# An Improved Algorithm for Incremental DFS Tree in Undirected Graphs

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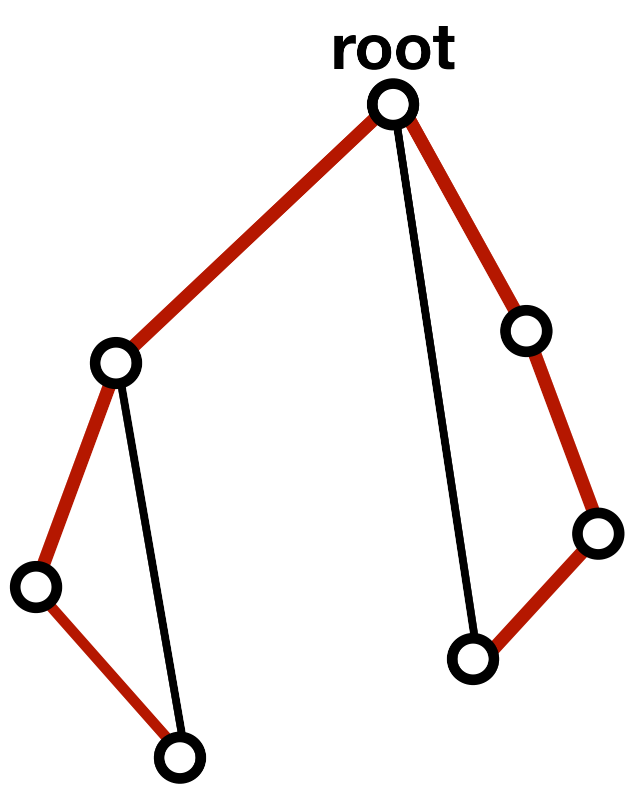
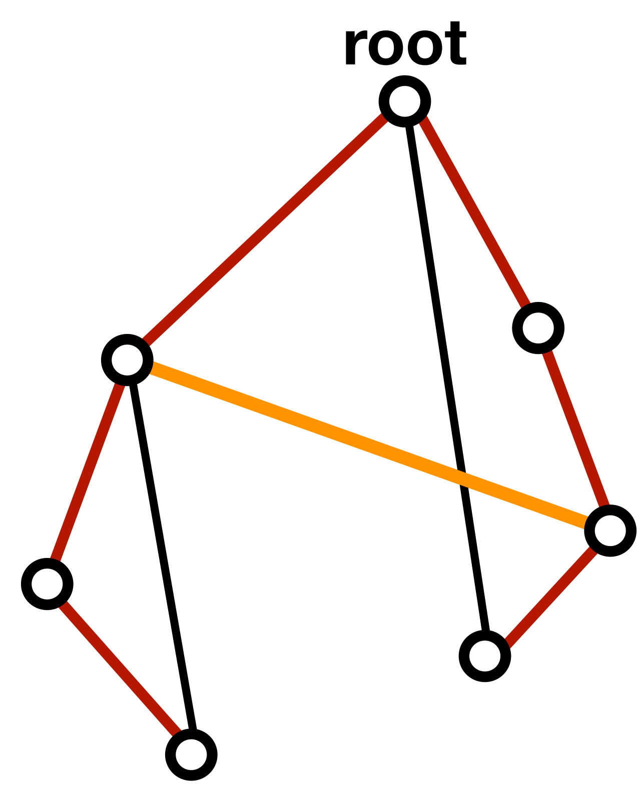
# Definition: DFS tree

- Given an undirected graph $G = (V, E)$ with a designated root
- DFS tree: a maximal tree containing the root, where every non-tree edge connects an ancestor and a descendant




**A DFS tree**




**Not a DFS tree**

# Definition: Incremental DFS tree

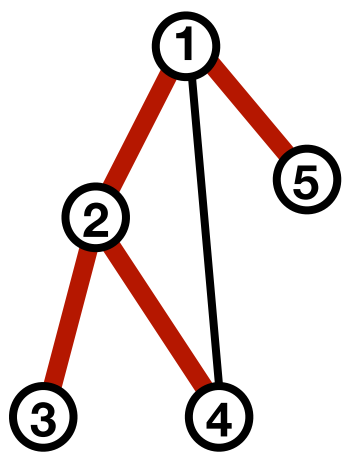
Data structure

- Maintain a DFS tree *T* in graph *G*

Update operations

- **Input:** insert an edge/vertex to *G*
- **Output:** print all edges of *T*

# Example: Incremental DFS tree

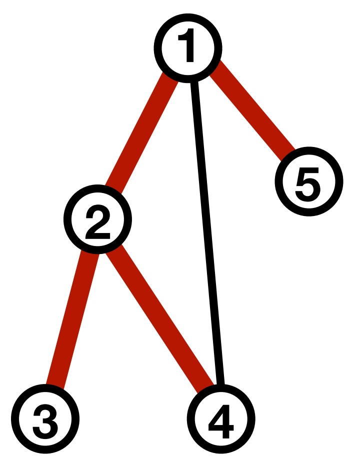
| | | | | |
|---|---|---|---|---|
| **Input:** Updates to *G* | | | | |
| **Picture** |   | | | |
| **Output:** Change in *T* | | | | |

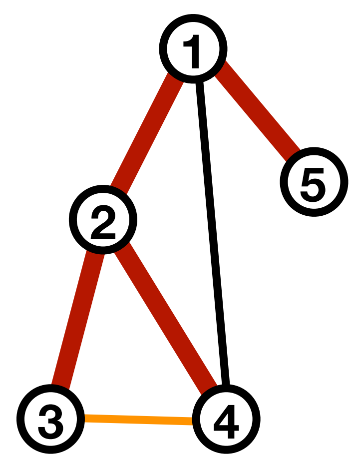
# Example: Incremental DFS tree

**Input:**
Updates to *G*

Insert(3,4)

**Picture**

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# Example: Incremental DFS tree

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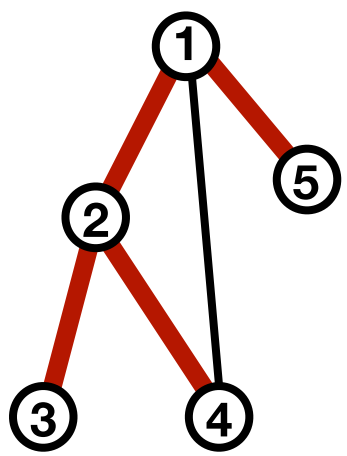

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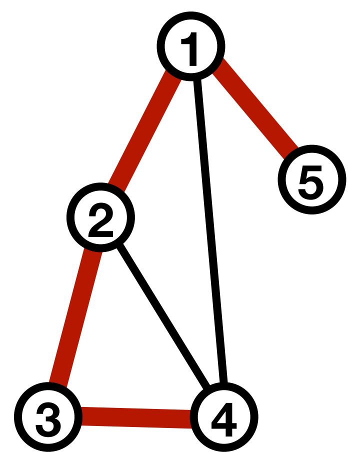

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# Example: Incremental DFS tree

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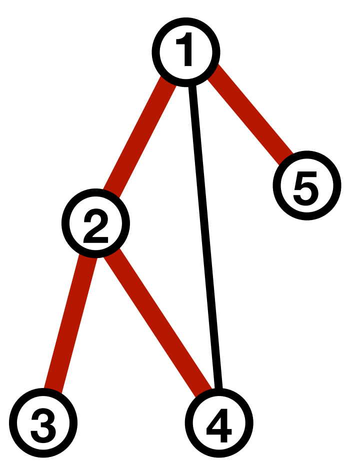
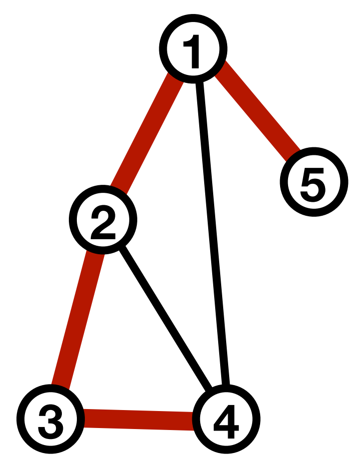
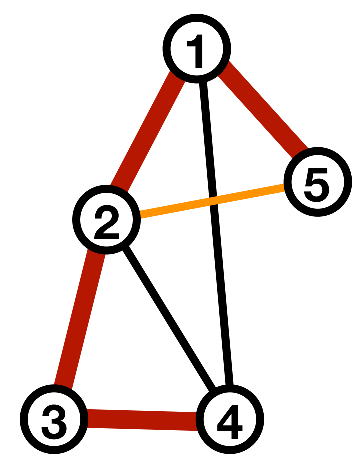
| **Input:** Updates to *G* | | Insert(3,4) | Insert(2,5) | |
| **Picture** |   |   |   | |
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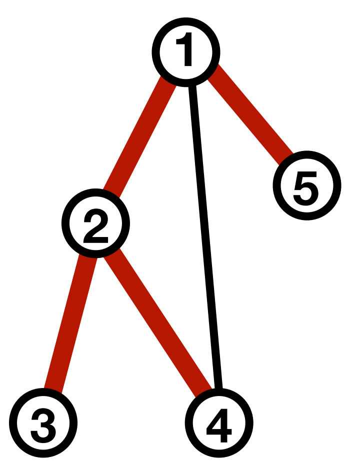
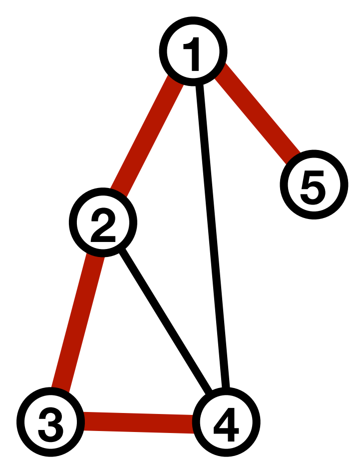
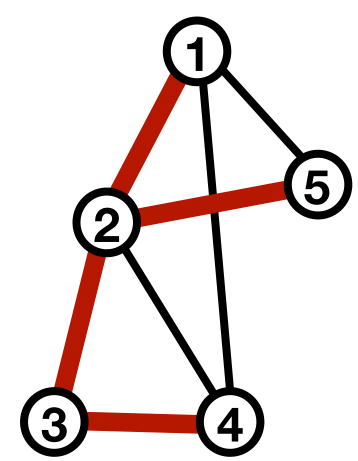
| **Picture** |  |  |  | |
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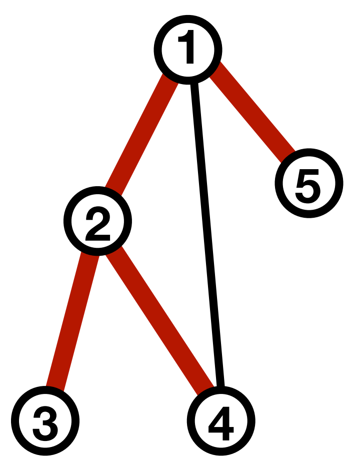
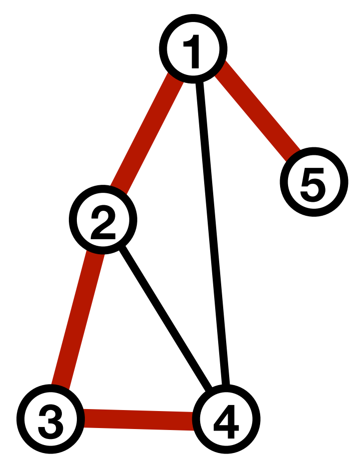
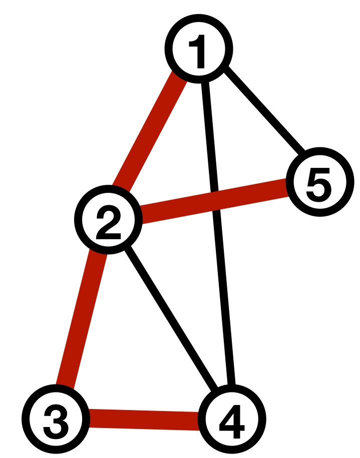
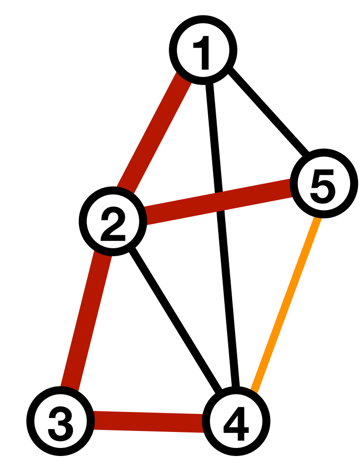
| **Picture** |  |  |  |  |
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# <u>Example</u>: Incremental DFS tree

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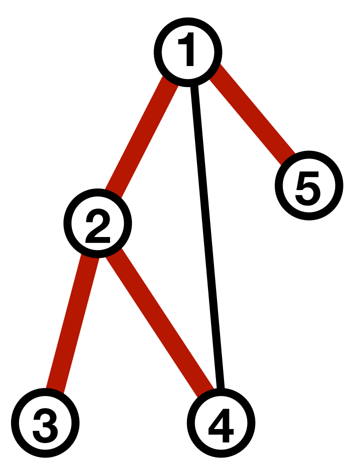
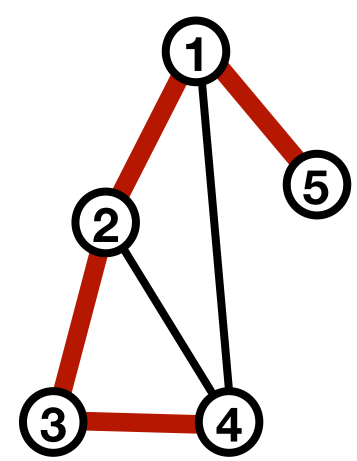
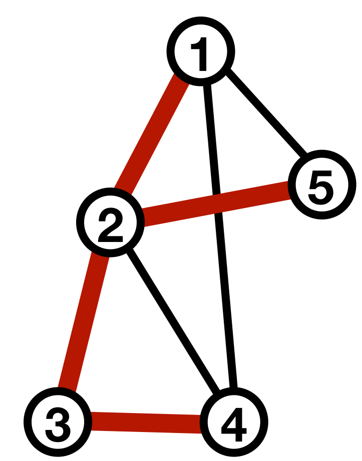
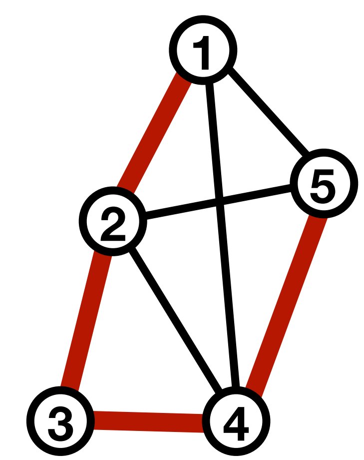
| **Picture** |  |  |  |  |
| **Output:**<br>Change in $T$ | | Delete(2,4)<br>Insert(3,4) | Delete(1,5)<br>Insert(2,5) | Delete(2,5)<br>Insert(4,5) |

