Faster Min-Plus Product for Monotone Instances

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Structured Min-Plus Problems

Min-Plus Product

- Given integral $n \times n$ matrices *A* and *B*, compute $(\min, +)$ product:

$$(A \star B)_{i,j} = \min_k \{A_{i,k} + B_{k,j}\}$$

- Min-Plus Product is equivalent to All-Pairs Shortest Paths [FM, 1971]
- Fastest runtime for $(\min, +)$ or APSP: $n^3/2^{\Omega(\sqrt{\log n})}$ [Williams, 2018]
- Hardness conjecture: $(\min, +)$ product requires $n^{3-o(1)}$ time

Structured Min-Plus Product

- If A and B have bounded entries $\in \{-W, \cdots, W, \infty\}$ then $A \star B$ can be computed in $\tilde{O}(Wn^{\omega})$ time [Alon et al., 1997]
- Inputs matrices with more structures:
  - Bounded-difference matrices [Bringmann et al., 2016]
  - Monotone matrices [Vassilevska Williams and Xu, 2020]

Bounded-Difference Min-Plus Product

- Matrix X has bounded-difference if:

$$|X_{i,j} - X_{i,j+1}|, |X_{i,j} - X_{i+1,j}| \leq 1$$

- Want to compute $A \star B$ when A and B both have bounded-difference
- Many applications in string problems:
  - Language edit distance, RNA folding [Bringmann et al., 2016]
  - Tree edit distance [Mao, 2021]
  - Dyck edit distance [Fried et al., 2022]

Monotone Min-Plus Product

- Matrix X is monotone if:

$$0 \leq X_{i,j} \leq X_{i,j+1} \leq O(n)$$

- Want to compute $A \star B$ when B is monotone
- Generalization of bounded-difference min-plus product [GPWX, 2021]
- Further application in graph problems:
  - Single-source replacement paths with negative weights [GPWX, 2021]

Min-Plus Convolution

- Given integral arrays *A* and *B* of length *n*, compute $(\min, +)$ convolution:
  $$(A \diamond B)_k = \min_i \{A_{k-i} + B_i\}$$
- Fastest runtime for $(\min, +)$ conv: $n^2/2^{\Omega(\sqrt{\log n})}$ [Williams, 2018]
- Hardness conjecture: $(\min, +)$ conv requires $n^{2-o(1)}$ time
  - Stronger than APSP or 3SUM conjecture

Monotone Min-Plus Convolution

- Array X is monotone if:

$$0 \leq X_i \leq X_{i+1} \leq O(n)$$

- Want to compute $A \diamond B$ when A and B both are monotone

- Applications:

  - Histogram indexing, necklace alignment [BCD+, 2006] [ACLL, 2014]

Runtime for Structured Min-Plus Instances

reference	bounded-difference min-plus product	monotone min-plus product	monotone min-plus convolution
baseline	n^3	n^3	n^2
Chan and Lewenstein 2015			$n^{1.859}$
Bringmann et al. 2016	$n^{2.824}$		
V. Williams and Xu 2020		$n^{(15+\omega)/6}$	
Gu et al. 2021		$n^{(12+\omega)/5}$	
Chi, Duan and Xie 2022	$n^{2+\omega/3}$		
new	$n^{(3+\omega)/2}$	$n^{(3+\omega)/2}$	$n^{1.5}$

FMM exponent $\omega < 2.373$ [Alman and Vassilevska Williams, 2021]

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Monotone Min-Plus Product

with runtime $n^{2+\omega/3}$

Approximate Min-Plus

- For convenience, assume $\omega = 2$ for the rest
- Estimate $C = A \star B$ up to sub-linear additive errors
  A common step in previous works
- Rounding: $\tilde{A}_{i,j} = \lfloor A_{i,j}/n^{1/3} \rfloor$, $\tilde{B}_{i,j} = \lfloor B_{i,j}/n^{1/3} \rfloor$
  Compute: $\tilde{C} = \tilde{A} \star \tilde{B}$
- Approximation: $|\tilde{C}_{i,j} - C_{i,j}/n^{1/3}| = O(1)$
  Runtime: $\tilde{O}(n^{2+2/3})$

Quotient & Remainder

Basic idea: match quotients with $\tilde{C}$, then find minimum remainders

$$A_{i,k} = n^{1/3}\tilde{A}_{i,k} + \hat{A}_{i,k}$$

$$B_{k,j} = n^{1/3}\tilde{B}_{k,j} + \hat{B}_{k,j}$$

$$C_{i,j} = n^{1/3}\tilde{C}_{i,j} + ?$$

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quotients remainder

Only focus on $k \in [n]$ such that:

$$|\tilde{A}_{i,k} + \tilde{B}_{k,j} - \tilde{C}_{i,j}| = O(1)$$

and then minimize:

$$\hat{A}_{i,k} + \hat{B}_{k,j}$$

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quotients remainder

Using polynomials:

$$A_{i,k}(x, y) = x^{\hat{A}_{i,k}} \cdot y^{\tilde{A}_{i,k}}$$

$$B_{k,j}(x, y) = x^{\hat{B}_{k,j}} \cdot y^{\tilde{B}_{k,j}}$$

$$C_{i,j}(x, y) = \sum_{k=1}^{n} (A_{i,k} \cdot B_{k,j})(x, y)$$

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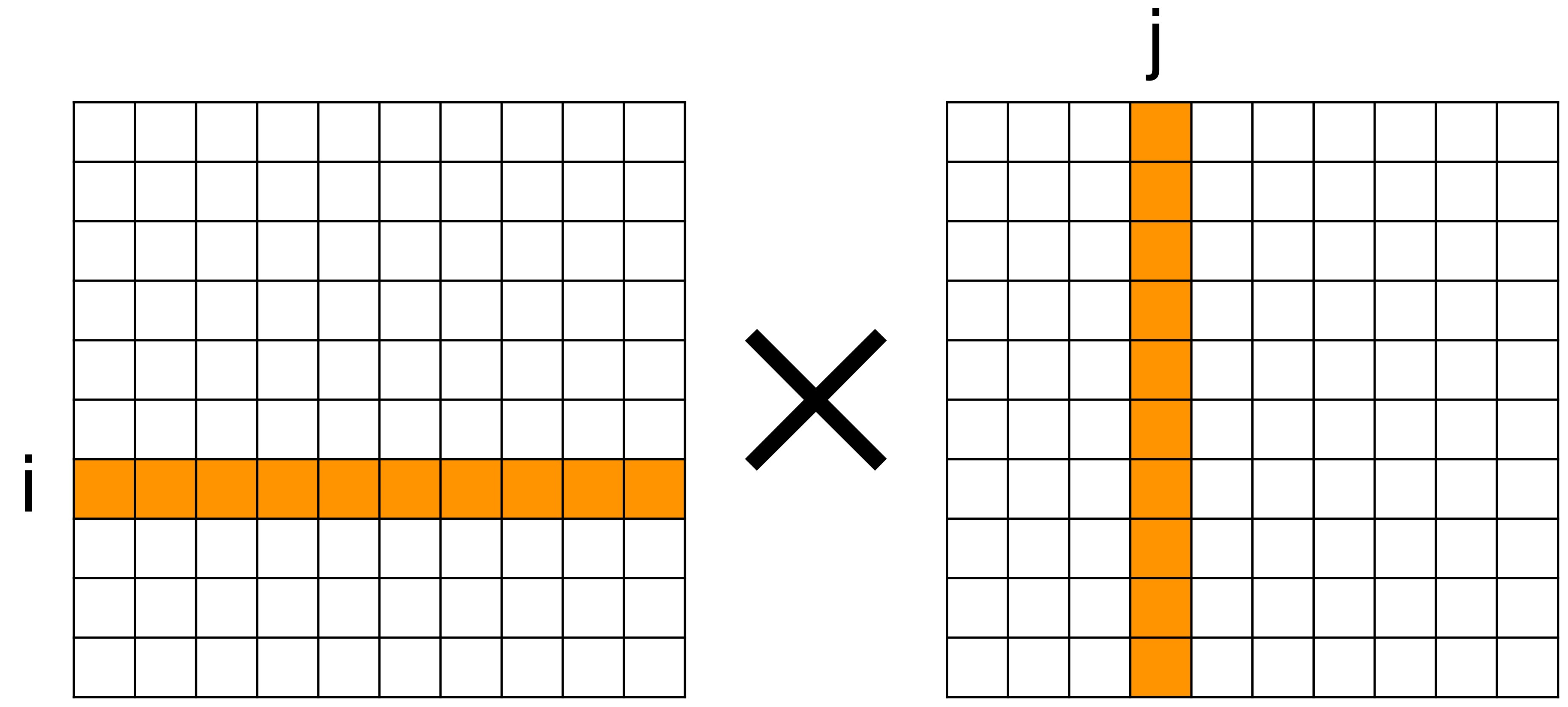

$$C_{i,j}(x, y) = \sum_{k=1}^{n} (A_{i,k} \cdot B_{k,j})(x, y)$$

$$= y^{\tilde{C}_{i,j}} F_0(x) + y^{\tilde{C}_{i,j}+1} F_1(x) + y^{\tilde{C}_{i,j}+2} F_2(x) + \cdots\cdots$$

