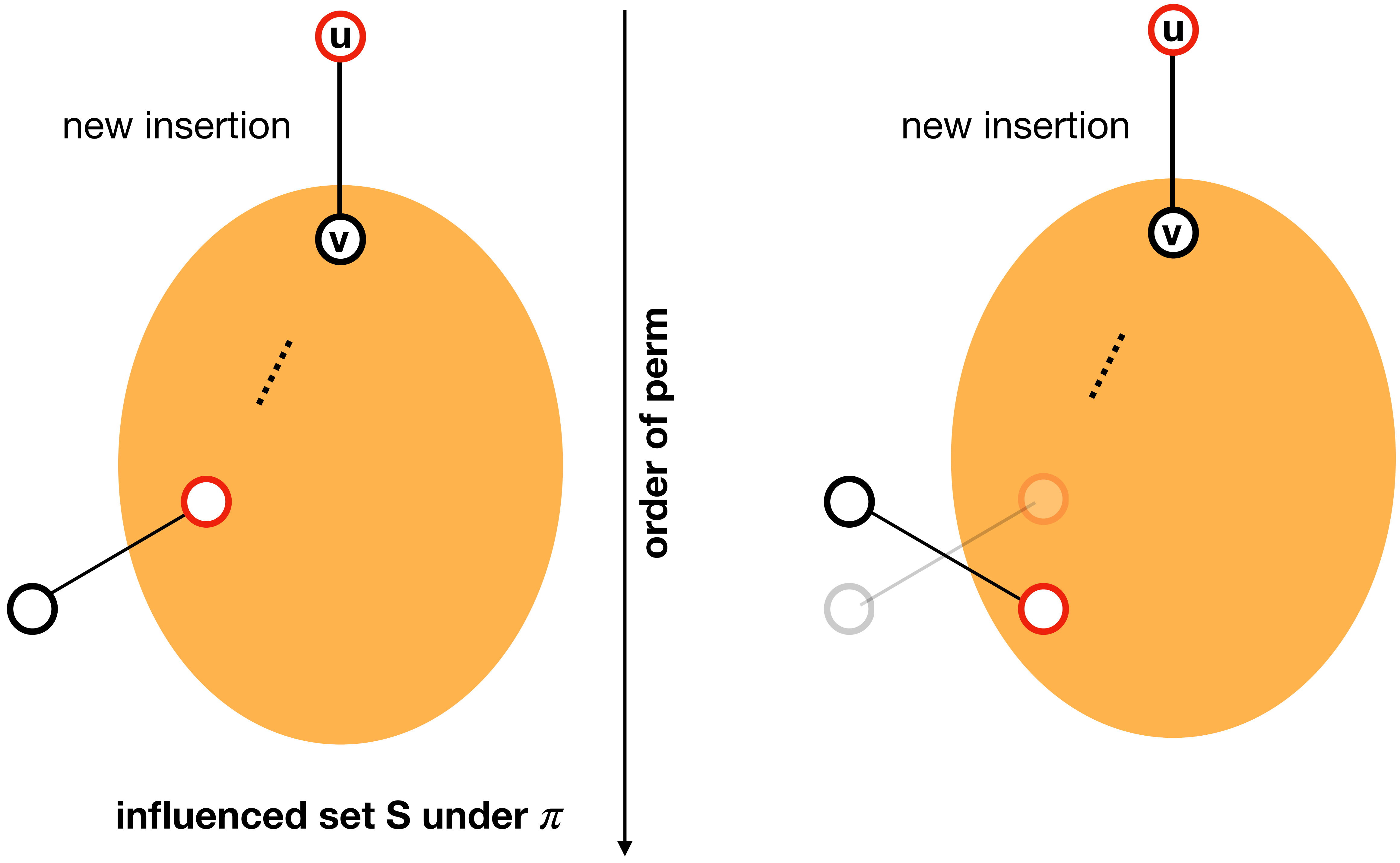


Prove S is still the influenced set under the new permutation σ