# Progress on incremental DFS

- *n* = # of vertices, *m* = # of edges

| Reference | Naïve | [BK'14] | [BCC+'16] | [NS'17] | New |
|---|---|---|---|---|---|
| Update time | $O(m+n)$ | $O(n)$ | $O(n \log^3 n)$ | $O(n \log n)$ | $O(n)$ |
| Space | $O(m+n)$ | $O(m+n)$ | $O(m \log n)$ | $O(m \log n)$ in bits | $O(m \log n)$ |
| Worst-case? | Yes | No | Yes | Yes | Yes |

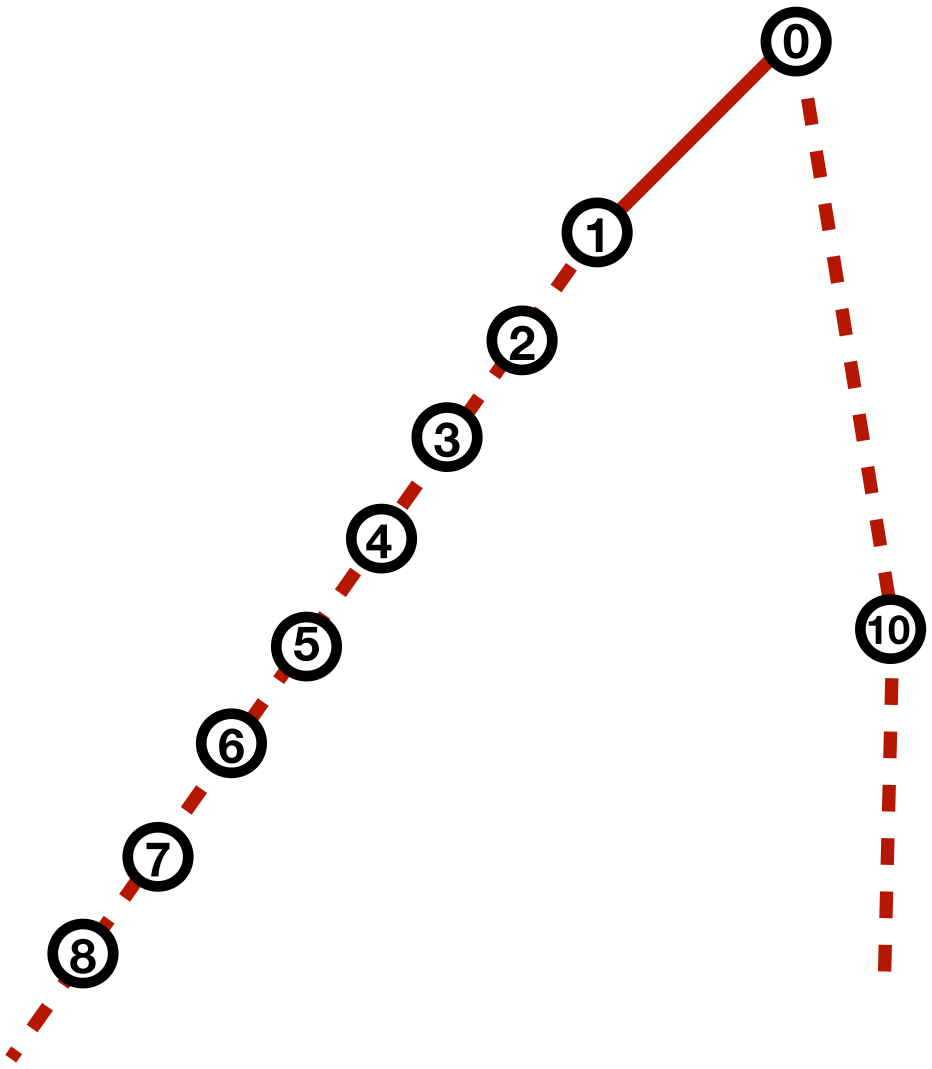
# Reduction to batch insertion

Main Theorem:

- Preprocess graph *G* in $O(\min\{m \log n, n^2\})$ time
- **Input:** a set *U* of *k* edge insertions
- **Output:** a DFS tree of *G*+*U* in $O(n+k)$

# Batch insertions [BCC+'16]

Revert & reroot



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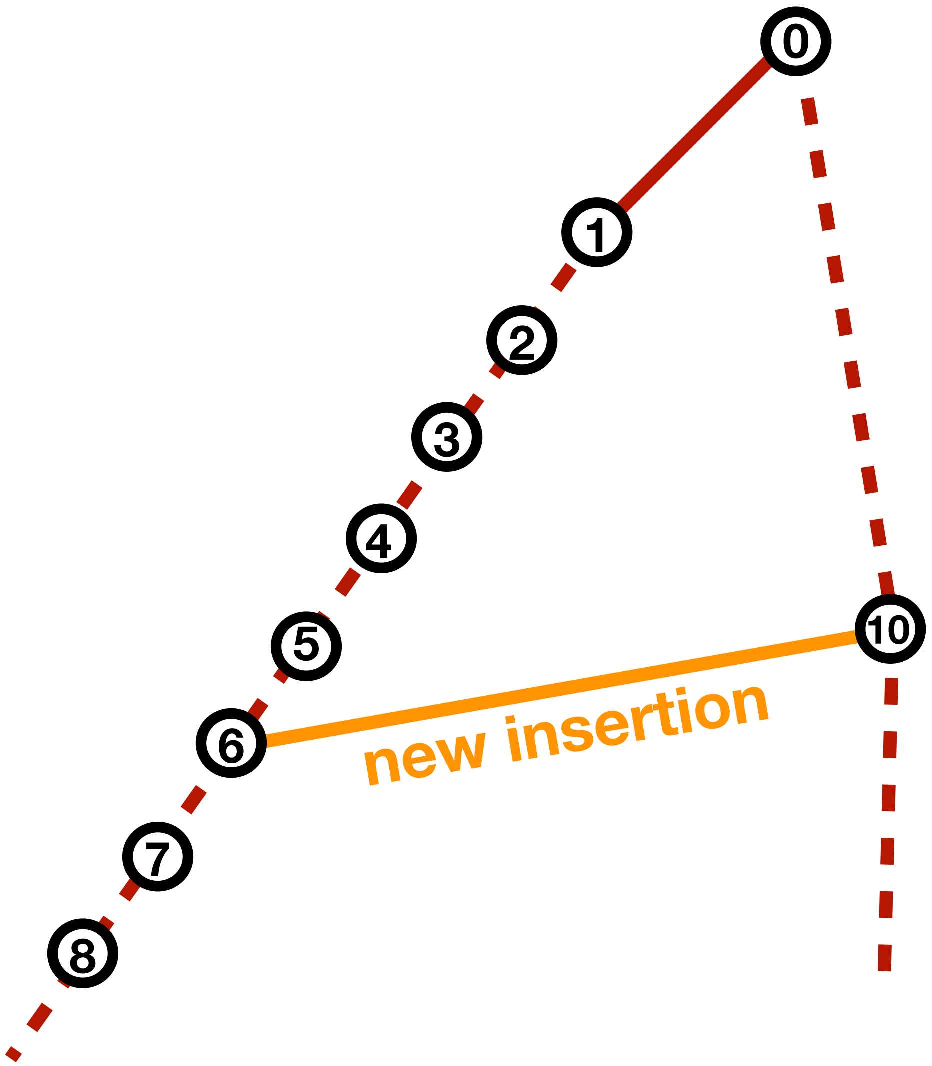

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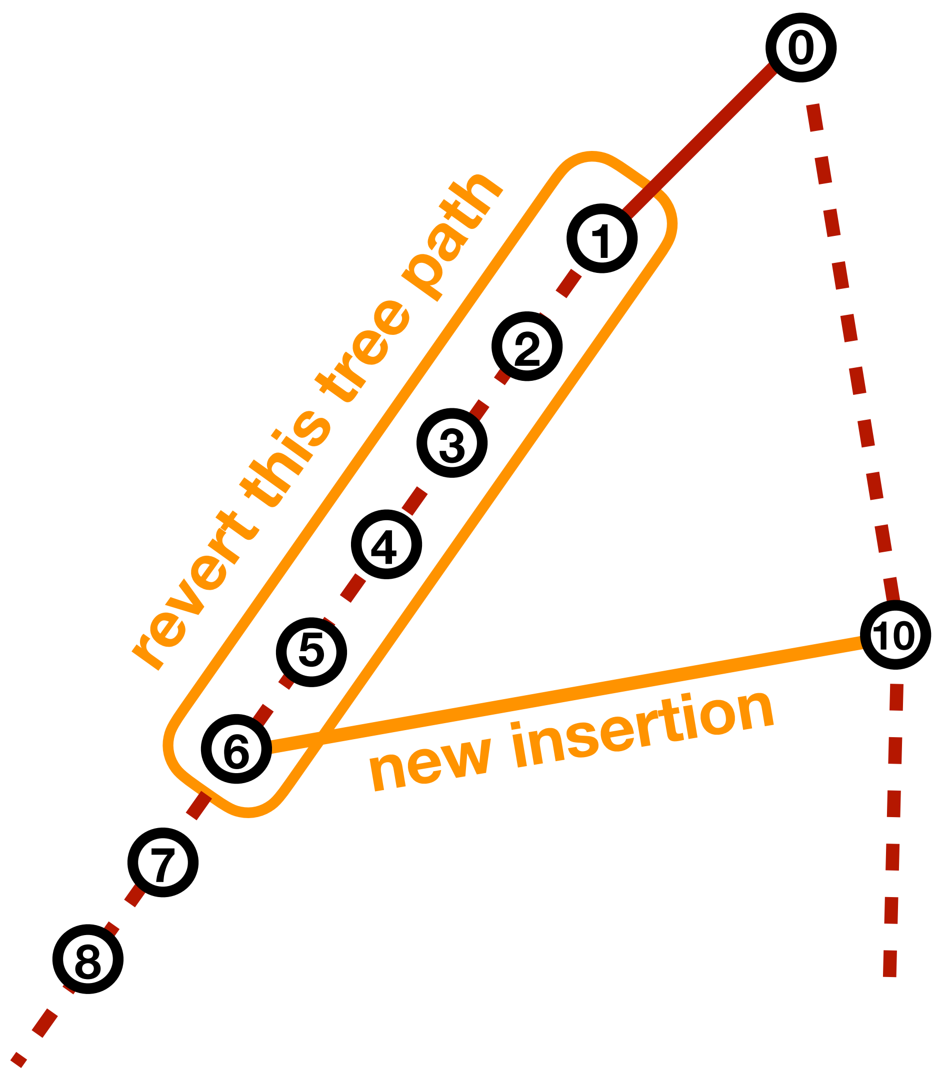

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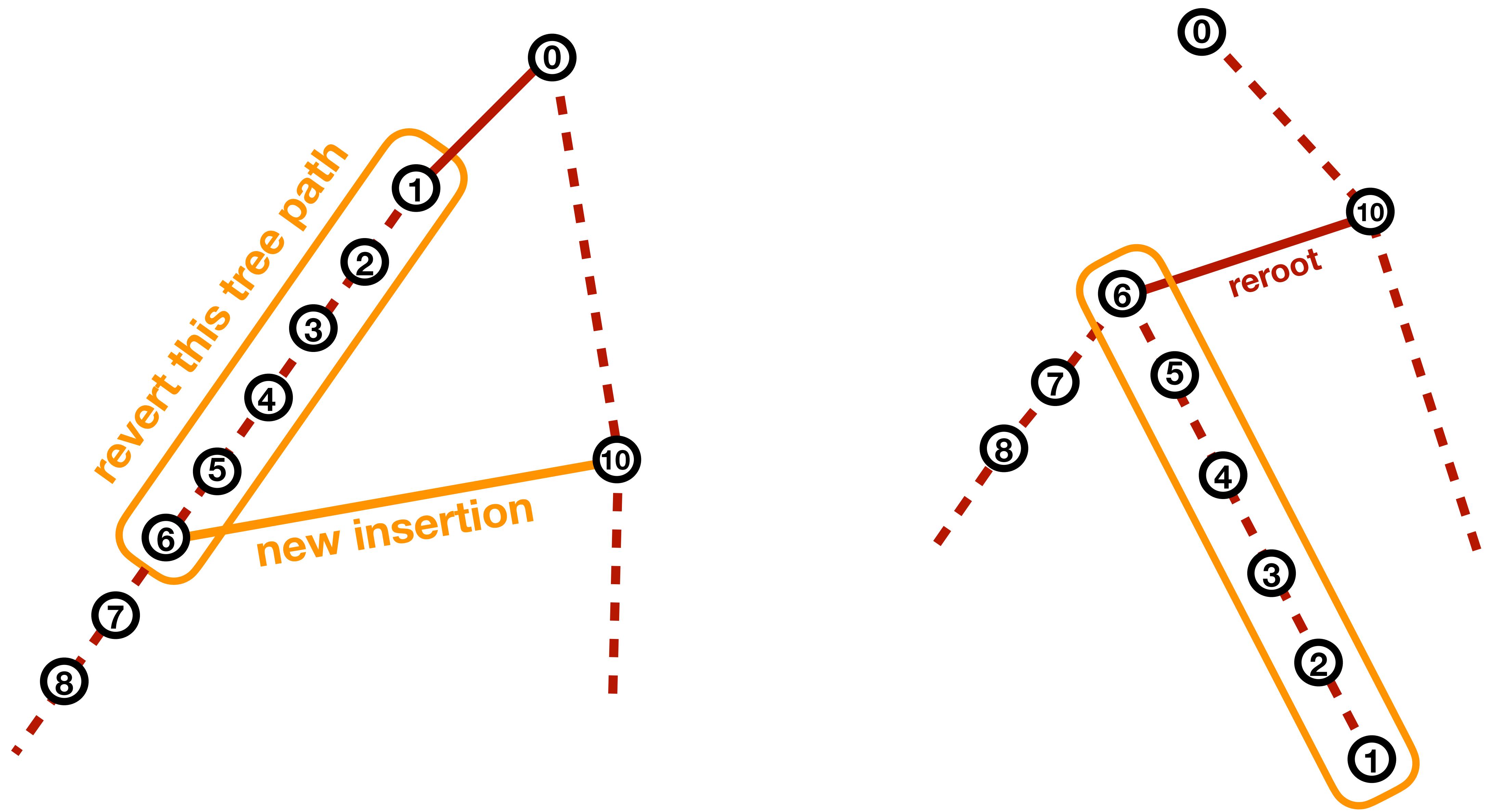