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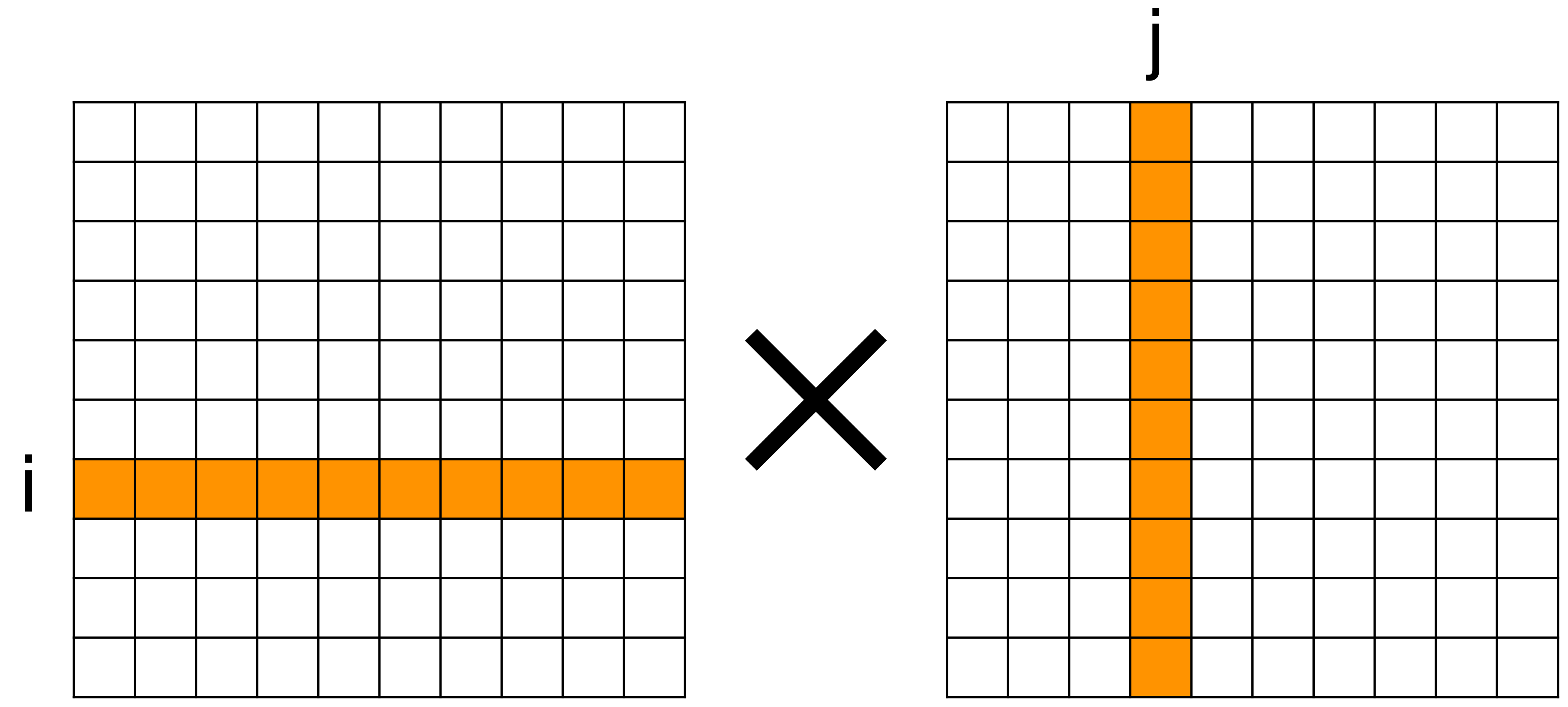

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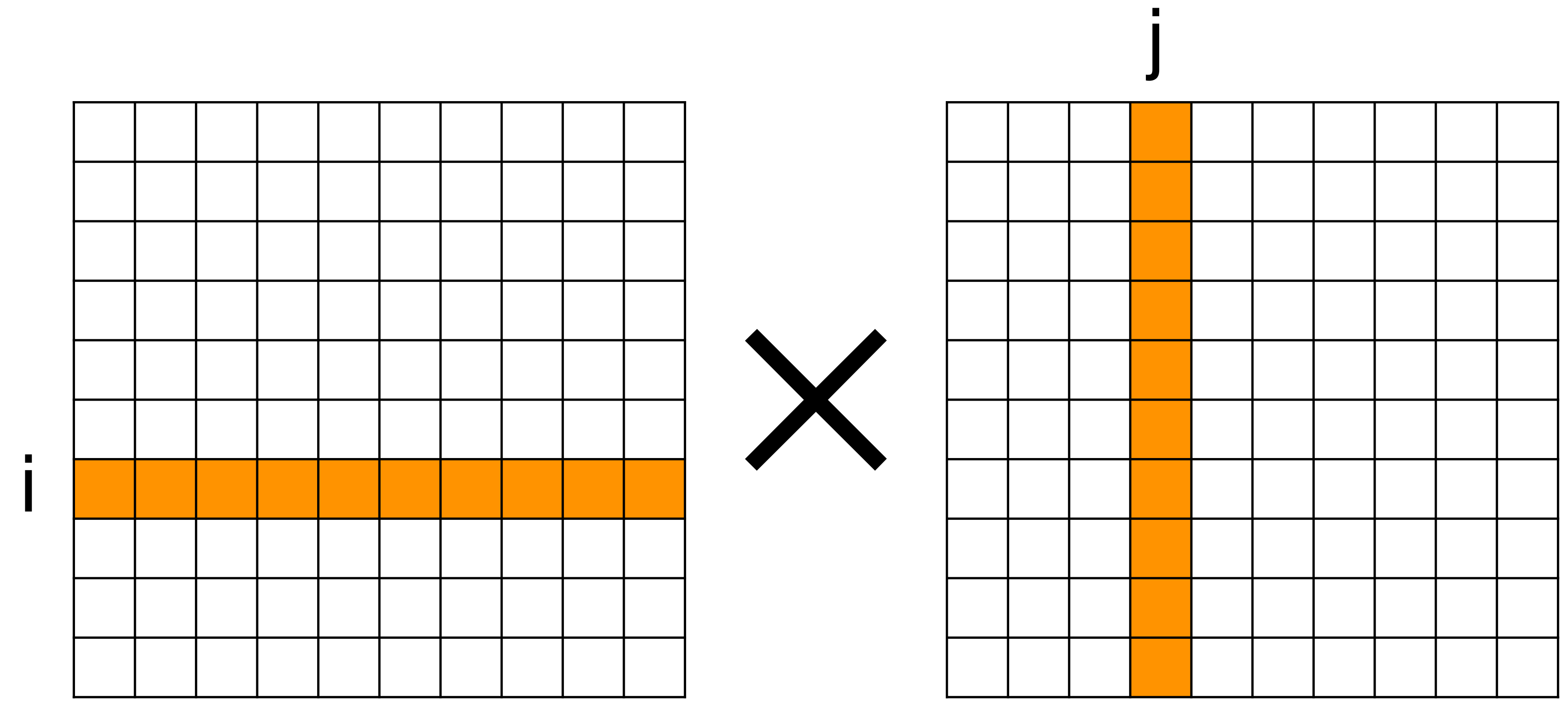

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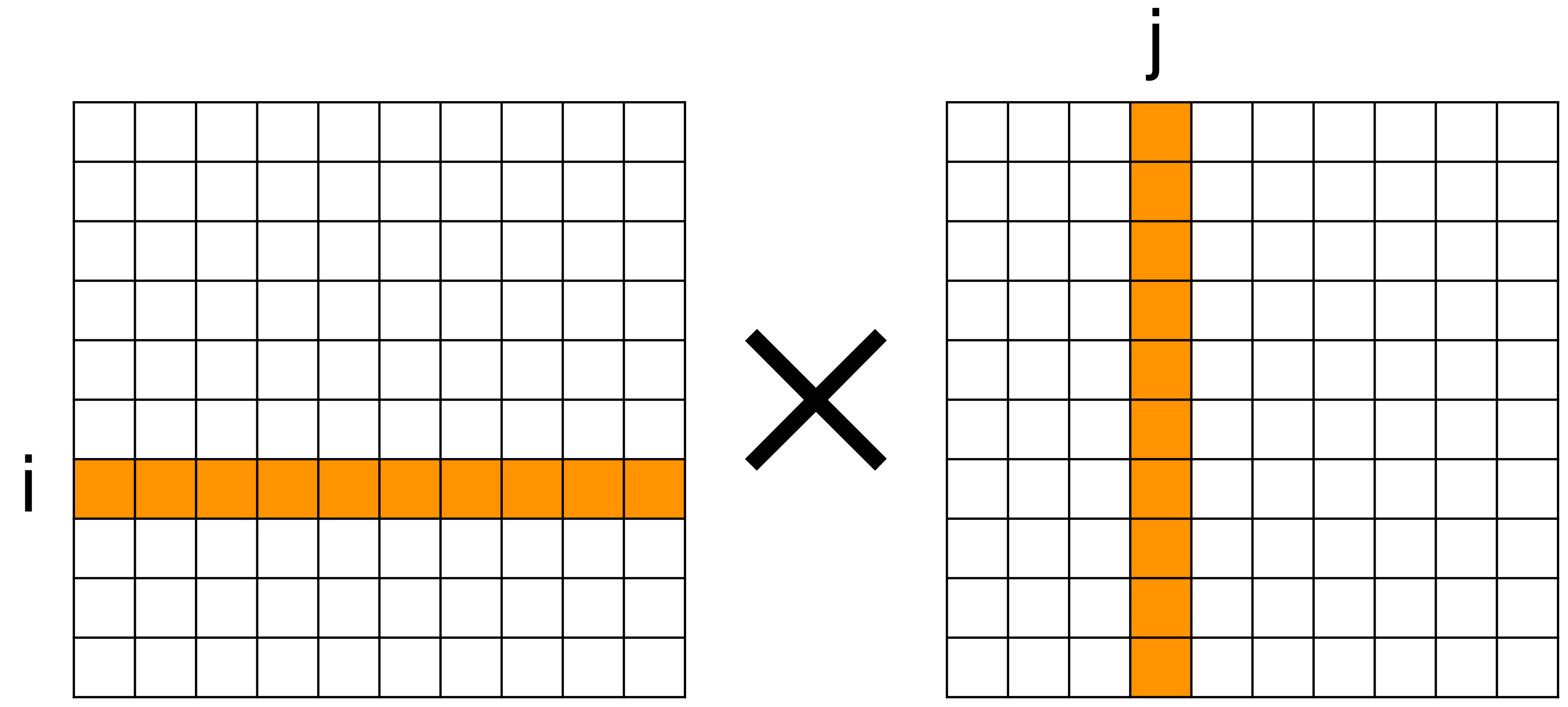

