Faster Min-Plus Product for Monotone Instances

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1. Definitions & Applications

Monotone Min-Plus Product

$$(A \star B)_{i,j} = \min_k \{A_{i,k} + B_{k,j}\}$$

$$0 \le B_{i,1} \le B_{i,2} \le \cdots \le B_{i,n} = O(n)$$

Monotone Min-Plus Convolution

$$(A \diamond B)_i = \min_i \{A_j + B_{i-j}\}$$

$$0 \le A_1 \le A_2 \le \cdots \le A_n \le O(n) \qquad 0 \le B_1 \le B_2 \le \cdots \le B_n \le O(n)$$

Applications in other algorithmic problems

- language edit distance
- Monotone Min-Plus Product & Convolution
- histogram indexing
- tree edit distance
- Dyck edit distance
- single source replacement paths

2. History & Results

reference	bounded-difference min-plus product	monotone min-plus product	monotone min-plus convolution
baseline	n^3	n^3	n^2
Chan and Lewenstein 2015			$n^{1.859}$
Bringmann et al. 2016	$n^{2.824}$		
V. Williams and Xu 2020		$n^{(15+\omega)/6}$	
Gu et al. 2021		$n^{(12+\omega)/5}$	
Chi, Duan and Xie 2022	$n^{2+\omega/3}$		
new	$n^{(3+\omega)/2}$	$n^{(3+\omega)/2}$	$n^{1.5}$

3. Faster Monotone Min-Plus Product Simplified

$$A_{i,k} = n^{1/3}\tilde{A}_{i,k} + R_{i,k}$$
$$B_{k,j} = n^{1/3}\tilde{B}_{k,j} + S_{k,j}$$
$$C_{i,j} = n^{1/3}\tilde{C}_{i,j} + ?$$

quotient remainder

Only focus on $k \in [n]$ such that:

$$|\tilde{A}_{i,k} + \tilde{B}_{k,j} - \tilde{C}_{i,j}| = O(1)$$

and then minimize:

$$R_{i,k} + S_{k,j}$$

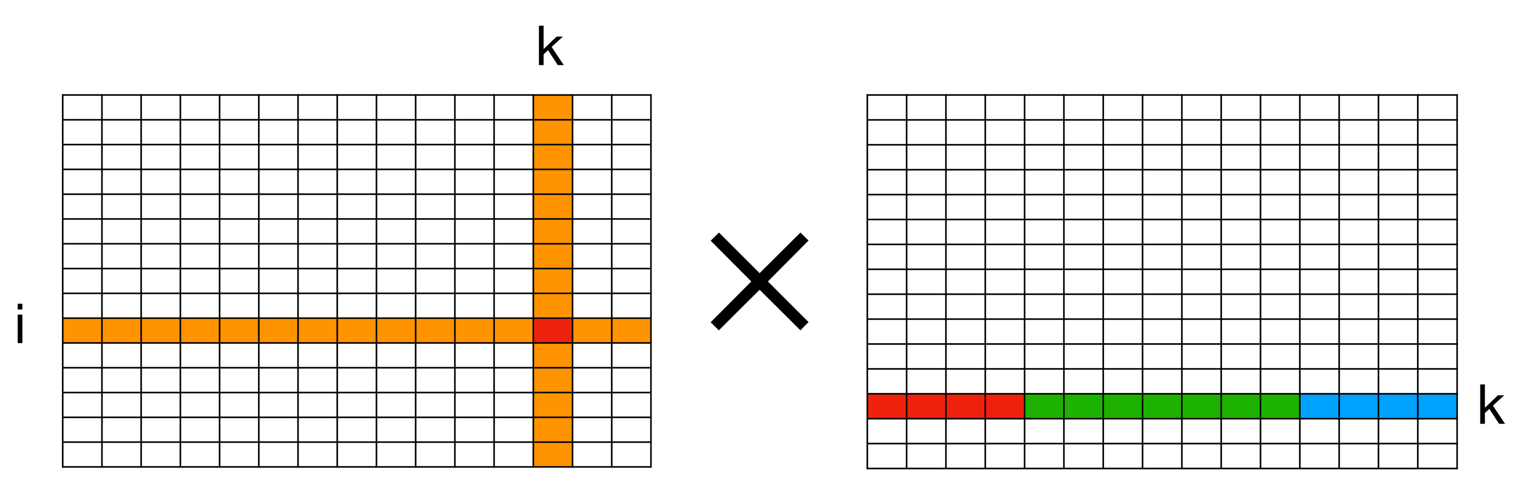
Degree reduction by modulo:

$$A^p_{i,k}(x,y) = x^{R_{i,k}} \cdot y^{\tilde{A}_{i,k} \mod p}$$
$$B^p_{k,j}(x,y) = x^{S_{k,j}} \cdot y^{\tilde{B}_{k,j} \mod p}$$
$$C^p_{i,j}(x,y) = \sum_{k=1}^{n} (A^p_{i,k} \cdot B^p_{k,j})(x,y)$$

$$C_{i,j}(x,y) = y^{\tilde{C}_{i,j}}F_0(x) + y^{\tilde{C}_{i,j}+1}F_1(x) + \cdots + y^{\tilde{C}_{i,j}+p}F_p(x) + y^{\tilde{C}_{i,j}+p+1}F_p(x) + \cdots$$