Running time analysis



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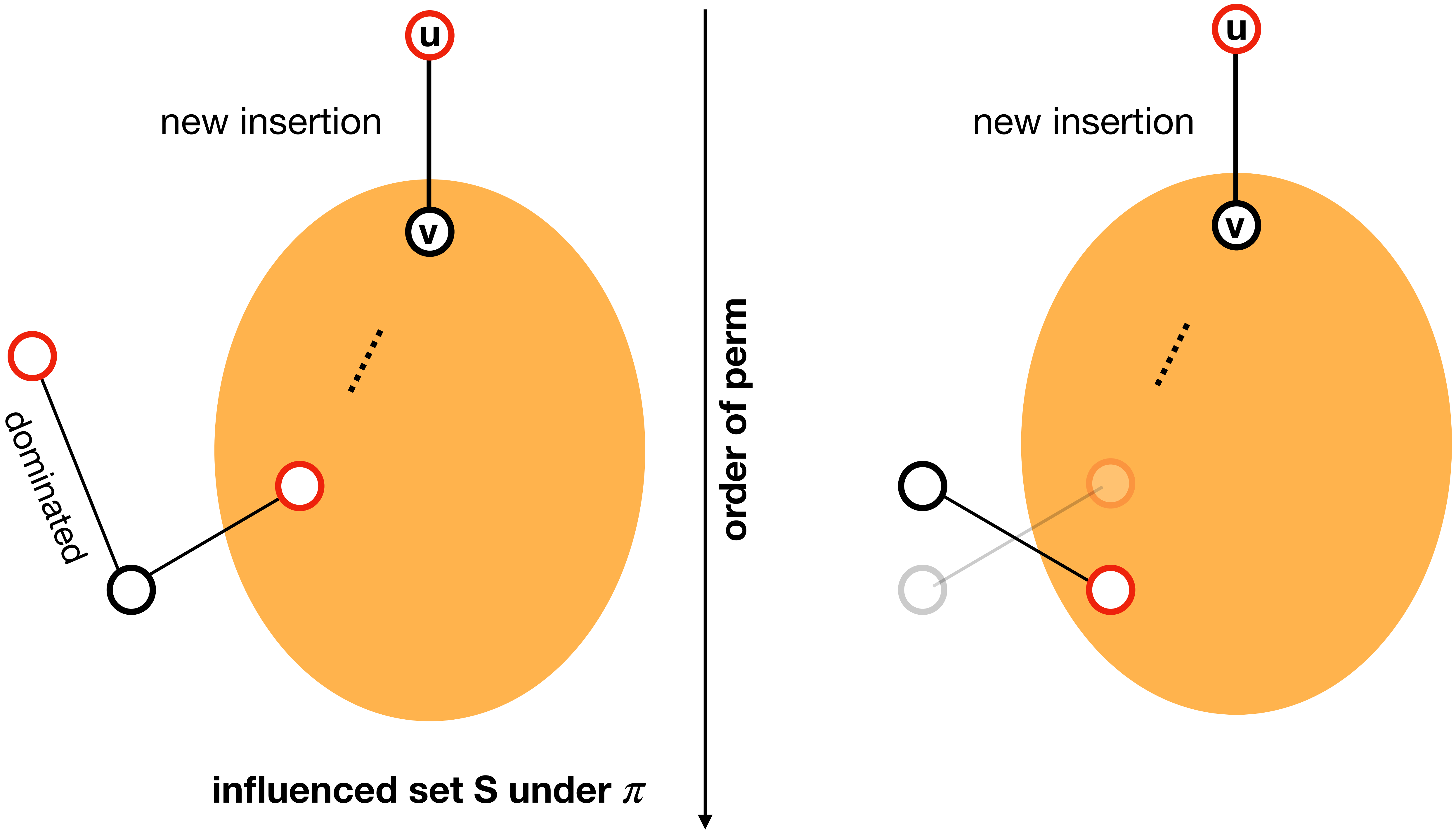