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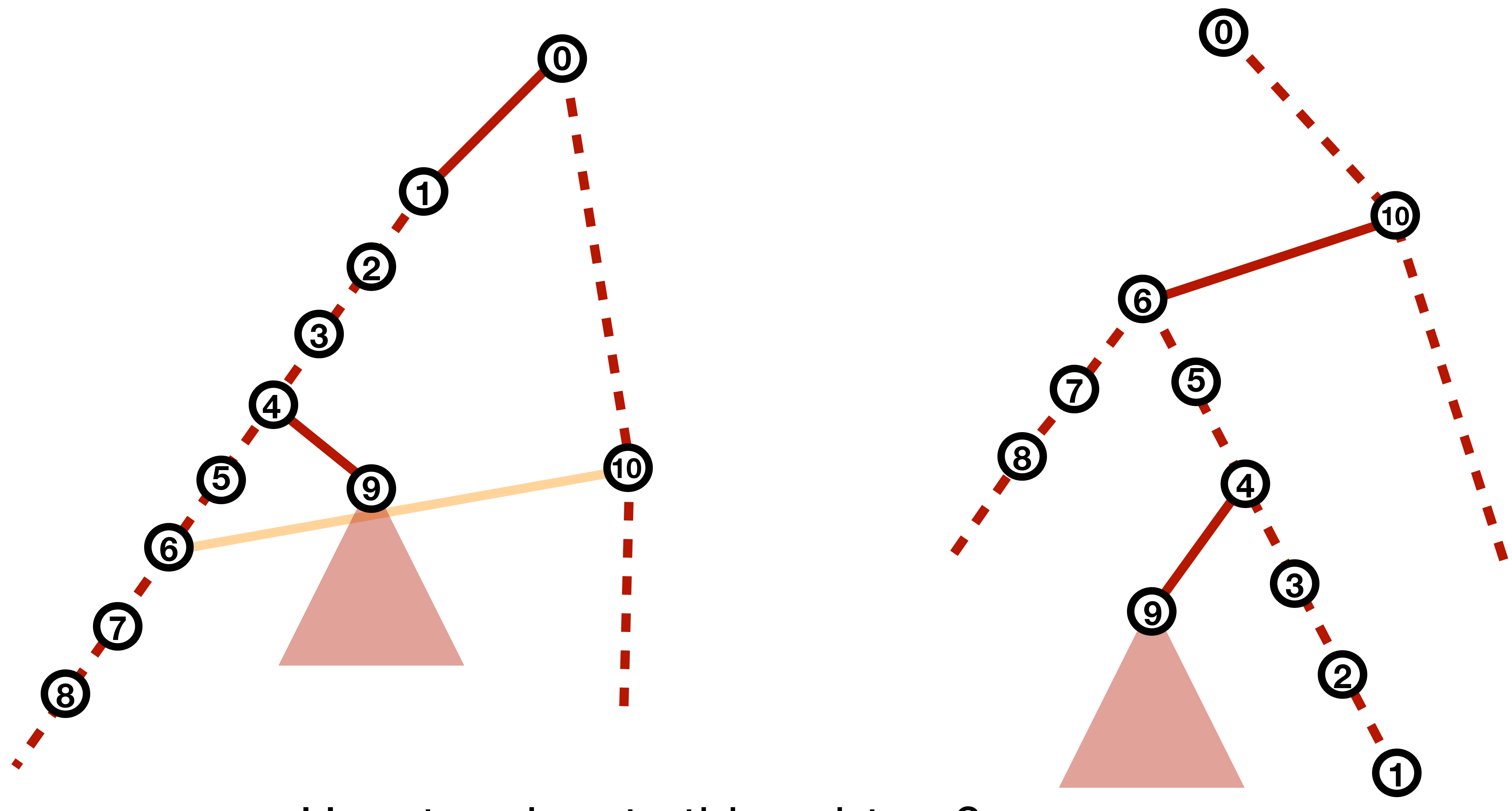


How to relocate this subtree?

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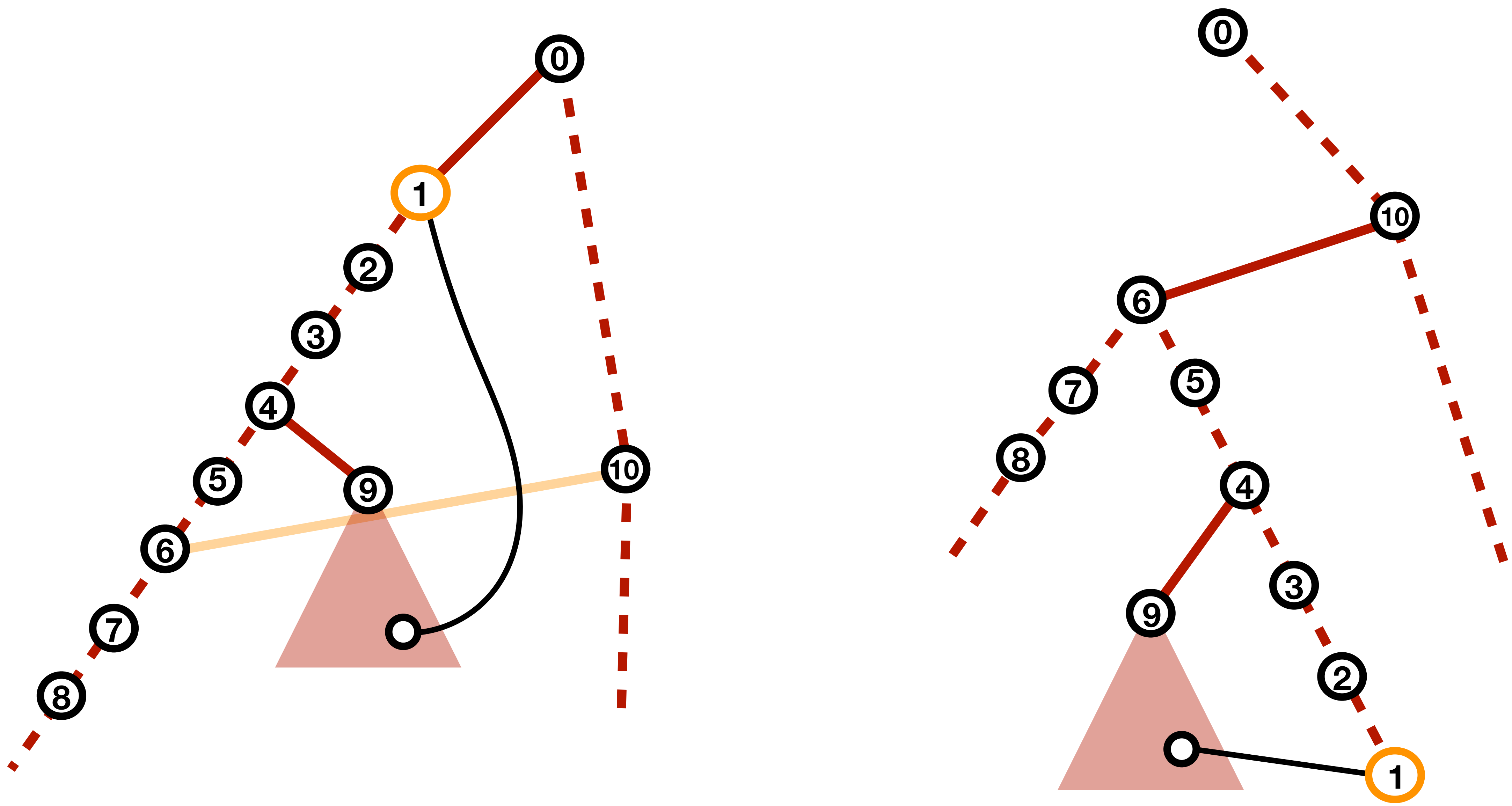


How to relocate this subtree?

Find the highest ancestor…

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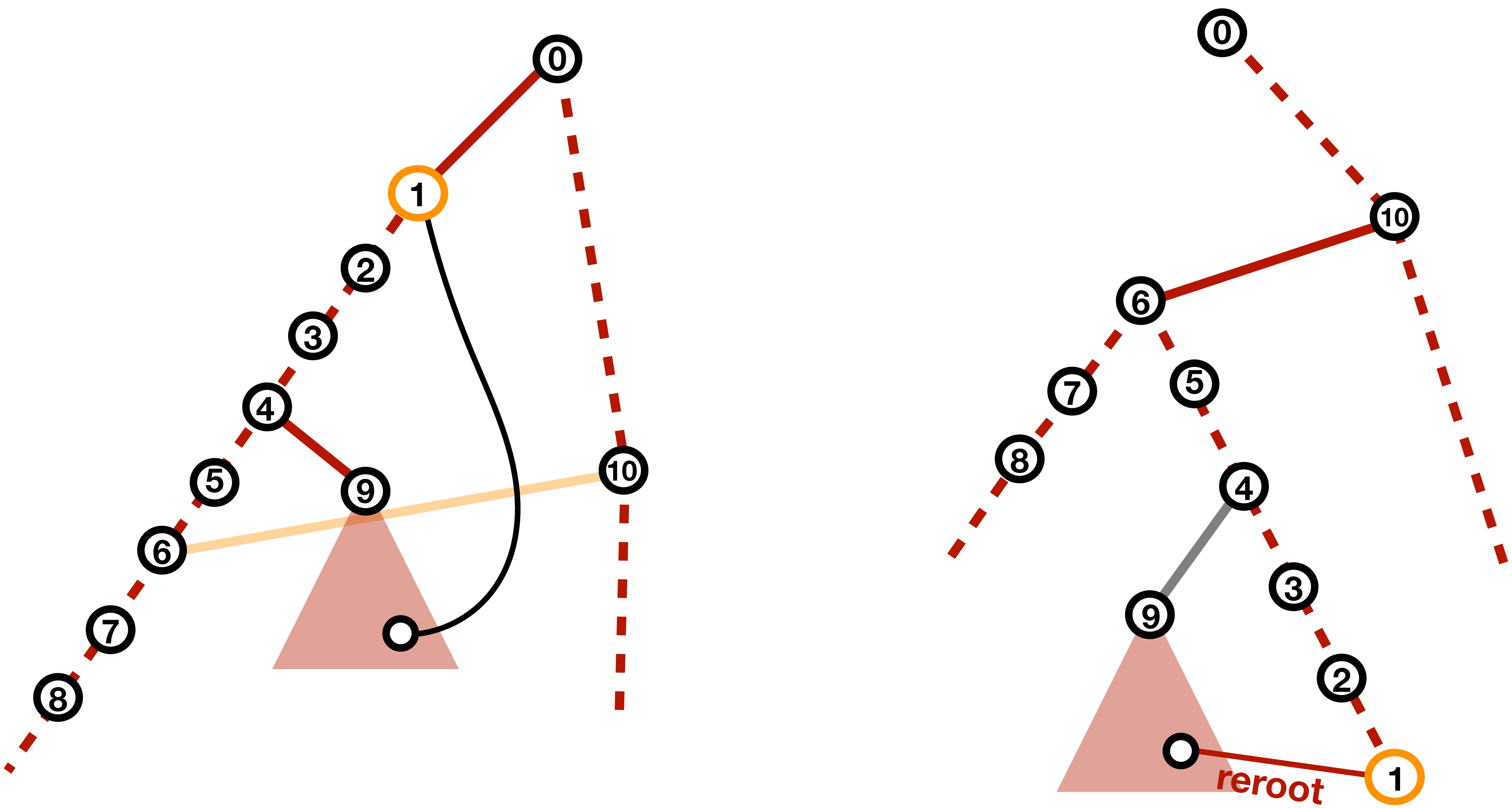


How to relocate this subtree?

Find the highest ancestor and reroot again.