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$C_{i,j} = n^{1/3}(\tilde{C}_{i,j} + b) +$ **min-deg of** $F_b(x)$

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$$C_{i,j}(x,y) = \sum_{k=1}^{n} (A_{i,k} \cdot B_{k,j})(x,y)$$

Runtime? No help

Multiplying polynomial matrices

$$C(x,y) = A(x,y) \cdot B(x,y)$$

takes time

$$n^{\omega=2} \cdot \underbrace{\deg_x}_{= n^{1/3}} \cdot \underbrace{\deg_y}_{= n^{2/3}} = n^3$$

Modulo on y-degrees

Key idea: apply modulo-p operations on degrees of y-variables

Using polynomials:

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Degree reduction by modulo:

$$A^{p}_{i,k}(x,y) = x^{\hat{A}_{i,k}} \cdot y^{\tilde{A}_{i,k} \mod p}$$

$$B^{p}_{k,j}(x,y) = x^{\hat{B}_{k,j}} \cdot y^{\tilde{B}_{k,j} \mod p}$$

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Runtime?

Multiplying polynomial matrices

$$C^{p}(x,y) = A^{p}(x,y) \cdot B^{p}(x,y)$$

takes time

$$n^{\omega=2} \cdot \underbrace{\deg_x}_{=\,n^{1/3}} \cdot p = n^{2+1/3} p$$

Modulo on y-degrees

Key idea: apply modulo-p operations on degrees of y-variables

Only focus on $k \in [n]$ s.t. $|\tilde{A}_{i,k} + \tilde{B}_{k,j} - \tilde{C}_{i,j}| = O(1)$

$$C_{i,j}(x, y) = y^{\tilde{C}_{i,j}} F_0(x) + y^{\tilde{C}_{i,j}+1} F_1(x) + y^{\tilde{C}_{i,j}+2} F_2(x) + \cdots + y^{\tilde{C}_{i,j}+p} F_p(x) + y^{\tilde{C}_{i,j}+p+1} F_p(x) + \cdots\cdots$$

Taking modulo-p adds many erroneous x-monomials

$$C^p_{i,j}(x, y) = y^{\tilde{C}_{i,j} \mod p}[F_0(x) + F_p(x)] + y^{\tilde{C}_{i,j}+1 \mod p}[F_1(x) + F_{1+p}(x)] + y^{\tilde{C}_{i,j}+2 \mod p} F_2(x) + \cdots\cdots$$

In other words, many x-monomials are hashed to the same y-bucket

Bounding total errors

The number of erroneous x-terms:

If $p \in [n^{1/3}, 2n^{1/3}]$ is a random prime, then total #errors $= \tilde{O}(n^{3-1/3})$

Proof:

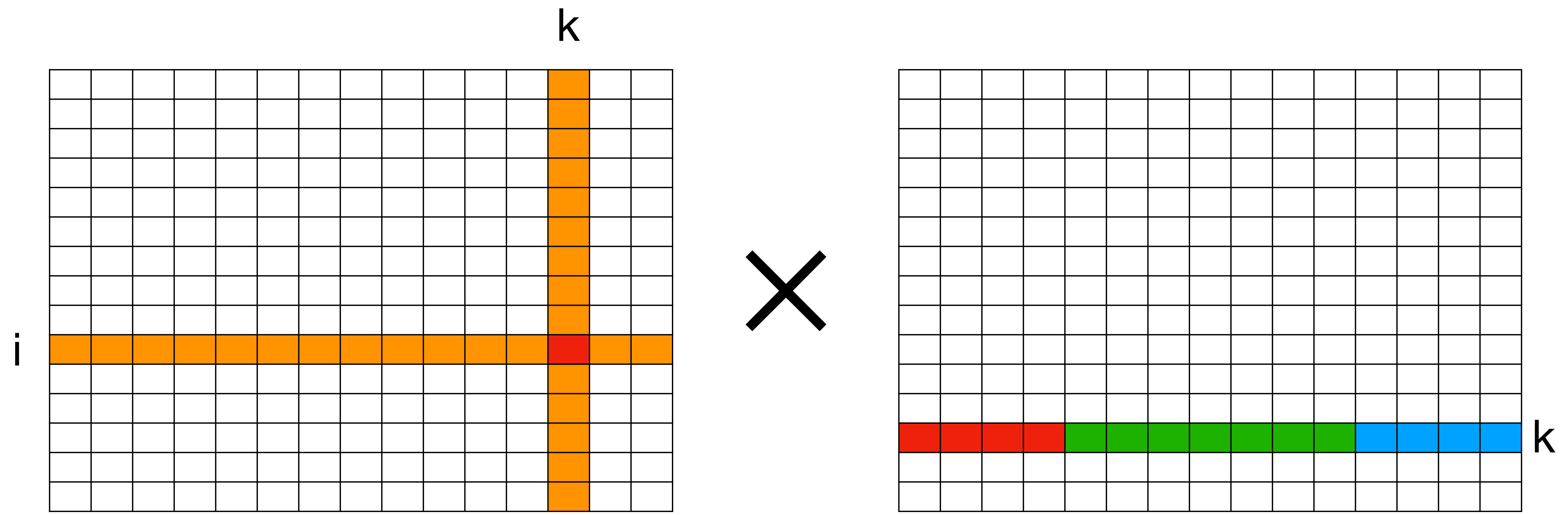
- Fix any $i, j, k \in [n]$ s.t. $|\tilde{A}_{i,k} + \tilde{B}_{k,j} - \tilde{C}_{i,j}| \neq O(1)$
  What is the probability that $\tilde{A}_{i,k} + \tilde{B}_{k,j} - \tilde{C}_{i,j} \equiv O(1) \mod p$
- As $p \in [n^{1/3}, 2n^{1/3}]$ is random, the probability is $\tilde{O}(n^{-1/3})$

Finding all error monomials

Goal:
Fix a prime $p \in [n^{1/3}, 2n^{1/3}]$, find all $(i, j, k) \in [n]^3$ such that:

$$|\tilde{A}_{i,k} + \tilde{B}_{k,j} - \tilde{C}_{i,j}| \neq O(1)$$

$$\tilde{A}_{i,k} + \tilde{B}_{k,j} - \tilde{C}_{i,j} \equiv O(1) \mod p$$



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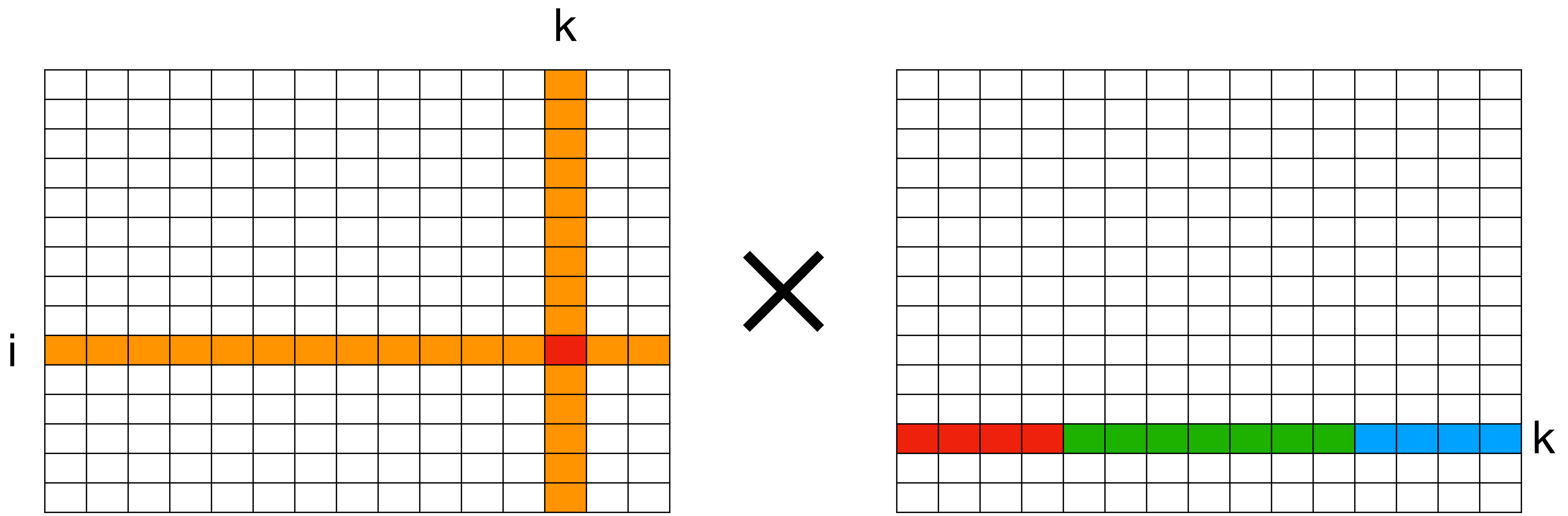