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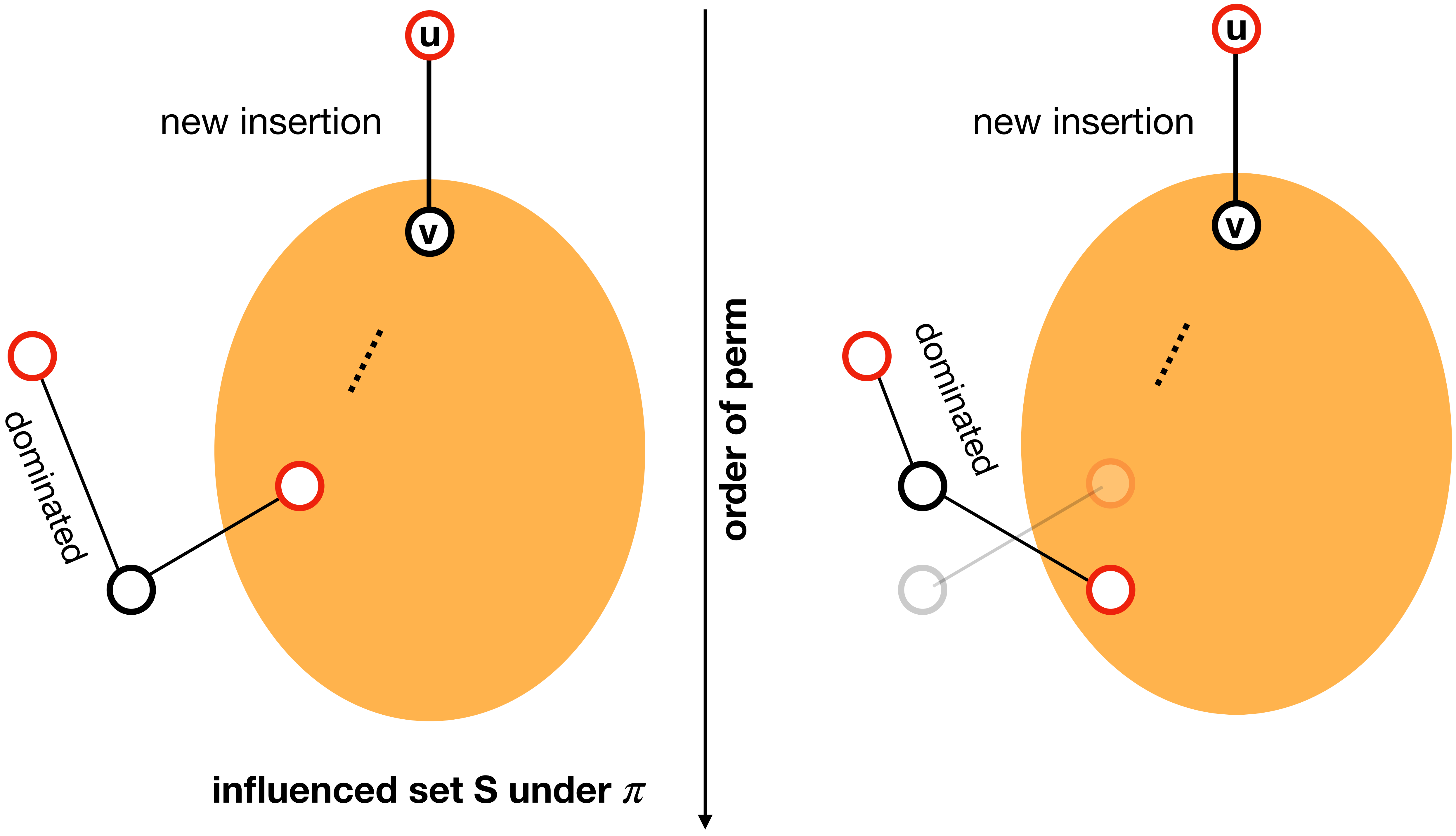