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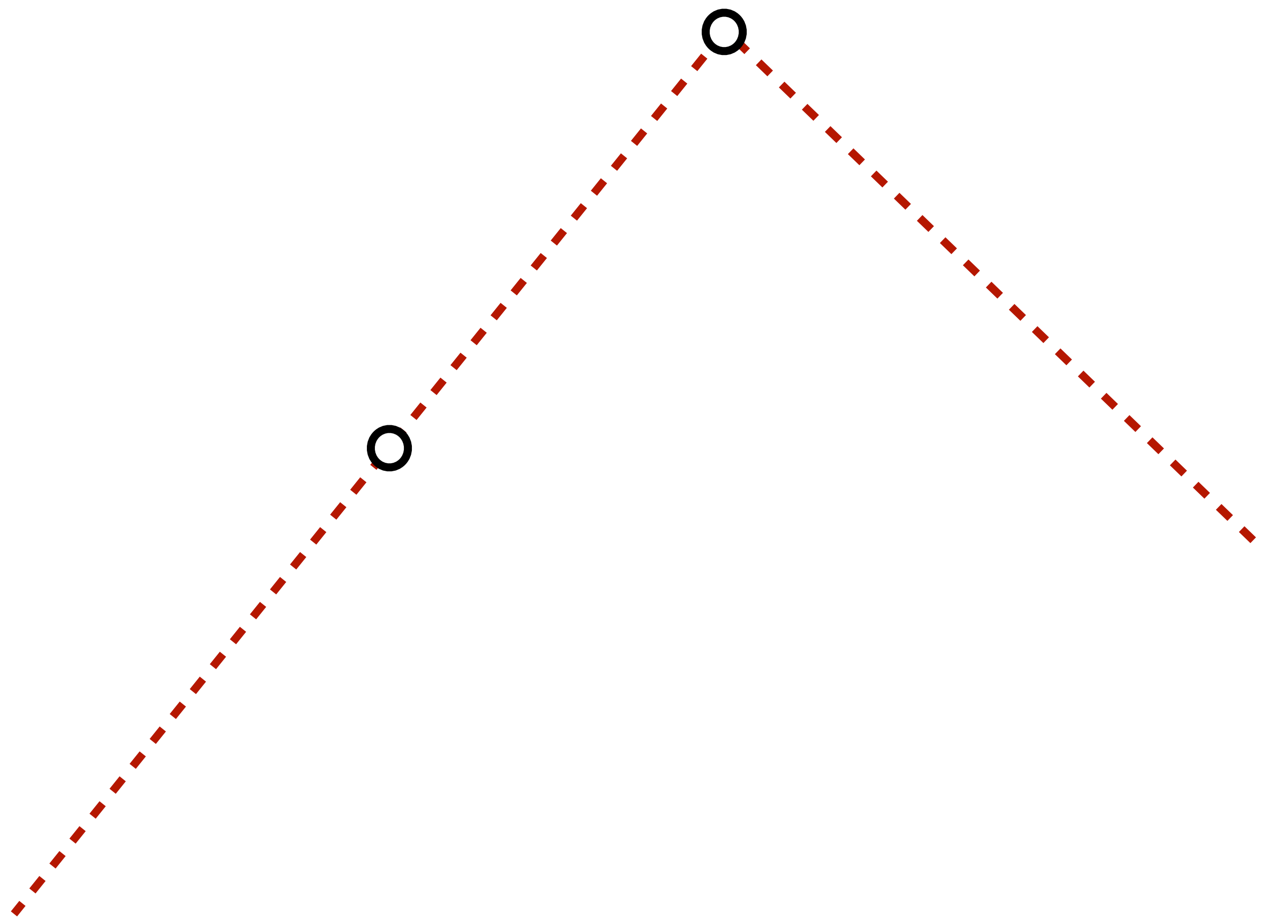
Recursively revert & reroot



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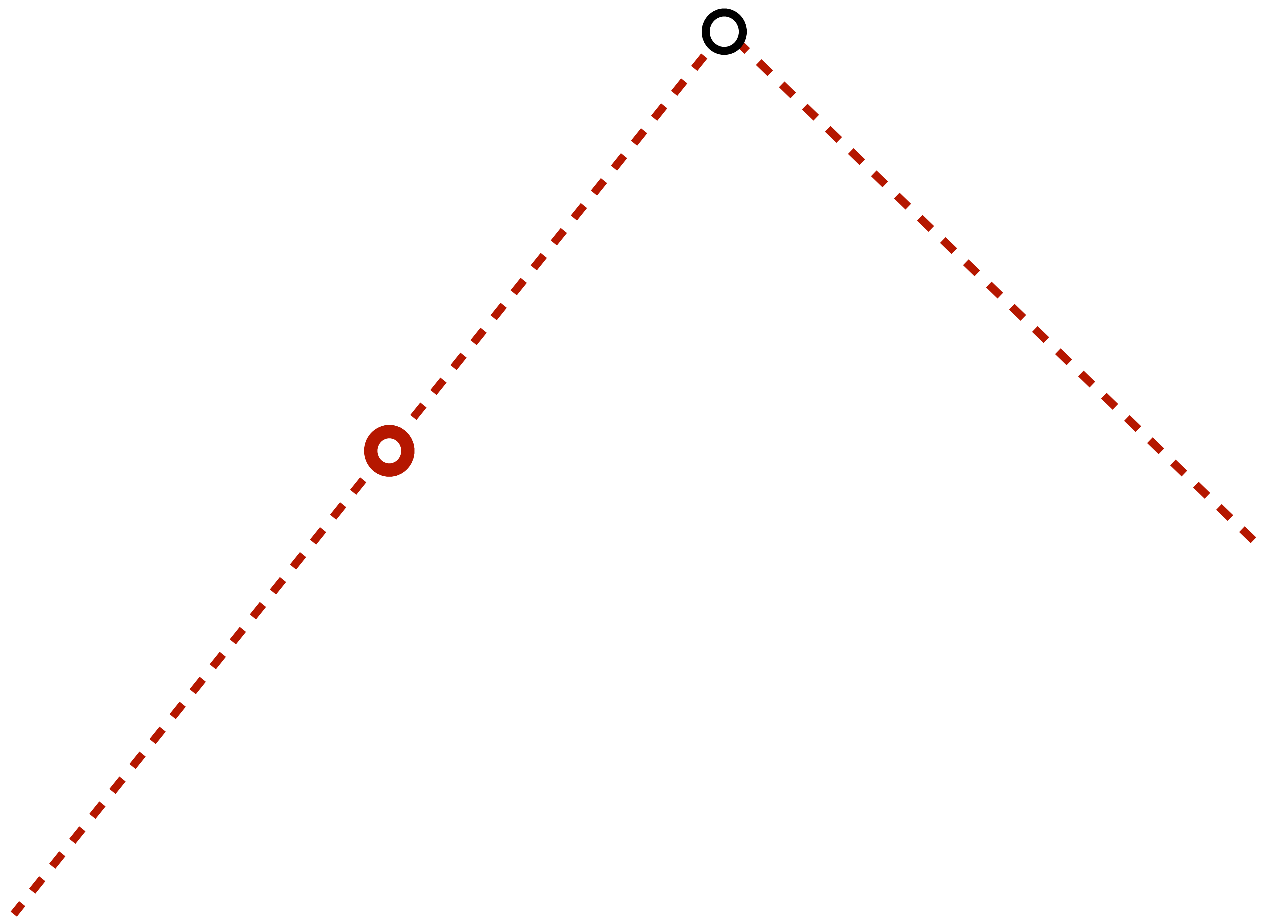

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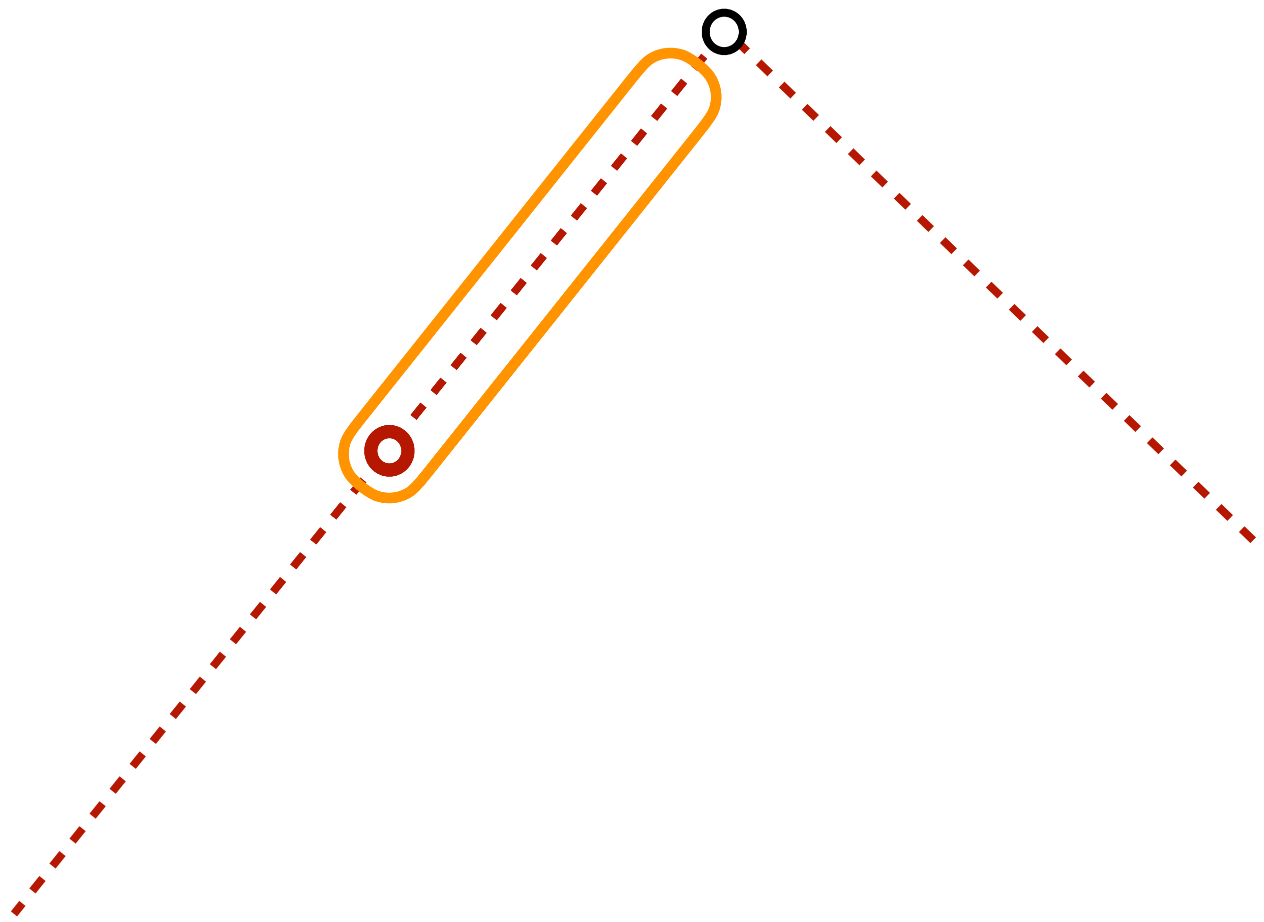

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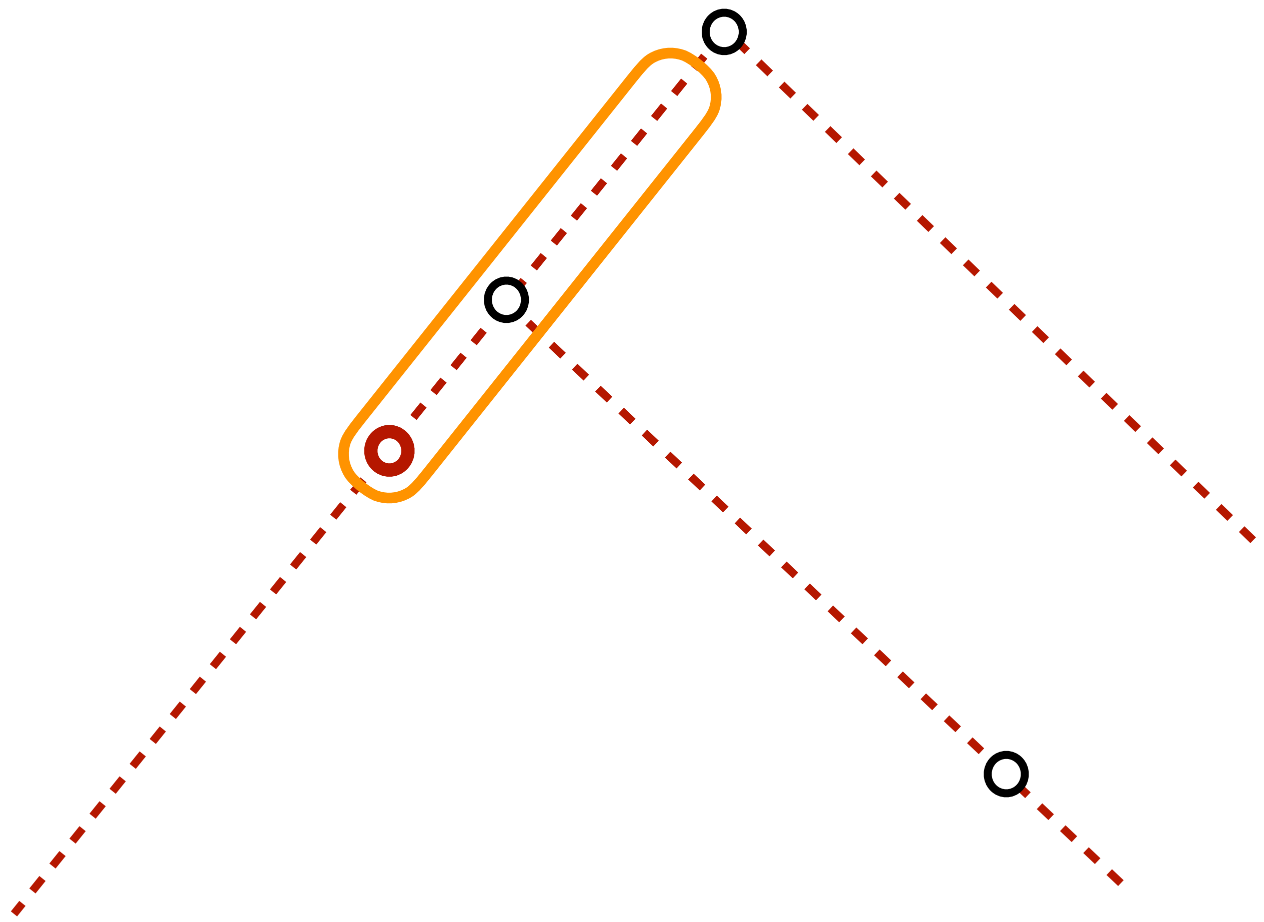