Divide each row into intervals

Entries in the each interval are the same

Finding all error monomials

Goal:
Fix a prime
$p \in [n^{1/3}, 2n^{1/3}]$, find all $(i,j,k) \in [n]^3$ such that:

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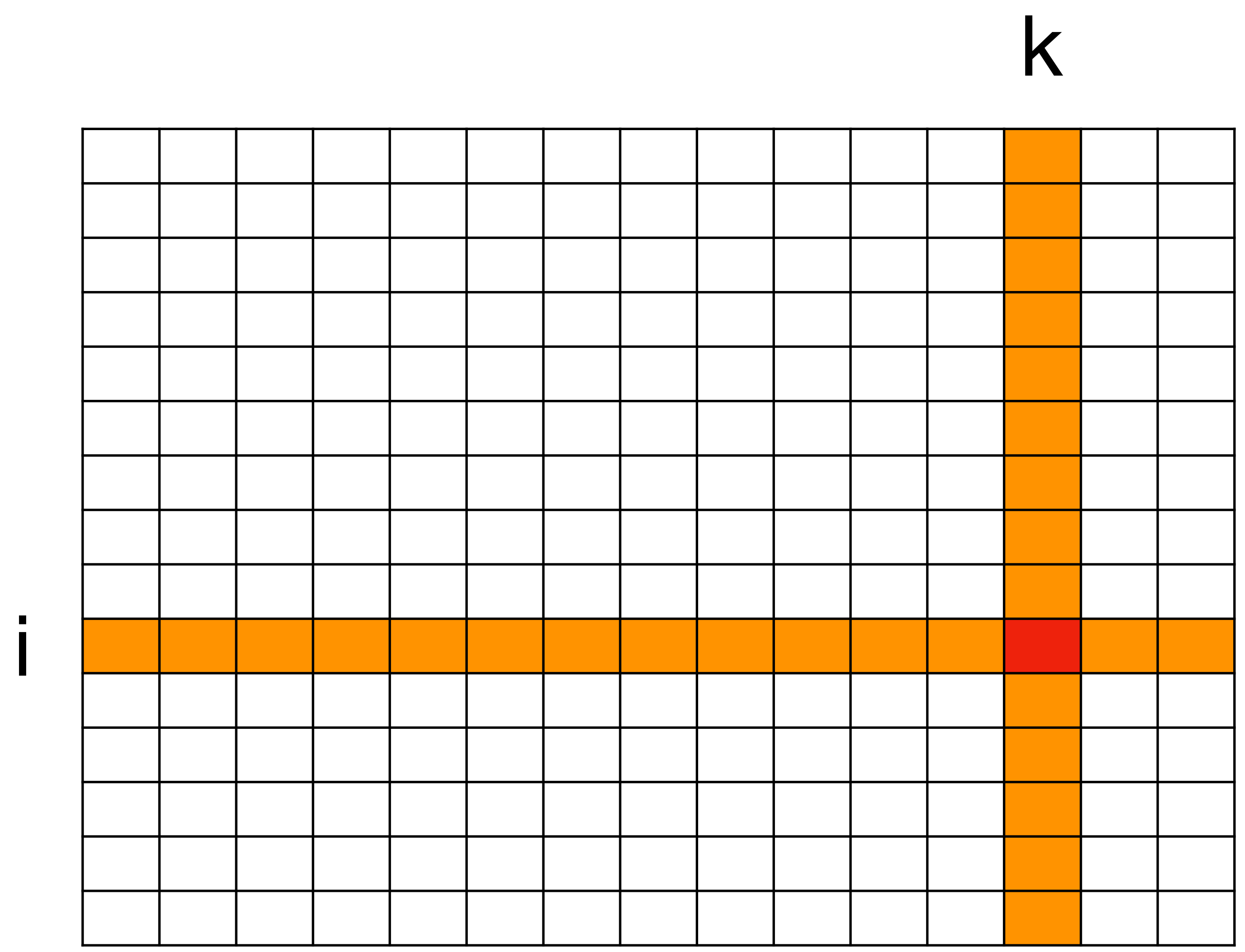
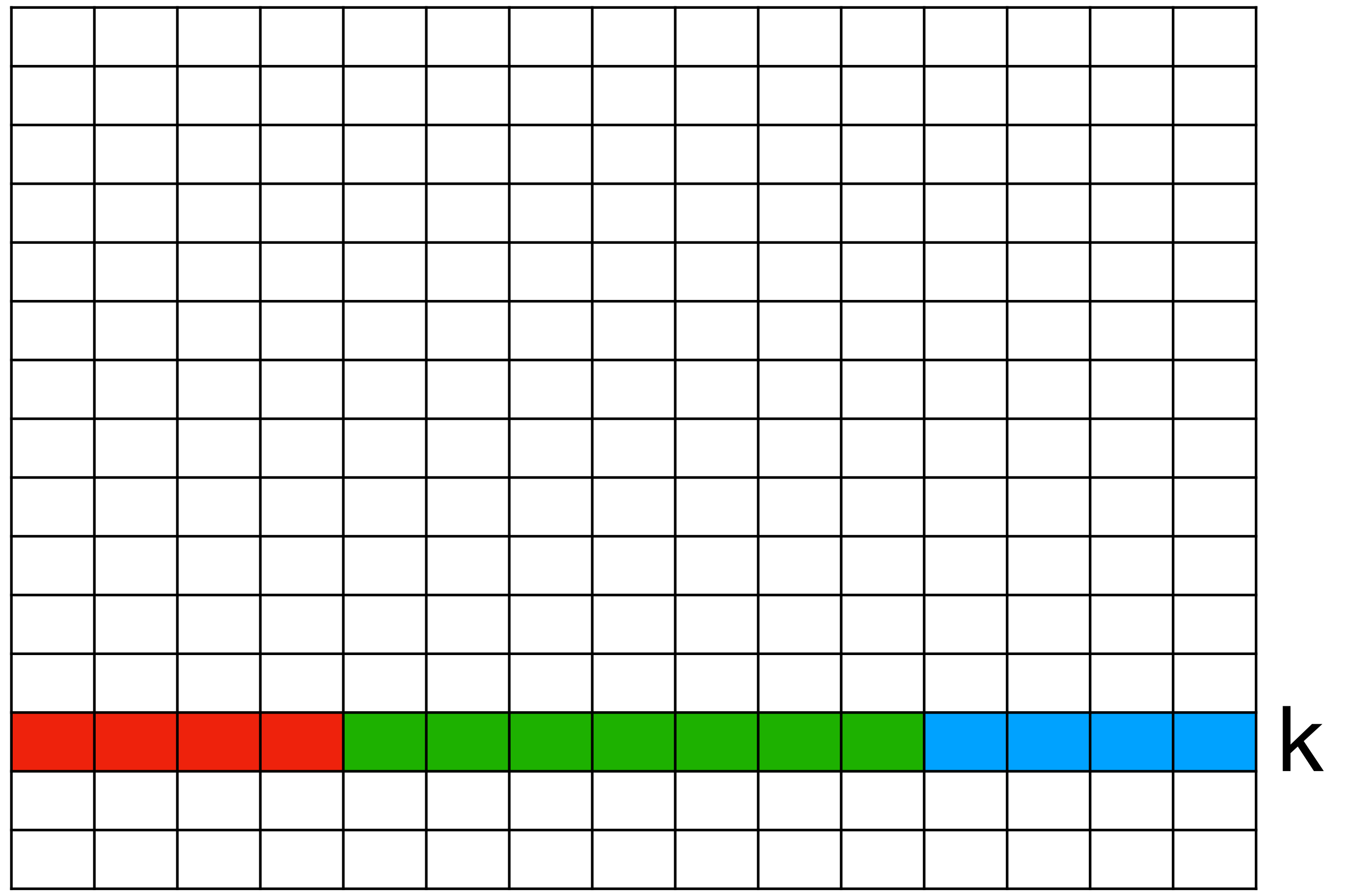


$\times$



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Since $\tilde{B}$ is monotone, there are at most $n^{2/3}$ intervals for each row

