1. Definitions & Applications

**Monotone Min-Plus Product**

\[(A \star B)_{i,j} = \min_k \{A_{i,k} + B_{k,j}\}\]

\[0 \leq B_{i,1} \leq B_{i,2} \leq \cdots \leq B_{i,n} = O(n)\]

**Monotone Min-Plus Convolution**

\[(A \circ B)_i = \min_j \{A_j + B_{i-j}\}\]

\[0 \leq A_1 \leq A_2 \leq \cdots \leq A_n = O(n)\]

\[0 \leq B_1 \leq B_2 \leq \cdots \leq B_k = O(n)\]

Applications in other algorithmic problems

- Language edit distance
- Monotone Min-Plus Product & Convolution
- Histogram indexing
- Tree edit distance
- Dyck edit distance
- Single source replacement paths

3. Faster Monotone Min-Plus Product

\[A_{i,k} = n^{1/3} \tilde{A}_{i,k} + R_{i,k}\]

\[B_{i,j} = n^{1/3} \tilde{B}_{i,j} + S_{i,j}\]

\[C_{i,j} = n^{1/3} \tilde{C}_{i,j} + ?\]

**Degree reduction by modulo:**

\[A_{i,j}(x,y) = x^{R_{i,j}} \cdot y^{\tilde{A}_{i,j}} \mod p\]

\[B_{i,j}(x,y) = x^{S_{i,j}} \cdot y^{\tilde{B}_{i,j}} \mod p\]

\[C_{i,j}(x,y) = \sum_{k=1}^{n} (A_{i,k} \cdot B_{k,j})(x,y)\]

Only focus on \(k \in [n]\) such that:

\[|\tilde{A}_{i,k} + \tilde{B}_{k,j} - \tilde{C}_{i,j}| = O(1)\]

and then minimize:

\[R_{i,k} + S_{j,k}\]

4. Faster Monotone Min-Plus Convolution

**Degree reduction by modulo:**

\[A_j(x,y) = x^{R_j} \cdot y^{\tilde{A}_j} \mod p, z\]

\[B_j(x,y) = x^{S_j} \cdot y^{\tilde{B}_j} \mod p, z\]

\[C_j(x,y) = \sum_{i=1}^{n} (A_i \cdot B_i)(x,y)\]

Only focus on \(j \in [n]\) such that:

\[|\tilde{A}_j + \tilde{B}_j - \tilde{C}_j| = O(1)\]

and then minimize:

\[R_j + S_{i,j}\]