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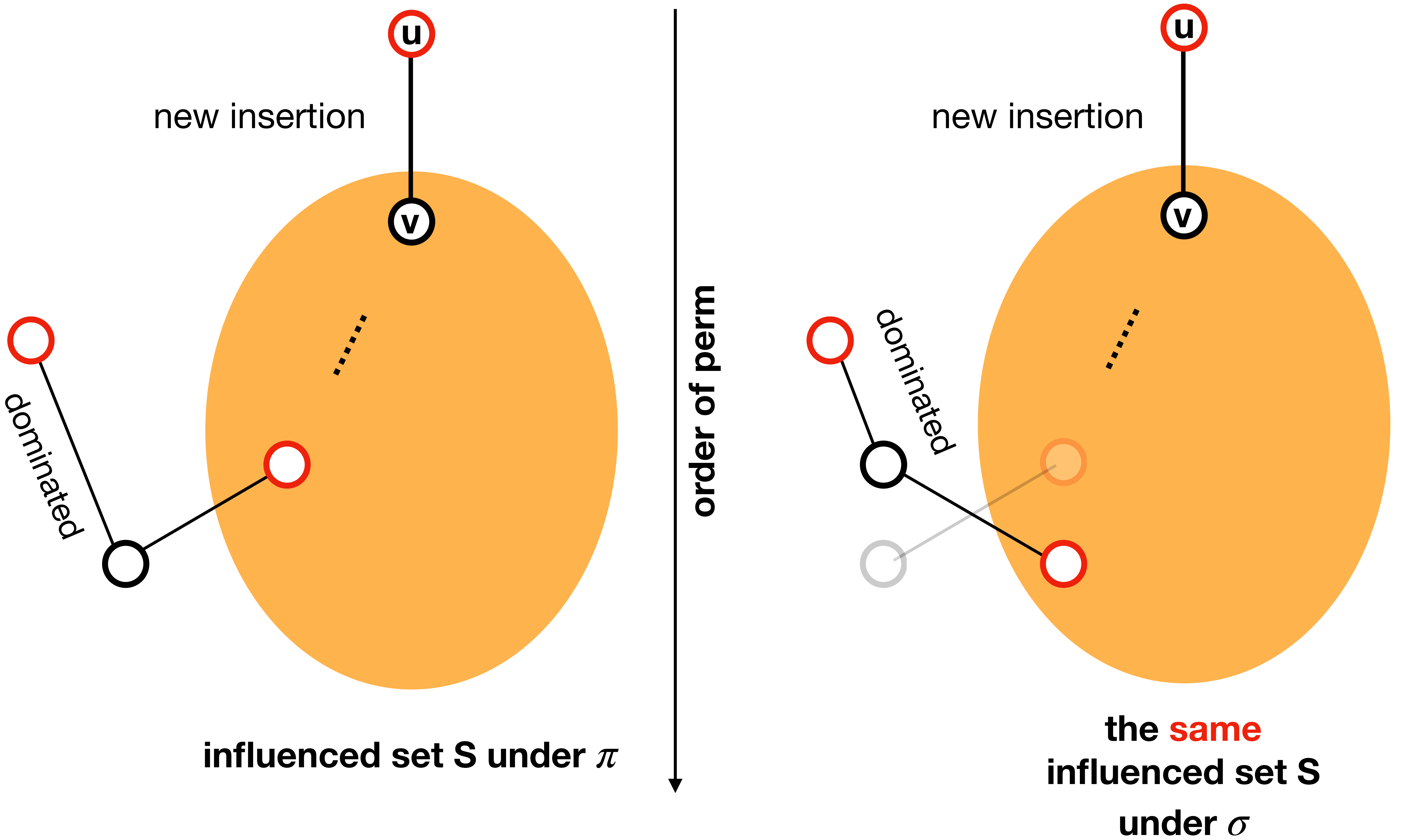

Thanks