Finding all error monomials

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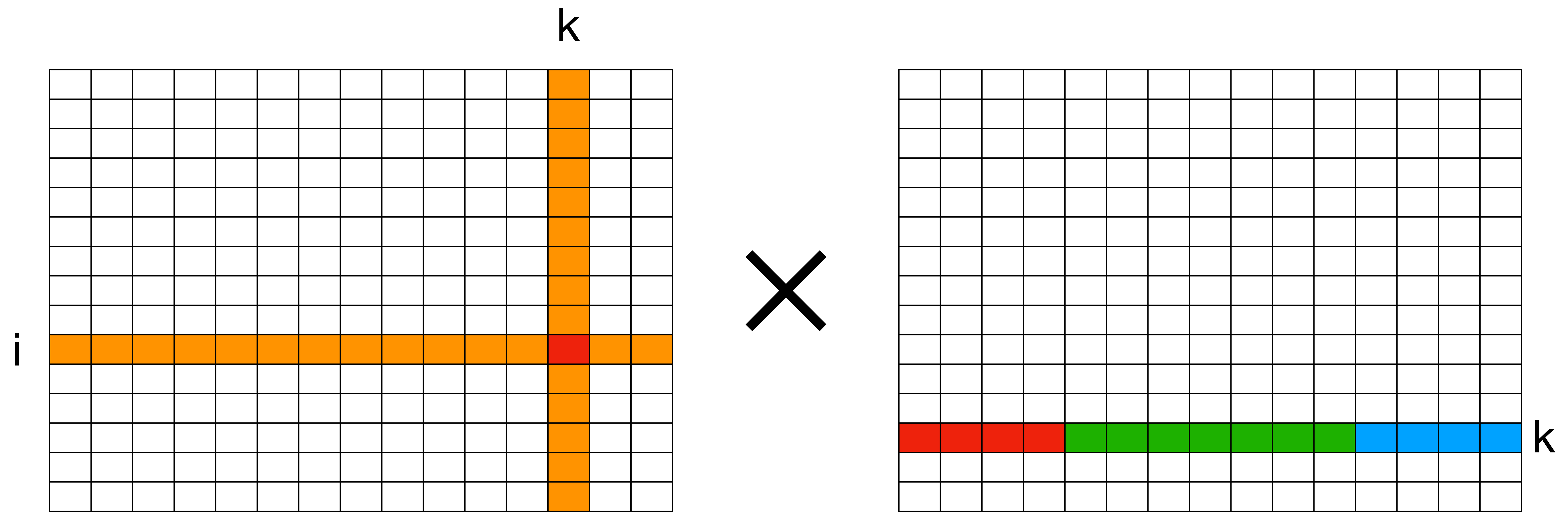

Algorithm:
- Fix any $i, k \in [n]$ and any interval of k-th row of B
- List all j in the interval such that
$$\tilde{C}_{i,j} \neq \tilde{A}_{i,k} + \tilde{B}_{k,j} - O(1)$$
$$\tilde{C}_{i,j} \equiv \tilde{A}_{i,k} + \tilde{B}_{k,j} - O(1) \mod p$$
- Total runtime = $\tilde{O}(n^{2+2/3})$

Divide each row into **intervals**

Entries in the each interval are the same

Since $\tilde{B}$ is monotone, there are at most $n^{2/3}$ intervals for each row

Monotone Min-Plus Product

with runtime $n^{(3+\omega)/2}$

Recursion: first attempt

<u>Basic idea:</u>

Instead of brute-force, compute the approx-matrix $\tilde{C} = \tilde{A} \star \tilde{B}$ by recursion

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$$\tilde{C} = \tilde{A} \star \tilde{B}, \text{ runtime} = n^{2+2/3}$$

$$C^p(x,y) = A^p(x,y) \cdot B^p(x,y), \text{ runtime} = n^{\omega=2} \cdot \deg_x \cdot p = n^{2+2/3}$$

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- Recursion: $A_{i,k} = 2\tilde{A}_{i,k} + \hat{A}_{i,k}\ ,\ \ B_{k,j} = 2\tilde{B}_{k,j} + \hat{B}_{k,j}$

  $\tilde{C} = \tilde{A} \star \tilde{B}$, by recursion

  $C^p(x,y) = A^p(x,y) \cdot B^p(x,y)$, runtime = $n^{\omega=2} \cdot \underbrace{\deg_x}_{=\,O(1)} \cdot p = n^2 p$

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  $C^p(x,y) = A^p(x,y) \cdot B^p(x,y)$, runtime = $n^{\omega=2} \cdot \underset{\substack{/\!/ \\ O(1)}}{\deg_x} \cdot p = n^2 p$

- Hopefully, we can choose $p = \Theta(n^{1/2})$, so total time = $n^{2.5}$

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- Recursion:

$$\lfloor \frac{A_{i,k}}{n^{1/2}} \rfloor, \lfloor \frac{2A_{i,k}}{n^{1/2}} \rfloor, \cdots \lfloor \frac{B_{k,j}}{n^{1/2}} \rfloor, \lfloor \frac{2B_{k,j}}{n^{1/2}} \rfloor, \cdots$$

$$A^{(l)}_{i,k}(x,y) = x^{\lfloor A_{i,k}/2^l \rfloor - 2\lfloor A_{i,k}/2^{l+1} \rfloor} \cdot y^{\lfloor A_{i,k}/2^{l+1} \rfloor} \mod p$$

$$B^{(l)}_{k,j}(x,y) = x^{\lfloor B_{k,j}/2^l \rfloor - 2\lfloor B_{k,j}/2^{l+1} \rfloor} \cdot y^{\lfloor B_{k,j}/2^{l+1} \rfloor} \mod p$$

Compute $C^{(l)}(x,y) = A^{(l)}(x,y) \cdot B^{(l)}(x,y)$

$$\lfloor \frac{C_{i,j}}{n^{1/2}} \rfloor, \lfloor \frac{2C_{i,j}}{n^{1/2}} \rfloor, \lfloor \frac{4C_{i,j}}{n^{1/2}} \rfloor, \lfloor \frac{8C_{i,j}}{n^{1/2}} \rfloor, \cdots\cdots$$

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- Error terms:

$$A_{i,k}^{(l)}(x,y) = x^{\lfloor A_{i,k}/2^l \rfloor - 2\lfloor A_{i,k}/2^{l+1} \rfloor} \cdot y^{\lfloor A_{i,k}/2^{l+1} \rfloor} \mod p$$

$$B_{k,j}^{(l)}(x,y) = x^{\lfloor B_{k,j}/2^l \rfloor - 2\lfloor B_{k,j}/2^{l+1} \rfloor} \cdot y^{\lfloor B_{k,j}/2^{l+1} \rfloor} \mod p$$

Compute $C^{(l)}(x,y) = A^{(l)}(x,y) \cdot B^{(l)}(x,y)$

find triples $(i,j,k) \in [n]^3$ s.t.

$$\lfloor A_{i,k}/2^l \rfloor + \lfloor B_{k,j}/2^l \rfloor - \lfloor C_{i,j}/2^l \rfloor \neq O(1)$$

$$\lfloor A_{i,k}/2^l \rfloor + \lfloor B_{k,j}/2^l \rfloor - \lfloor C_{i,j}/2^l \rfloor \equiv O(1) \mod p$$

Inductively maintain all such triples

Recursion: first attempt

Basic idea:

Instead of brute-force, compute the approx-matrix $\tilde{C} = \tilde{A} \star \tilde{B}$ by recursion

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$$\lfloor A_{i,k}/2^l \rfloor + \lfloor B_{k,j}/2^l \rfloor - \lfloor C_{i,j}/2^l \rfloor \equiv O(1) \mod p$$

Inductively maintain all such triples

- **Issues with induction:**

- Want to find $(i,j,k) \in [n]^3$

$$\lfloor A_{i,k}/2^l \rfloor + \lfloor B_{k,j}/2^l \rfloor - \lfloor C_{i,j}/2^l \rfloor \equiv 1 \mod p$$

Need to know triples:

$$\lfloor A_{i,k}/2^{l+1} \rfloor + \lfloor B_{k,j}/2^{l+1} \rfloor - \lfloor C_{i,j}/2^{l+1} \rfloor \equiv 2^{-1} \mod p$$

But only have triples:

$$\lfloor A_{i,k}/2^{l+1} \rfloor + \lfloor B_{k,j}/2^{l+1} \rfloor - \lfloor C_{i,j}/2^{l+1} \rfloor \equiv O(1) \mod p$$

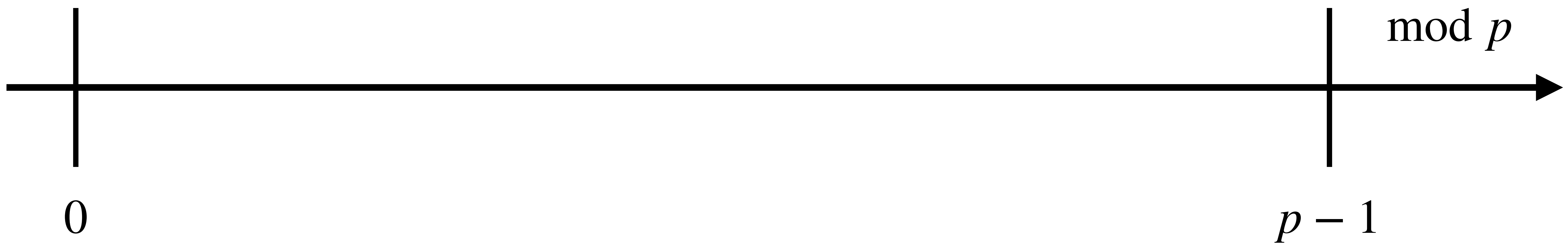
Recursion: first attempt

<u>Basic idea:</u>

Instead of brute-force, compute the approx-matrix $\tilde{C} = \tilde{A} \star \tilde{B}$ by recursion

<u>Maintaining erroneous terms during recursions:</u>

- **<u>Issue:</u>**
  division by 2 under modulo-p is not necessarily shrinking, could jump around
  $O(1)$ and 2^{-1} are far away under modulo-p



$$\Delta_l = \lfloor A_{i,k}/2^l \rfloor + \lfloor B_{k,j}/2^l \rfloor - \lfloor C_{i,j}/2^l \rfloor$$