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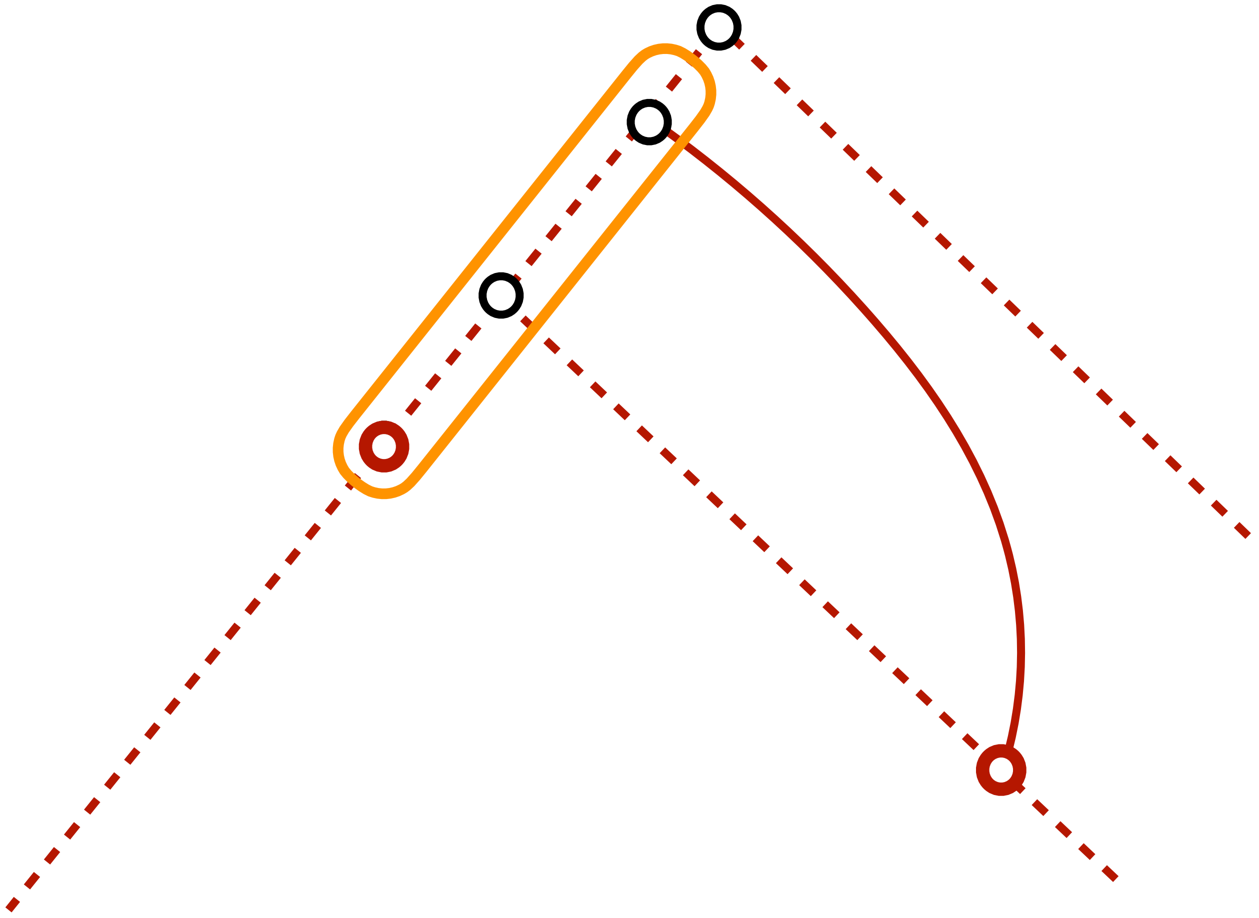

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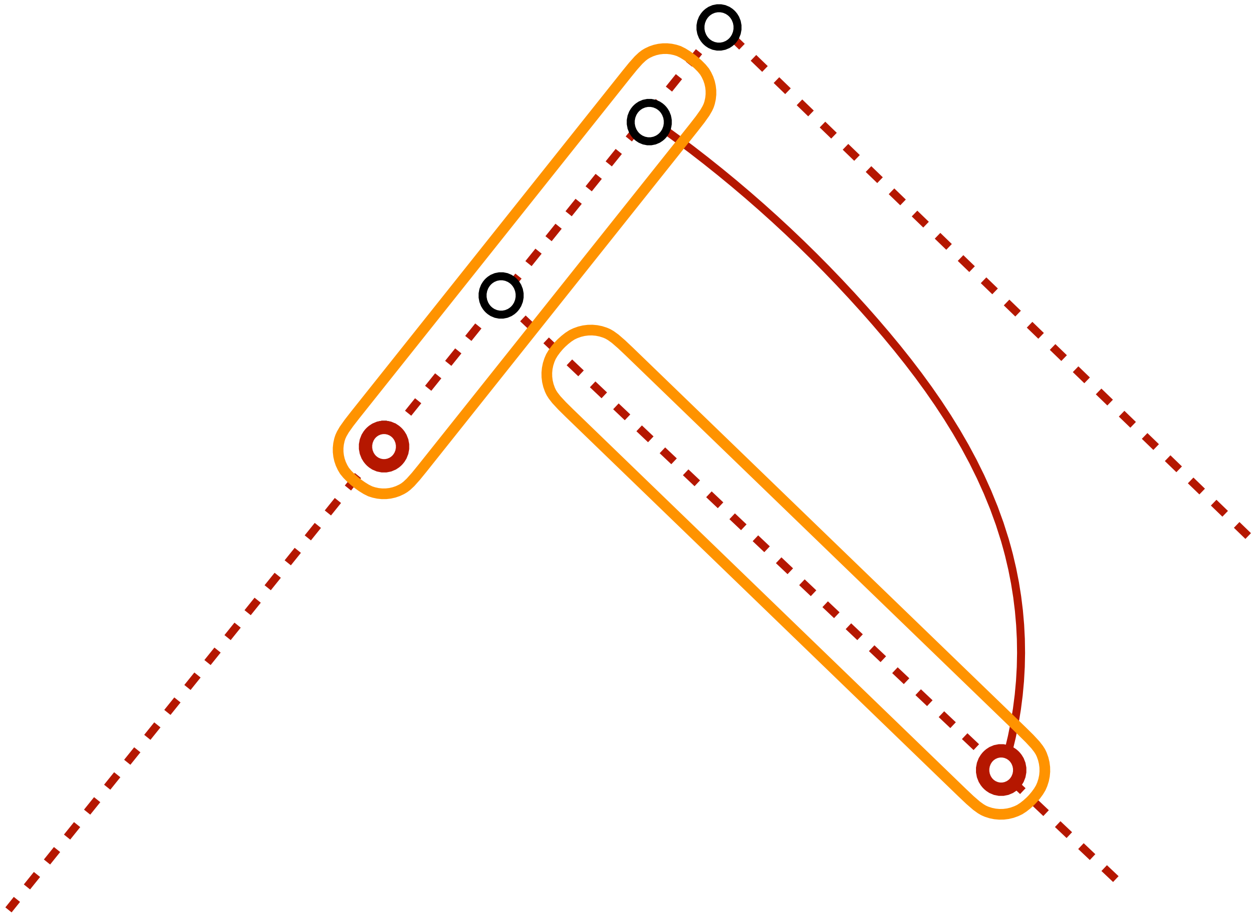

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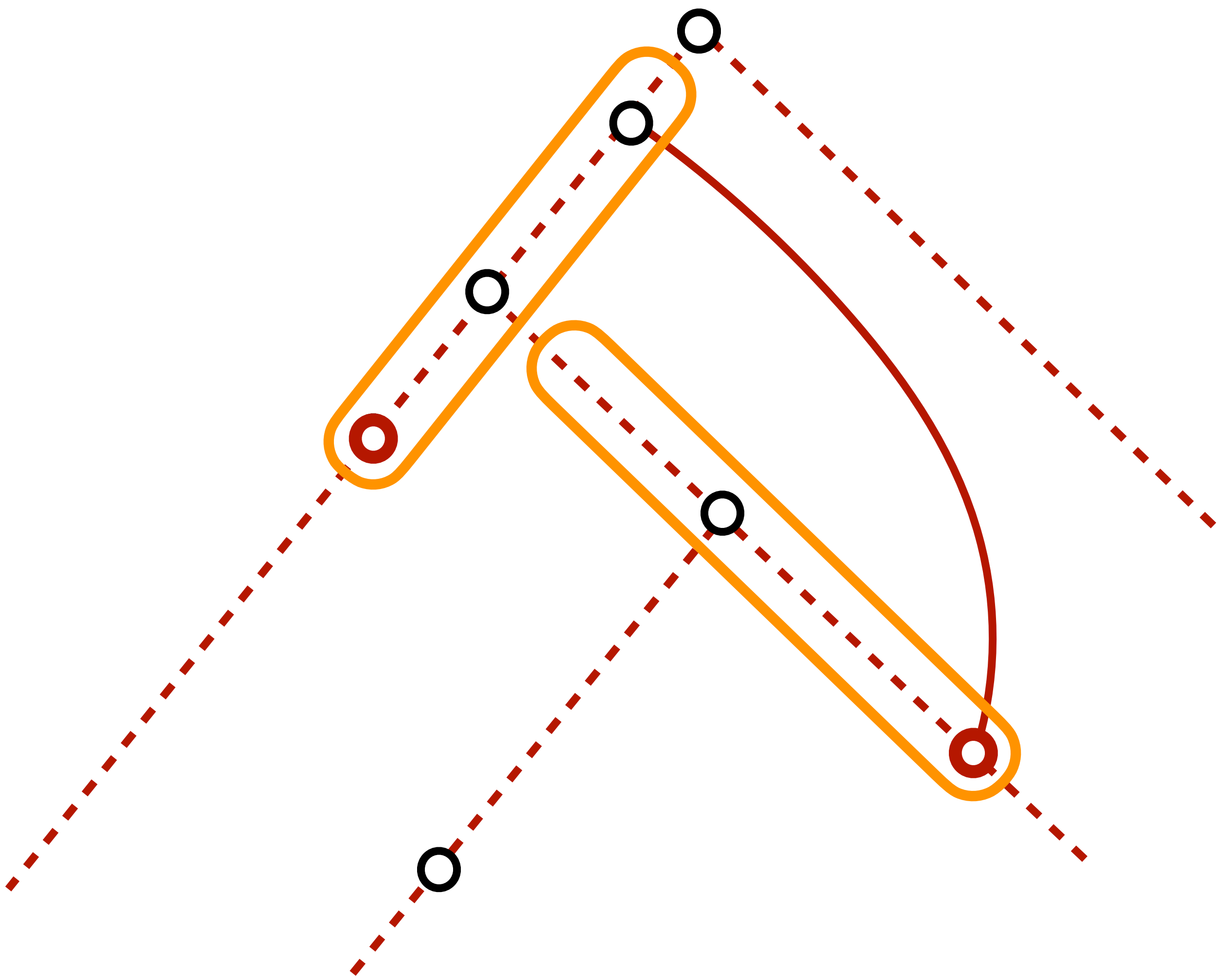

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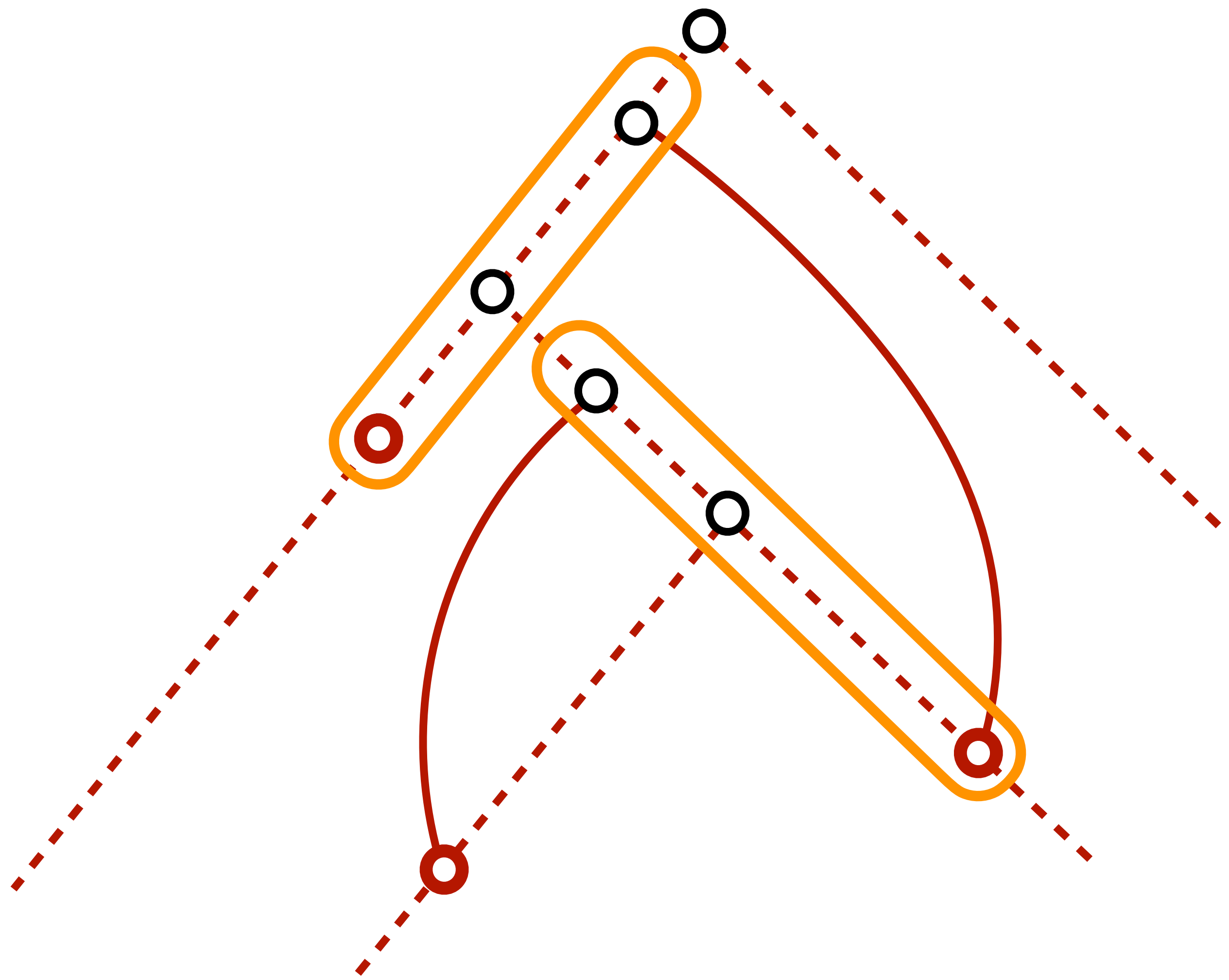