Modulo adds noise to output

$$C^p_{i,j}(x,y) = y^{\tilde{C}_{i,j} \mod p}\left(F_0(x) + F_p(x)\right) + y^{\tilde{C}_{i,j}+1 \mod p}\left(F_1(x) + F_p(x)\right) + \cdots$$



Finding all noisy terms:

- Fix any $i, k \in [n]$ List all j in the interval such that
  $\tilde{C}_{i,j} \ne \tilde{A}_{i,k} + \tilde{B}_{k,j} - O(1)$
  $\tilde{C}_{i,j} \equiv \tilde{A}_{i,k} + \tilde{B}_{k,j} - O(1) \mod p$
- Total runtime = $\tilde{O}(n^{2+2/3})$ $(\omega = 2)$

Divide each row into intervals
Entries in the each interval are the same

Since $\tilde{B}$ is monotone, there are at most $n^{2/3}$ intervals for each row

4. Faster Monotone Min-Plus Convolution Simplified

$$A_i = n^{1/5}\tilde{A}_i + R_i$$
$$B_i = n^{1/5}\tilde{B}_i + S_i$$
$$C_i = n^{1/5}\tilde{C}_i + ?$$

quotient remainder

Only focus on $j \in [n]$ such that:

$$|\tilde{A}_j + \tilde{B}_{i-j} - \tilde{C}_j| = O(1)$$

and then minimize:

$$R_j + S_{i-j}$$

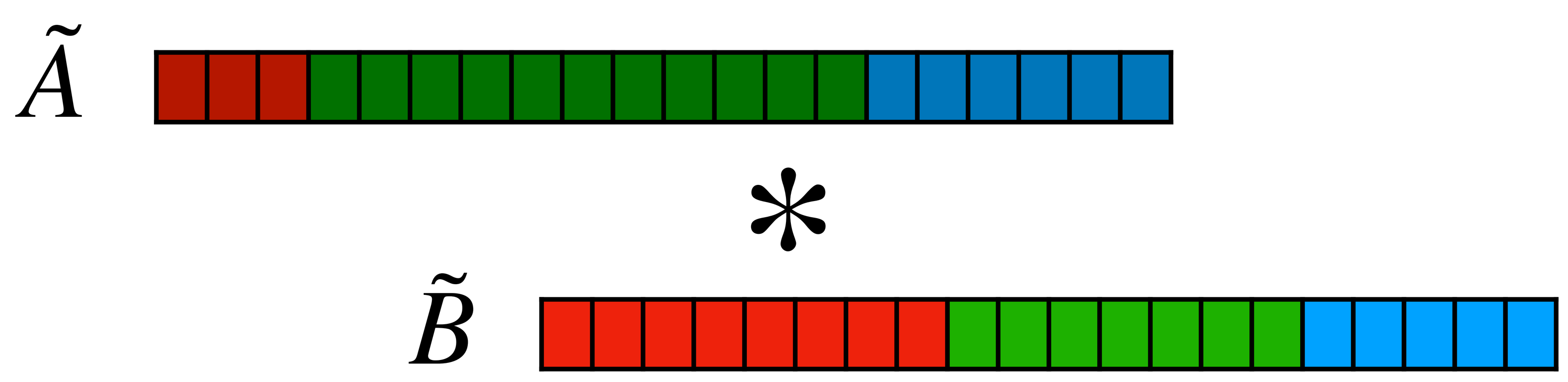
Degree reduction by modulo:

$$A^p_i(x,y,z) = x^{R_i} \cdot y^{\tilde{A}_i \mod p} \cdot z^i$$
$$B^p_i(x,y,z) = x^{S_i} \cdot y^{\tilde{B}_i \mod p} \cdot z^i$$
$$C^p_i(x,y,z) = \sum_{j=1}^{i-1} (A^p_j \cdot B^p_{i-j})(x,y,z)$$

$$C_i(x,y,z)/z^i = y^{\tilde{C}_{i,j}}F_0(x) + y^{\tilde{C}_{i,j}+1}F_1(x) + \cdots + y^{\tilde{C}_{i,j}+p}F_p(x) + y^{\tilde{C}_{i,j}+p+1}F_p(x) + \cdots$$

Modulo adds noise to output

$$C^p_i(x,y,z)/z^i = y^{\tilde{C}_{i,j} \mod p}\left(F_0(x) + F_p(x)\right) + y^{\tilde{C}_{i,j}+1 \mod p}\left(F_1(x) + F_p(x)\right) + \cdots$$



Finding all noisy terms:

- Fix any pairs of intervals of $[i_1, i_2], [j_1, j_2]$, find all i, j s.t.
  $\tilde{C}_{i+j} \ne \tilde{A}_i + \tilde{B}_j - O(1)$
  $\tilde{C}_{i+j} \equiv \tilde{A}_i + \tilde{B}_j - O(1) \mod p$
- Total runtime = $\tilde{O}(n^{8/5})$

Divide both $\tilde{A}, \tilde{B}$ into intervals
Entries in the each interval are the same

Since $\tilde{A}, \tilde{B}$ is monotone, there are at most $n^{4/5}$ intervals