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Hopefull all Δ_l
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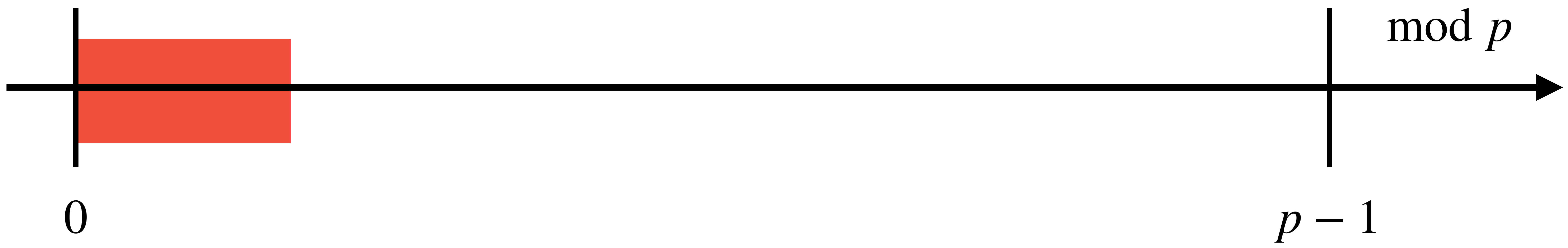

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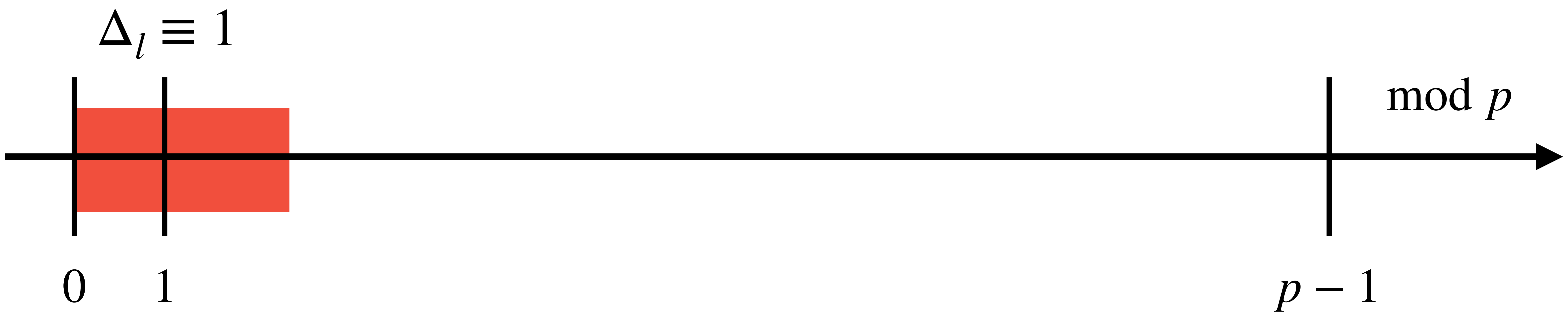

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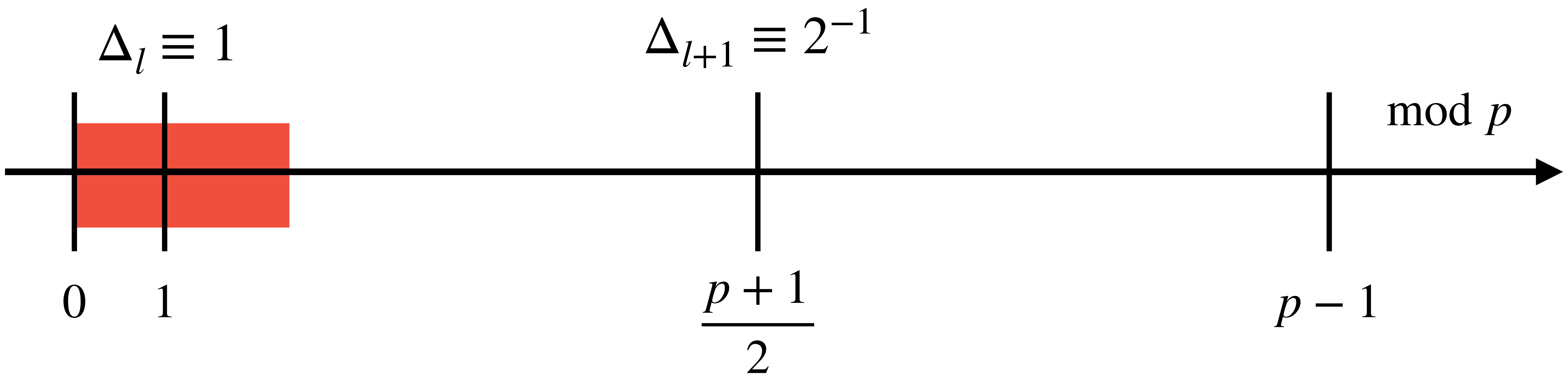

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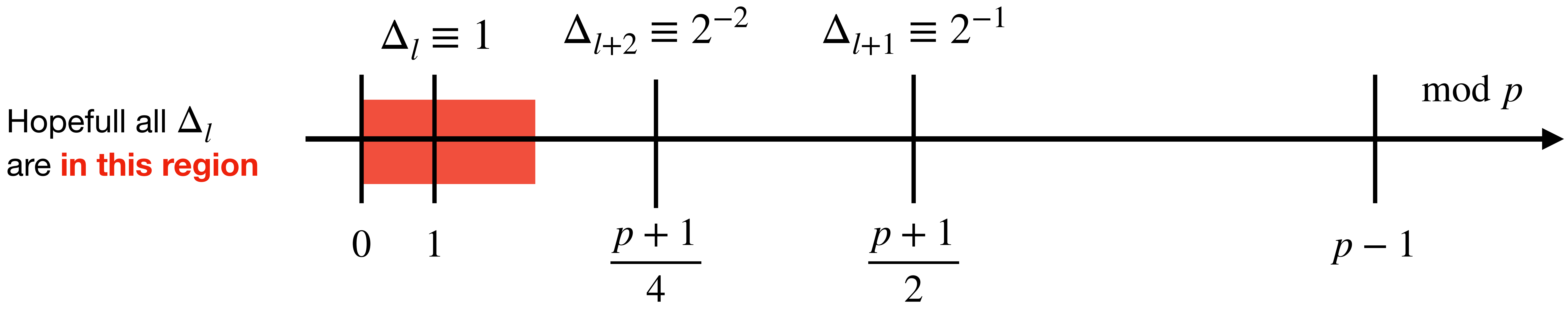

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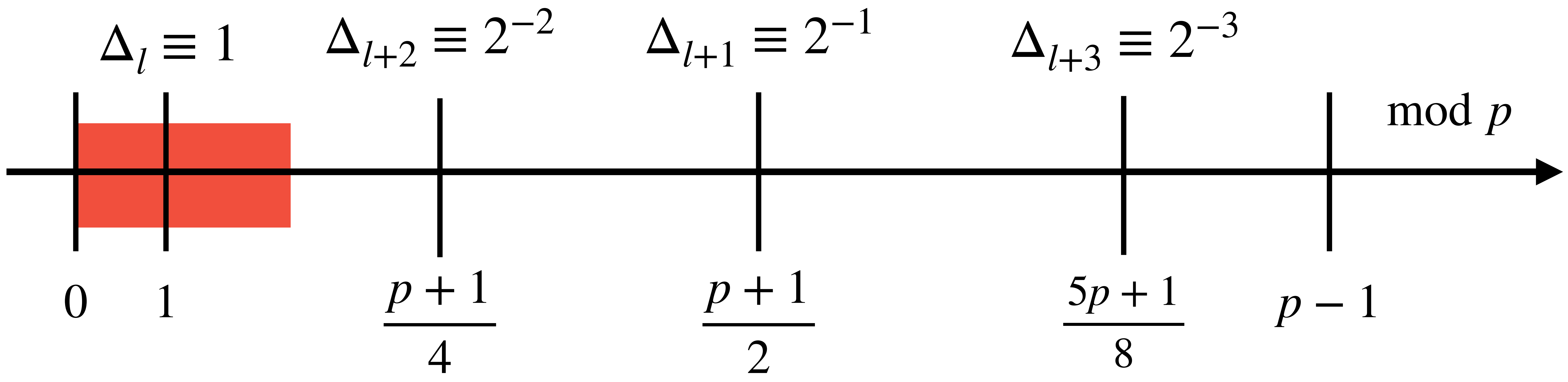

$$\Delta_l = \lfloor A_{i,k}/2^l \rfloor + \lfloor B_{k,j}/2^l \rfloor - \lfloor C_{i,j}/2^l \rfloor$$

Recursion: scaling residues

Key idea:

Instead of scaling entry values, only **scale the residues** modulo-p

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$$A^{(l)}_{i,k}(x,y) = x^{\{0,1\}} \cdot y^{\lfloor A_{i,k}/2^l \rfloor} \mod p \qquad B^{(l)}_{k,j}(x,y) = x^{\{0,1\}} \cdot y^{\lfloor B_{k,j}/2^l \rfloor} \mod p$$