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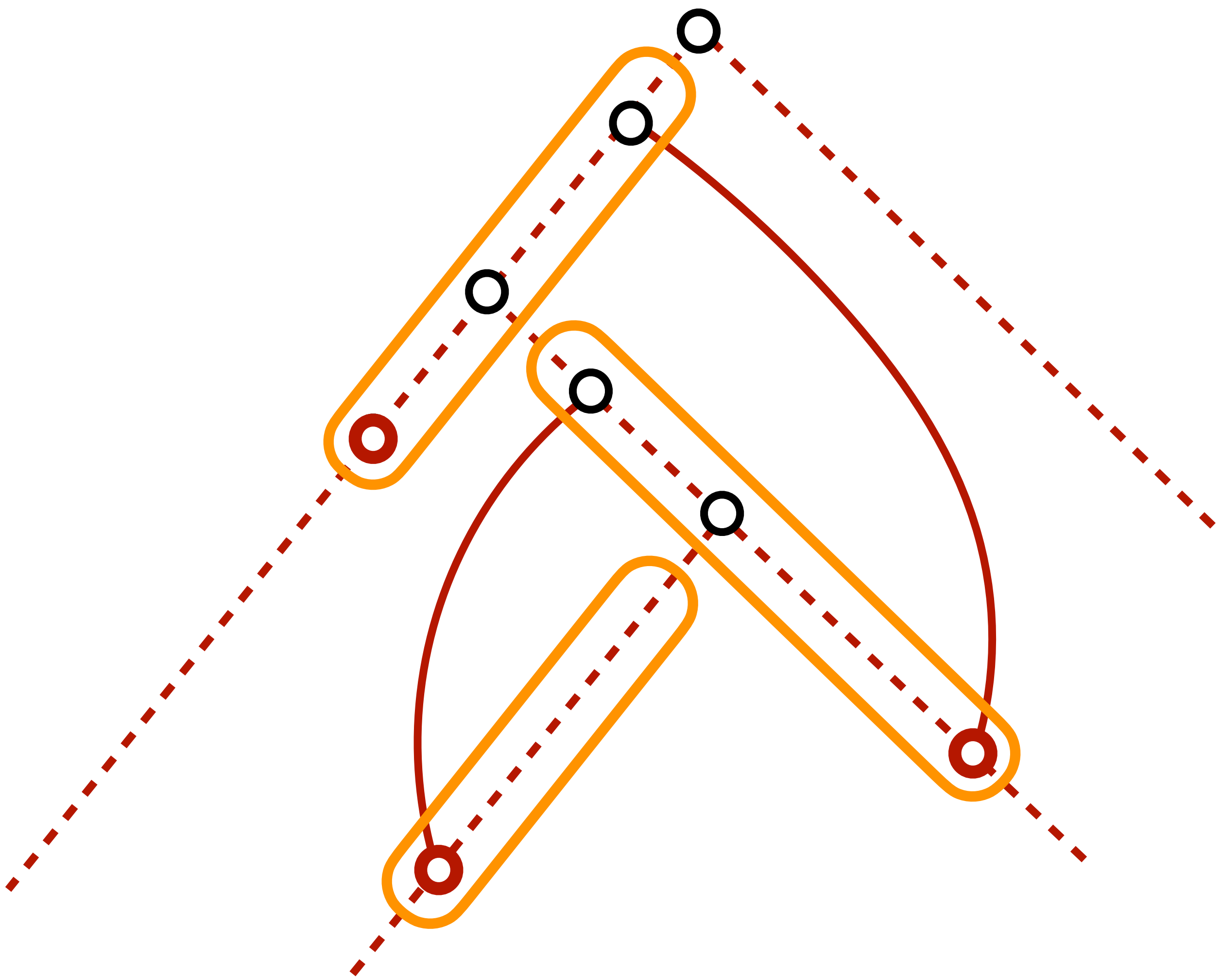

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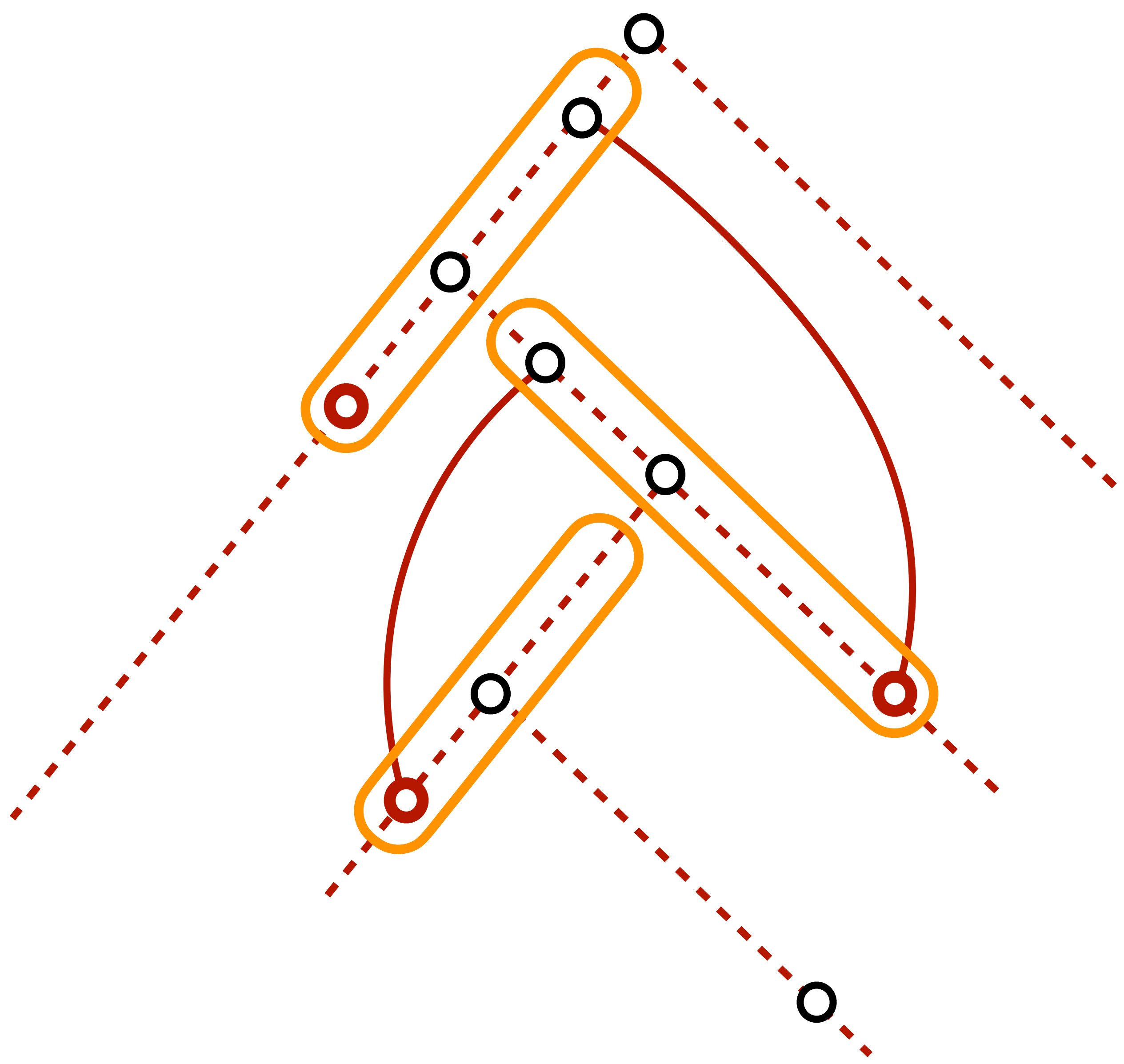

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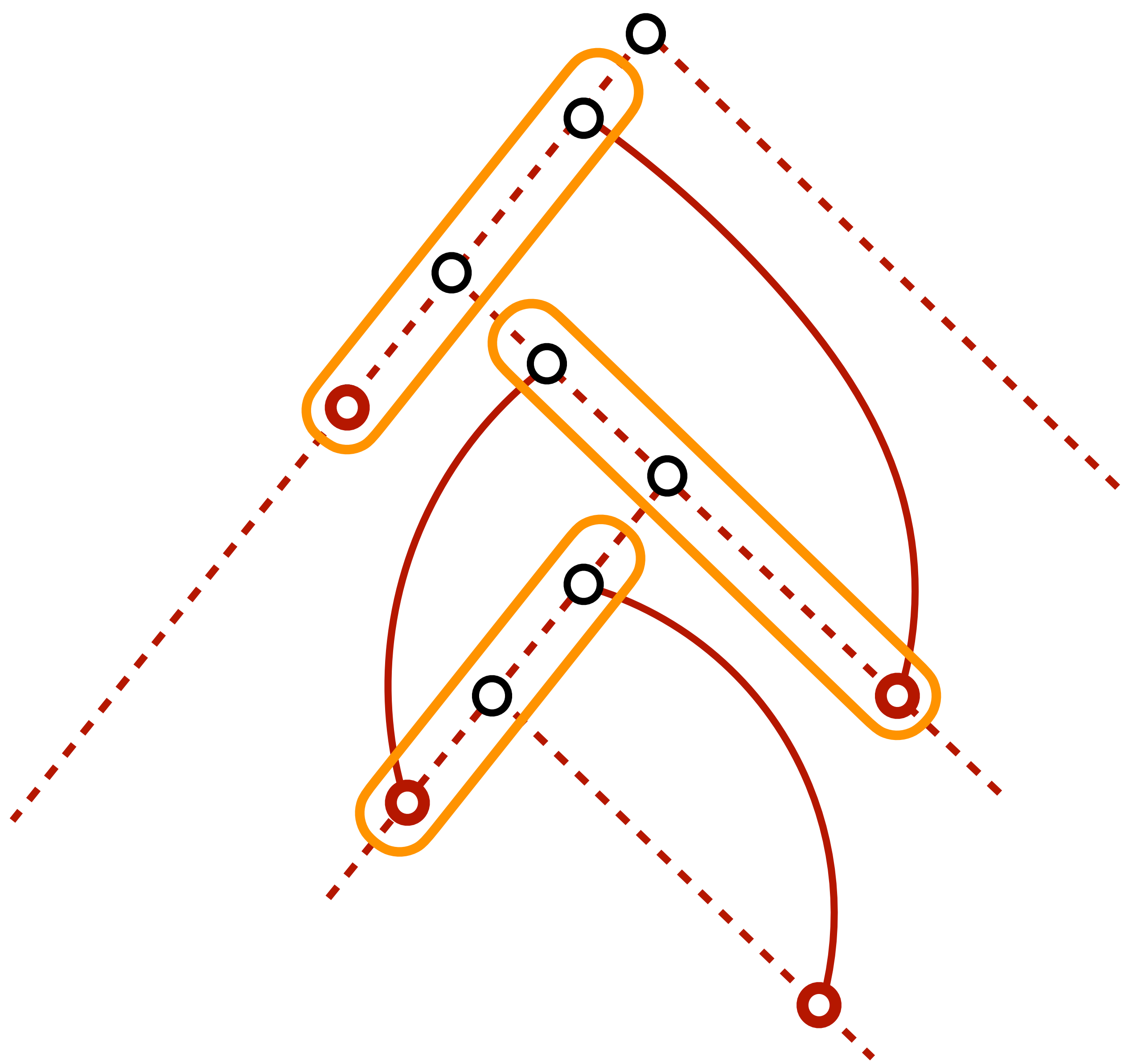

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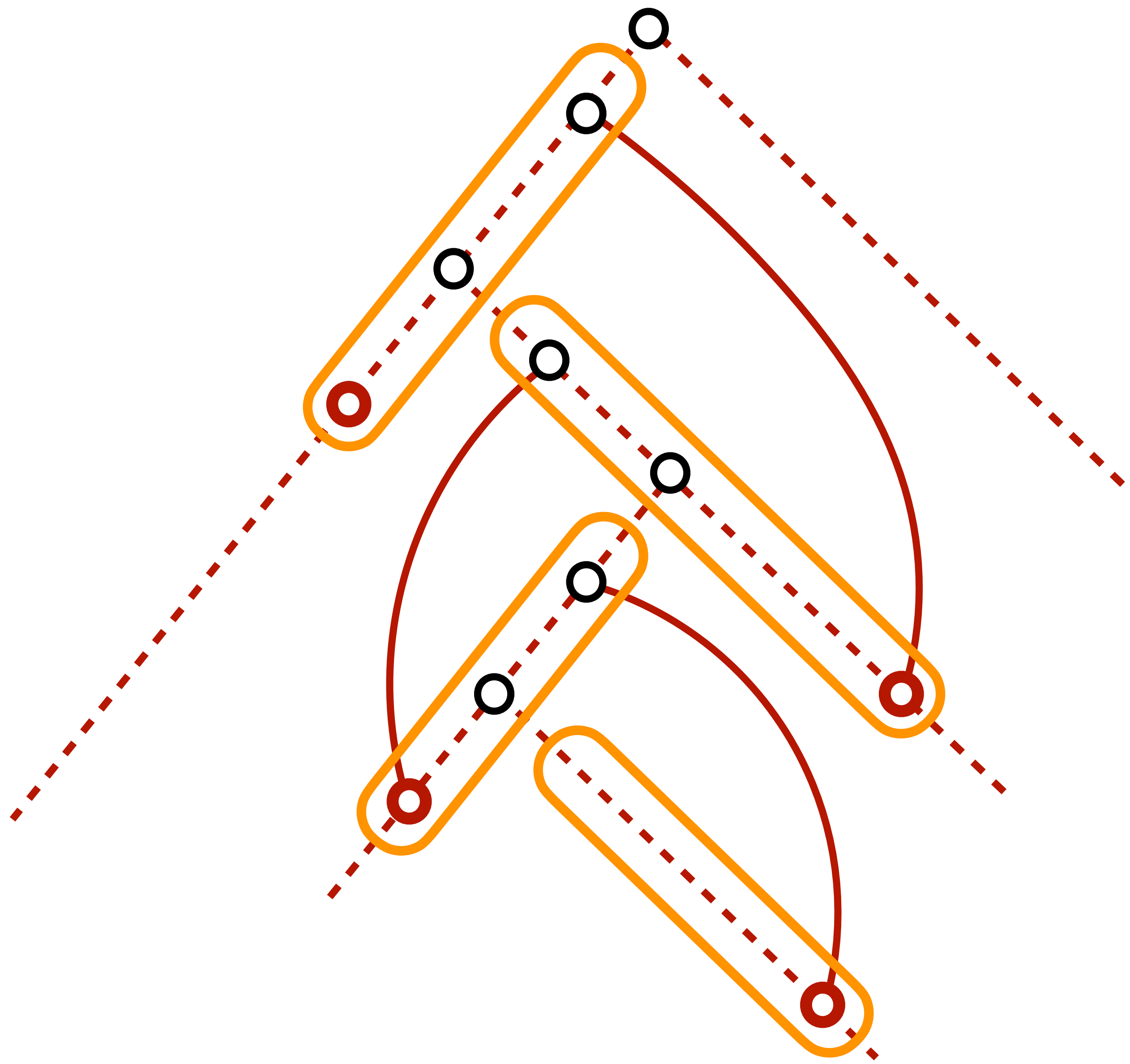