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- Now: $A_{i,k} = p\tilde{A}_{i,k} + R_{i,k} \qquad B_{k,j} = p\tilde{B}_{k,j} + S_{k,j}$

$$\lfloor \frac{R_{i,k}}{n^{1/2}} \rfloor, \lfloor \frac{2R_{i,k}}{n^{1/2}} \rfloor, \lfloor \frac{4R_{i,k}}{n^{1/2}} \rfloor, \lfloor \frac{8R_{i,k}}{n^{1/2}} \rfloor, \ldots\ldots, \lfloor \frac{S_{k,j}}{n^{1/2}} \rfloor, \lfloor \frac{2S_{k,j}}{n^{1/2}} \rfloor, \lfloor \frac{4S_{k,j}}{n^{1/2}} \rfloor, \lfloor \frac{8S_{k,j}}{n^{1/2}} \rfloor, \ldots\ldots$$

$$A^{(l)}_{i,k}(x,y) = x^{\lfloor R_{i,k}/2^l \rfloor - 2\lfloor R_{i,k}/2^{l+1} \rfloor} \cdot y^{\lfloor R_{i,k}/2^{l+1} \rfloor} \qquad B^{(l)}_{k,j}(x,y) = x^{\lfloor S_{k,j}/2^l \rfloor - 2\lfloor S_{k,j}/2^{l+1} \rfloor} \cdot y^{\lfloor S_{k,j}/2^{l+1} \rfloor}$$

Recursion: scaling residues

Key idea:
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$$A_{i,k}^{(l)}(x,y) = x^{\lfloor R_{i,k}/2^l \rfloor - 2\lfloor R_{i,k}/2^{l+1} \rfloor} \cdot y^{\lfloor R_{i,k}/2^{l+1} \rfloor} \qquad B_{k,j}^{(l)}(x,y) = x^{\lfloor S_{k,j}/2^l \rfloor - 2\lfloor S_{k,j}/2^{l+1} \rfloor} \cdot y^{\lfloor S_{k,j}/2^{l+1} \rfloor}$$

- Recursion: $C_{i,j} = p\tilde{C}_{i,j} + T_{i,j}$

$$\lfloor T_{i,j}/2^l \rfloor = \min_{1 \le k \le n} \{ \lfloor R_{i,k}/2^l \rfloor + \lfloor S_{k,j}/2^l \rfloor \mid |\tilde{A}_{i,k} + \tilde{B}_{k,j} - \tilde{C}_{i,j}| = O(1) \} \pm O(1)$$

Compute $C^{(l)}(x,y) = A^{(l)}(x,y) \cdot B^{(l)}(x,y)$, and then some extra work

Recursion: scaling residues

Key idea:

Instead of scaling entry values, only **scale the residues** modulo-p

- Error terms:

$$A^{(l)}_{i,k}(x,y) = x^{\lfloor R_{i,k}/2^l \rfloor - 2\lfloor R_{i,k}/2^{l+1} \rfloor} \cdot y^{\lfloor R_{i,k}/2^{l+1} \rfloor}$$

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Compute $C^{(l)}(x,y) = A^{(l)}(x,y) \cdot B^{(l)}(x,y)$

find triples $(i,j,k) \in [n]^3$ s.t.

$\lfloor T_{i,j}/2^l \rfloor = \lfloor R_{i,k}/2^l \rfloor + \lfloor S_{k,j}/2^l \rfloor \pm O(1)$

$|\tilde{A}_{i,k} + \tilde{B}_{k,j} - \tilde{C}_{i,j}| \neq O(1)$

Inductively maintain all such triples

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$|\tilde{A}_{i,k} + \tilde{B}_{k,j} - \tilde{C}_{i,j}| \neq O(1)$

Inductively maintain all such triples

- **Previous issues fixed:**

- Want to find $(i,j,k) \in [n]^3$

$\lfloor T_{i,j}/2^l \rfloor = \lfloor R_{i,k}/2^l \rfloor + \lfloor S_{k,j}/2^l \rfloor \pm O(1)$

Need to know triples:

$\lfloor T_{i,j}/2^{l+1} \rfloor = \lfloor R_{i,k}/2^{l+1} \rfloor + \lfloor S_{k,j}/2^{l+1} \rfloor \pm \frac{O(1)}{2}$

Already have maintained:

$\lfloor T_{i,j}/2^{l+1} \rfloor = \lfloor R_{i,k}/2^{l+1} \rfloor + \lfloor S_{k,j}/2^{l+1} \rfloor \pm O(1)$

Recursion: finding error terms

<u>Basic idea:</u>

Use the fact that $p \in [n^{1/2}, 2n^{1/2}]$ is a uniformly random prime

<u>Goal:</u>

- At each recursion level, find all $(i, j, k) \in [n]^3$ such that:
- $\lfloor T_{i,j}/2^l \rfloor = \lfloor R_{i,k}/2^l \rfloor + \lfloor S_{k,j}/2^l \rfloor \pm O(1)$
  $|\tilde{A}_{i,k} + \tilde{B}_{k,j} - \tilde{C}_{i,j}| \neq O(1)$

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Sanity check:

- Even without recursion (l=0), the probability that $T_{i,j} = R_{i,k} + S_{k,j}$ is $\tilde{O}(n^{-1/2})$
- Need recursion to locate all k's

Recursion: finding error terms

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Goal:

- At each recursion level, find all $(i, j, k) \in [n]^3$ such that:
- $\lfloor T_{i,j}/2^l \rfloor = \lfloor R_{i,k}/2^l \rfloor + \lfloor S_{k,j}/2^l \rfloor \pm O(1)$
  $|\tilde{A}_{i,k} + \tilde{B}_{k,j} - \tilde{C}_{i,j}| \neq O(1)$

Total number of error terms:

- $\lfloor T_{i,j}/2^l \rfloor = \lfloor R_{i,k}/2^l \rfloor + \lfloor S_{k,j}/2^l \rfloor \pm O(1)$
  or
  $|A_{i,k} + B_{k,j} - C_{i,j}| \equiv O(2^l) \mod p$
- Probability is at most $\tilde{O}(2^l \cdot n^{-1/2})$
  Total #errors = $\tilde{O}(2^l \cdot n^3/p)$

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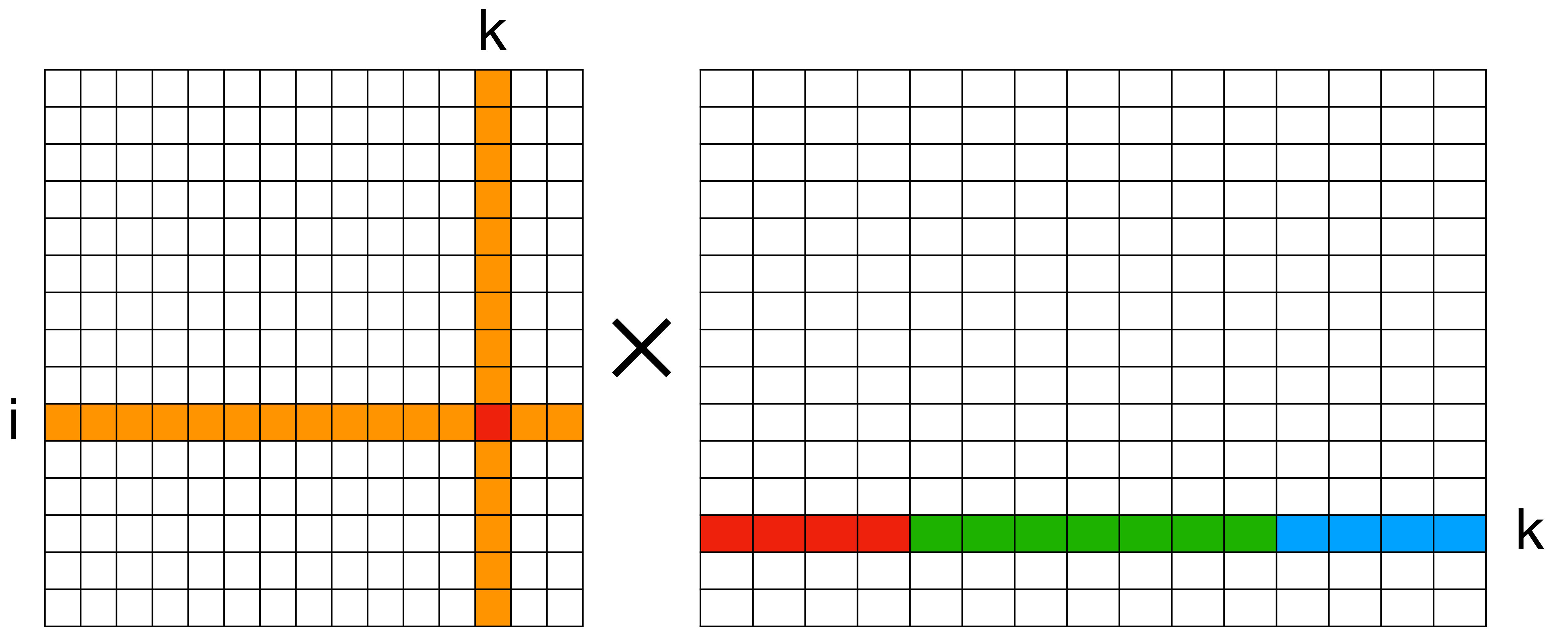
Extra factor of 2^l



Key observation:
Since B is monotone, we can partition each row $\{\lfloor S_{k,j}/2^l \rfloor\}_{1 \le j \le n}$ into $O(n/2^l)$ **intervals**, such that entries in the each interval are the same. Group error terms as **index-interval triples** $(i, k, [a, b])$

Recursion: finding error terms

Index-interval triples:

Use the fact that $p \in [n^{1/2}, 2n^{1/2}]$ is a uniformly random prime

Total #Index-interval triples:

- $|A_{i,k} + B_{k,j} - C_{i,j}| \equiv O(2^l) \mod p$
- Probability is at most $\tilde{O}(2^l \cdot n^{-1/2})$
  Total #error terms = $\tilde{O}(2^l \cdot n^3/p)$
- Partition each B-row into $O(n/2^l)$ intervals
- Total #index-interval triples = $\tilde{O}(n^3/p)$

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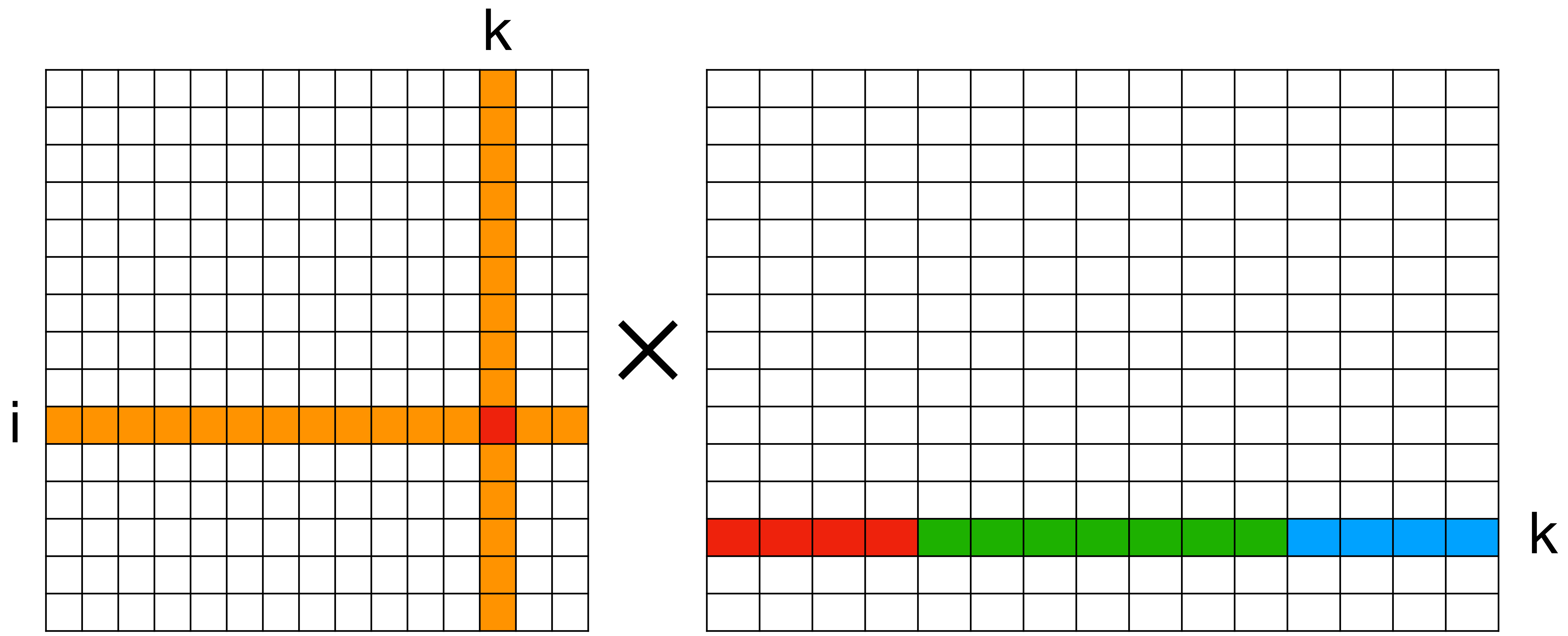

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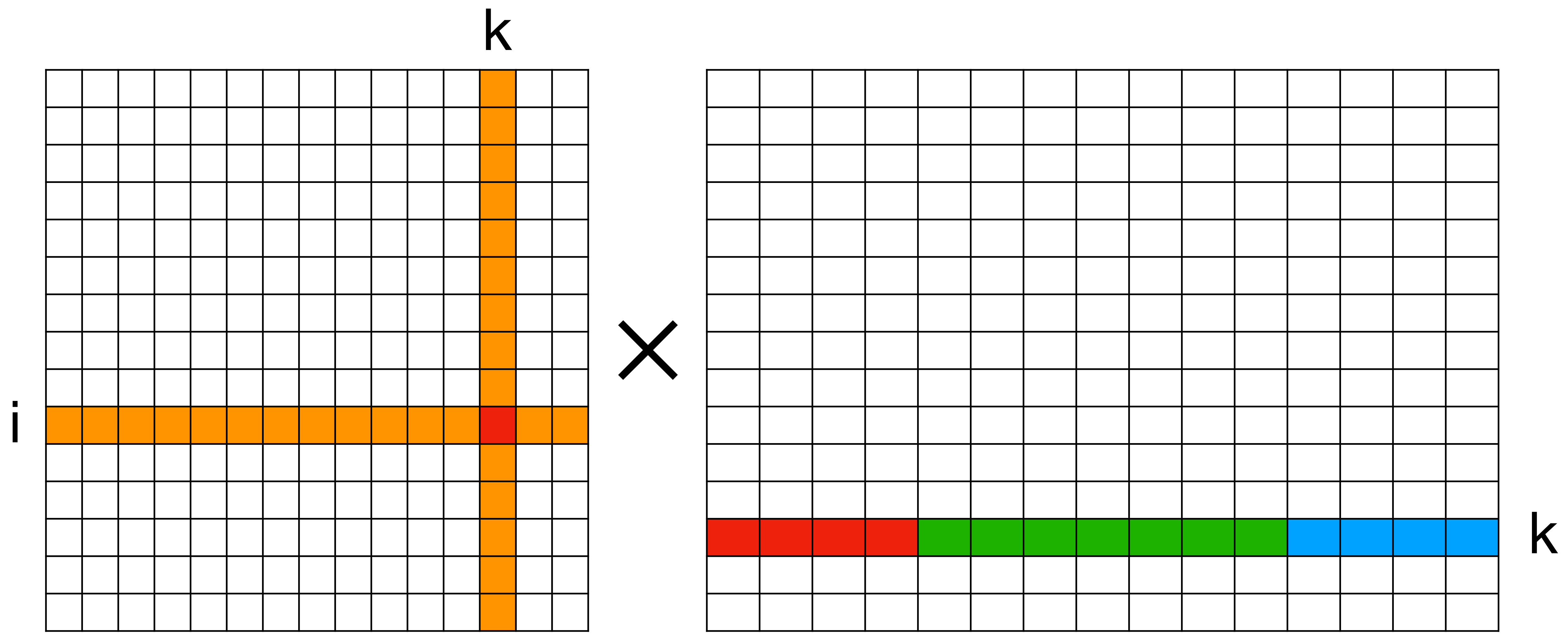

Subtracting error terms:

For any $(i, k, [a, b])$, use segment-tree data structures

$$C^{(l)}_{i,j}(x, y) \leftarrow C^{(l)}_{i,j}(x, y) - A^{(l)}_{i,k}(x, y) \cdot B^{(l)}_{k,j}(x, y)$$

simultaneously for all $j \in [a, b]$, as all $B^{(l)}_{k,j}$ are equal

Recursion: finding error terms

Index-interval triples:

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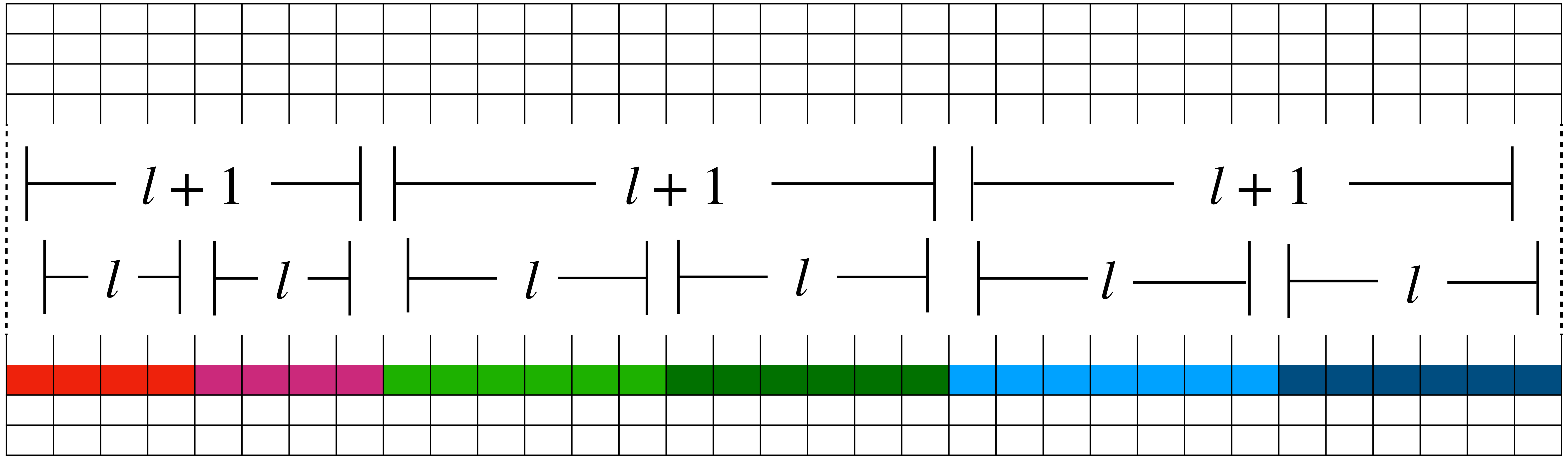

Maintaining index-interval triples recursively:

- Triples $\{(i, k, [a, b])\}$ on the l-level is a **refinement** of triples on the $(l+1)$-level

Thanks for listening