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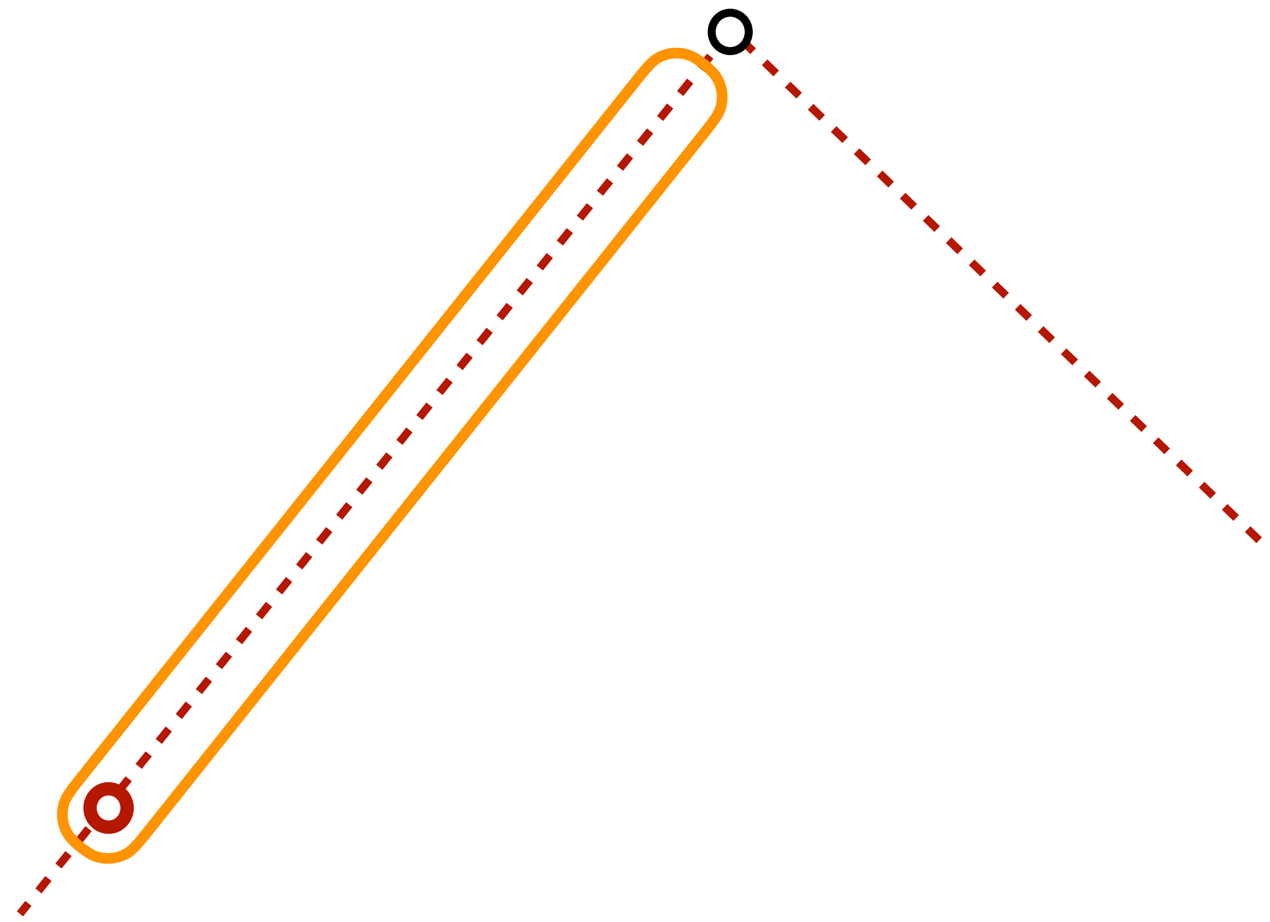
**Bottleneck:** finding highest ancestors on reverted paths



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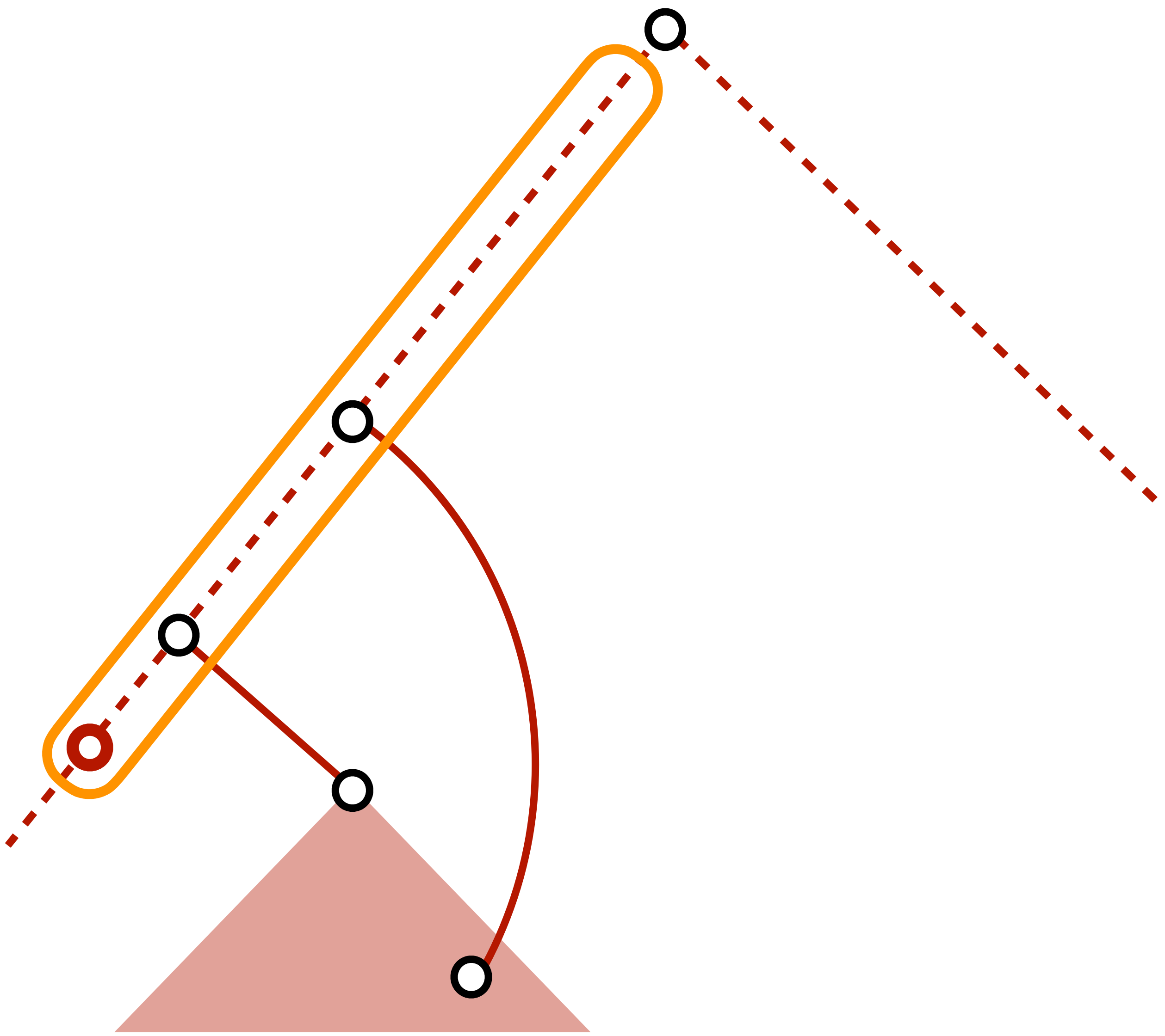

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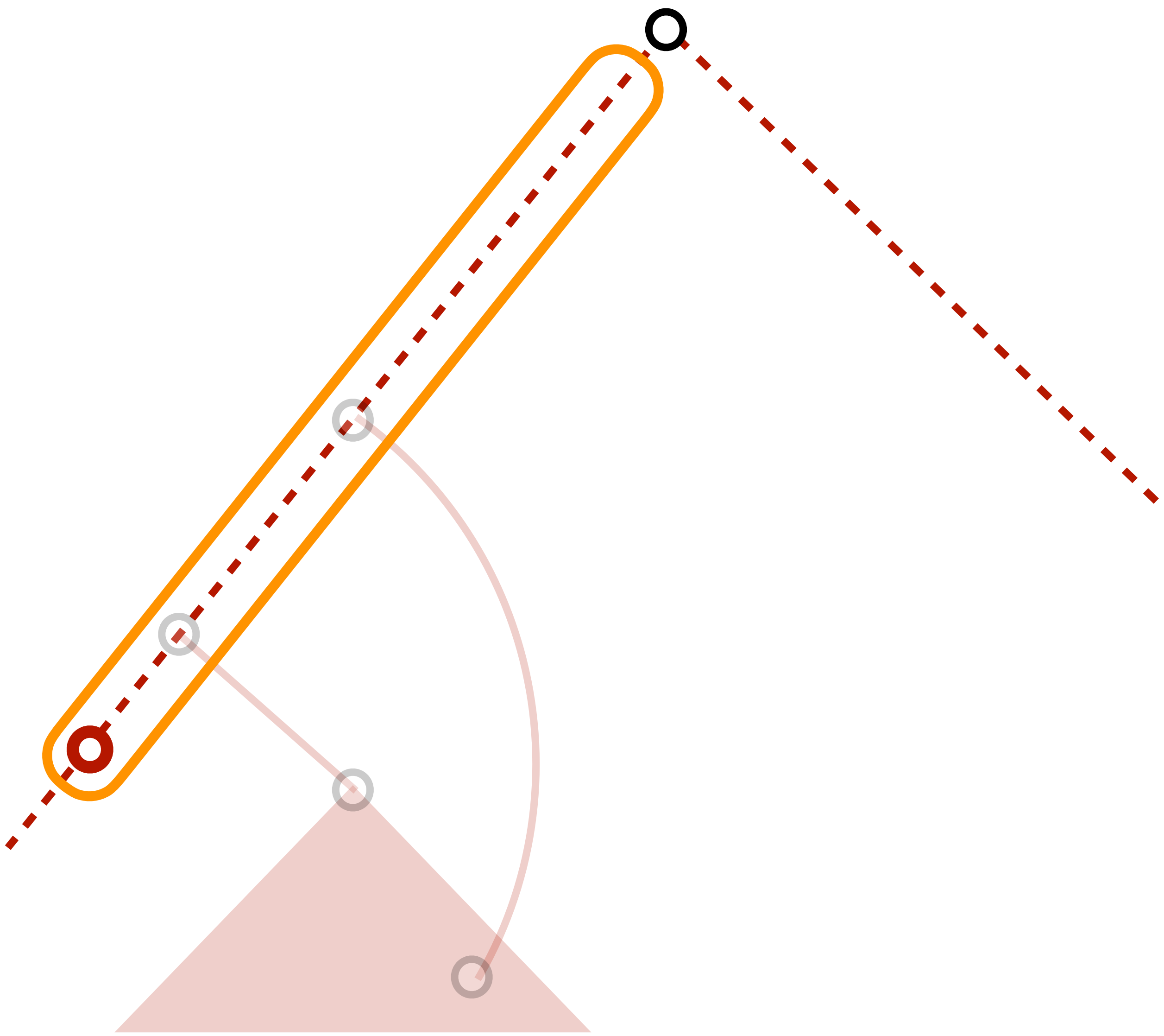

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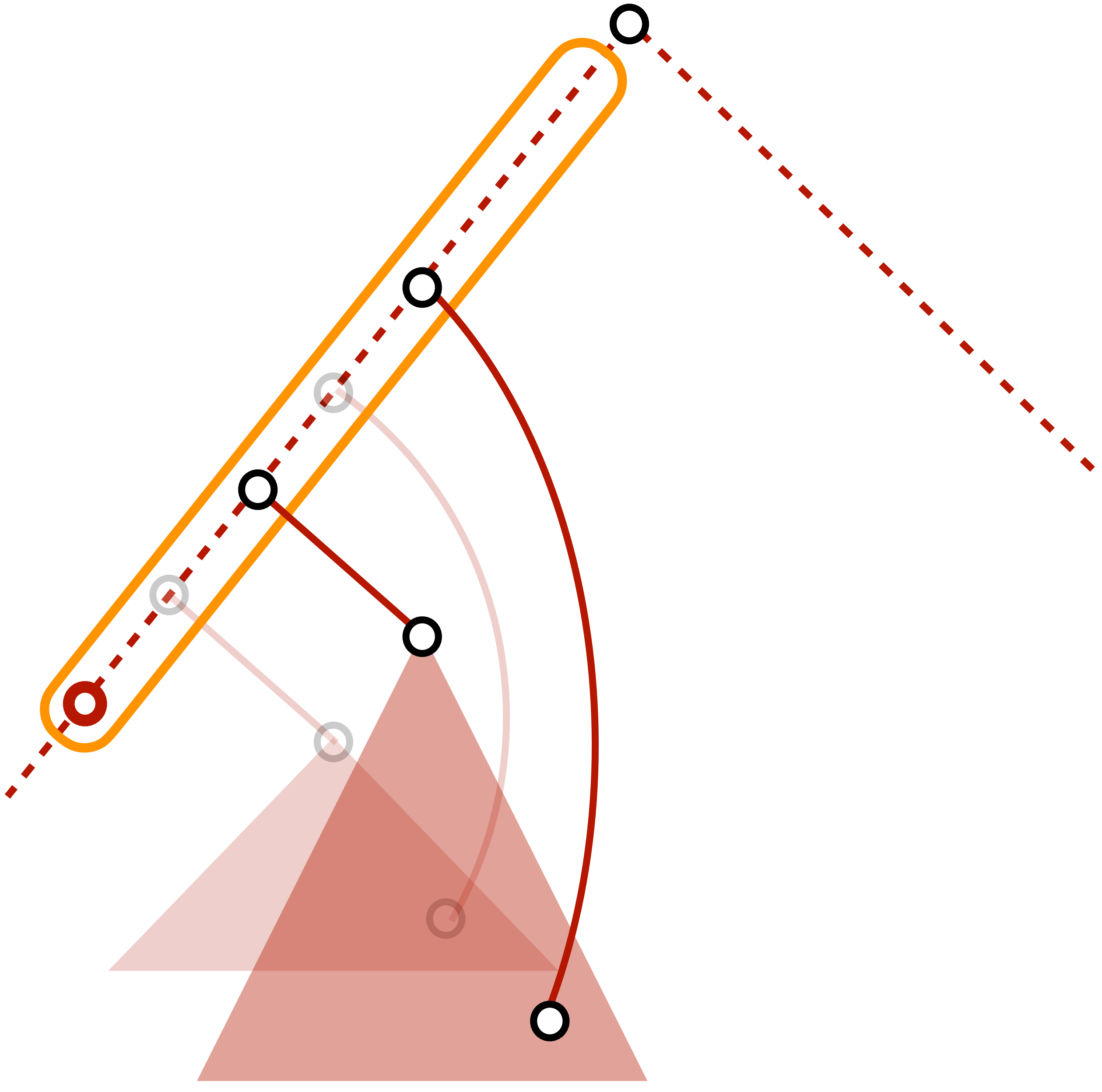

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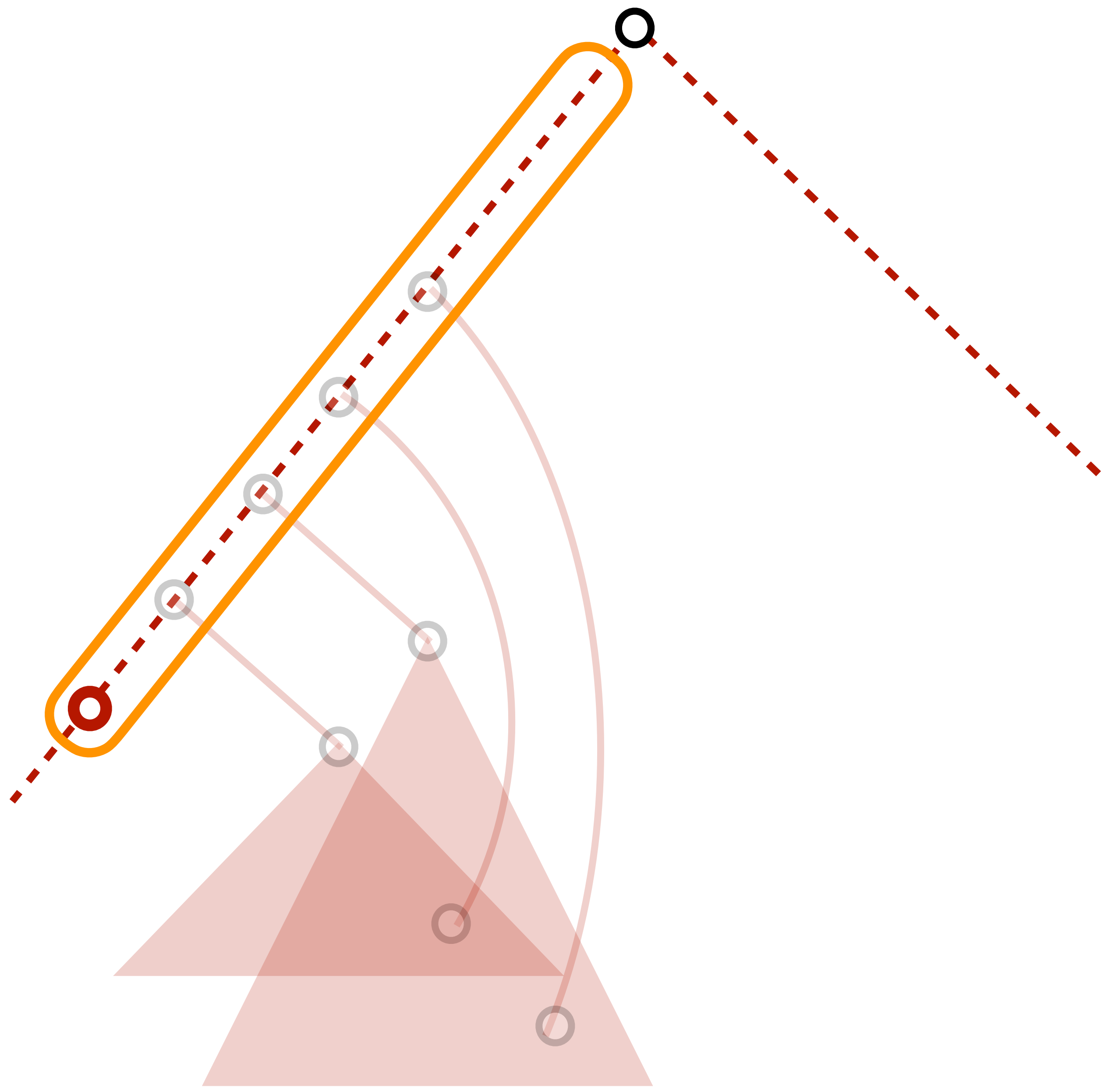

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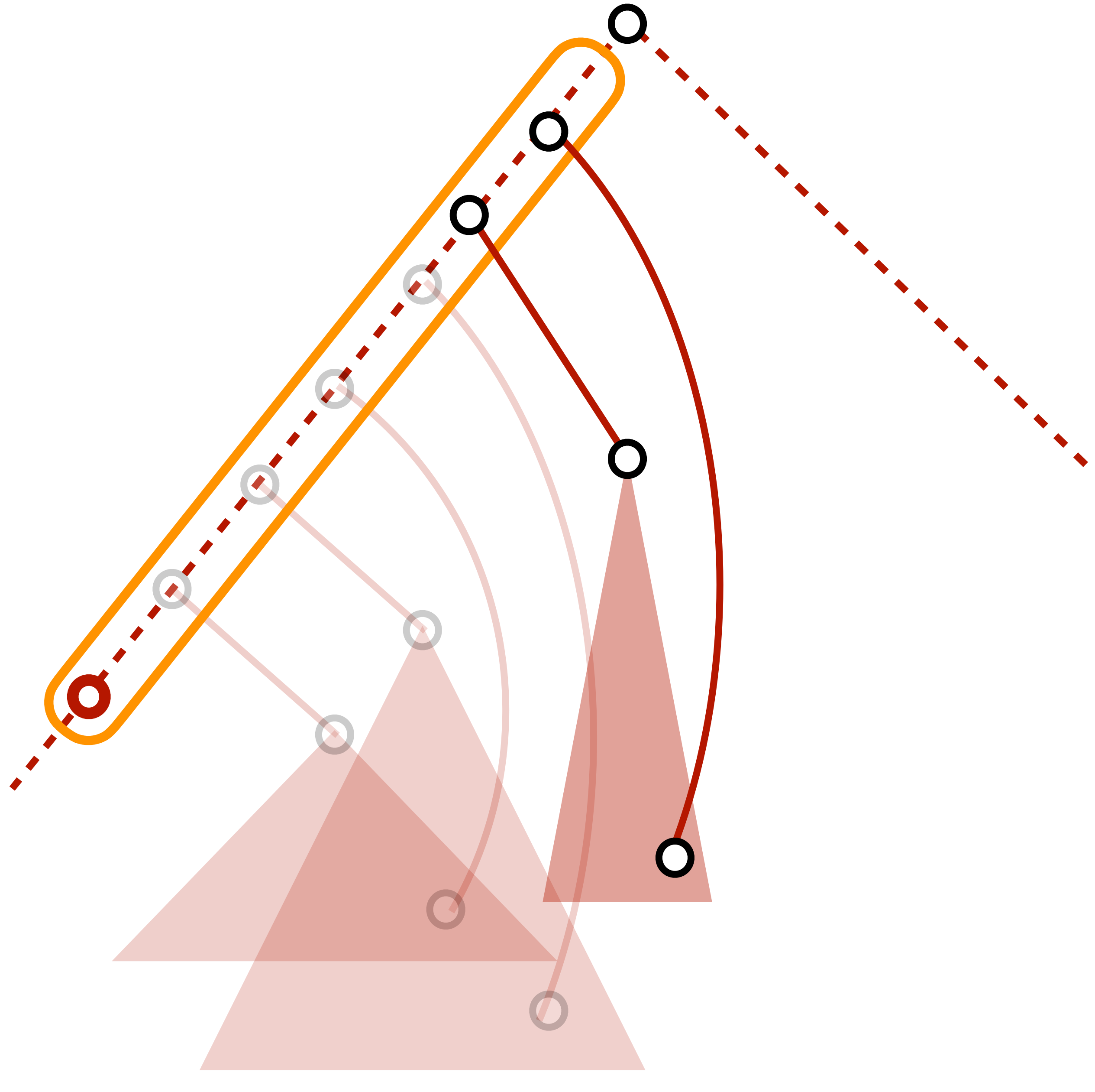

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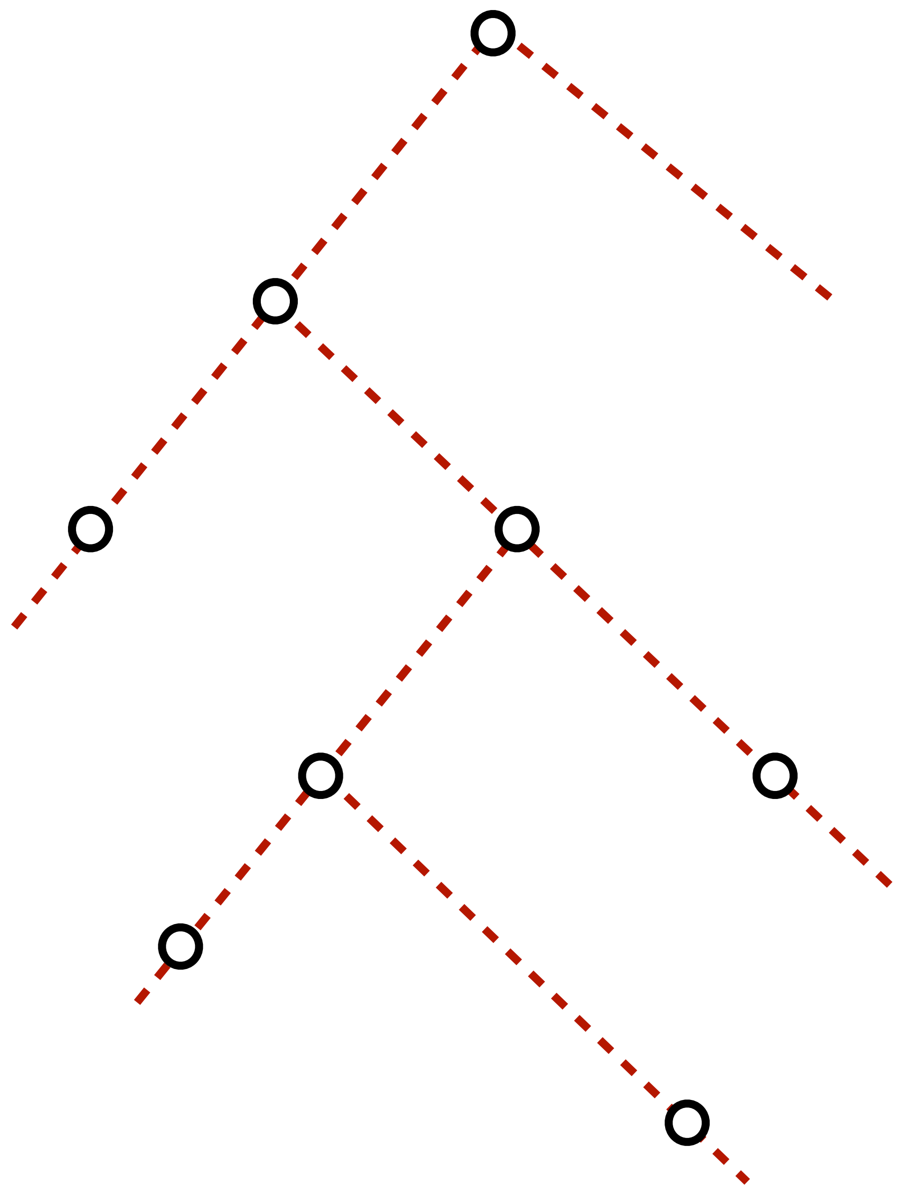

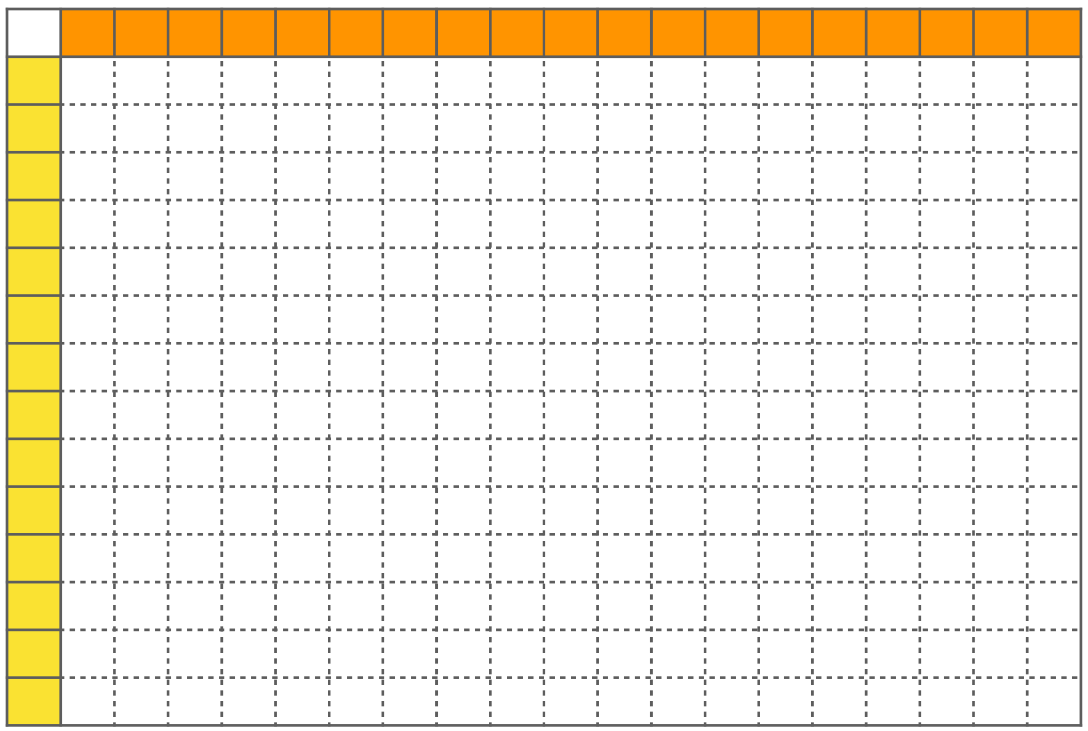

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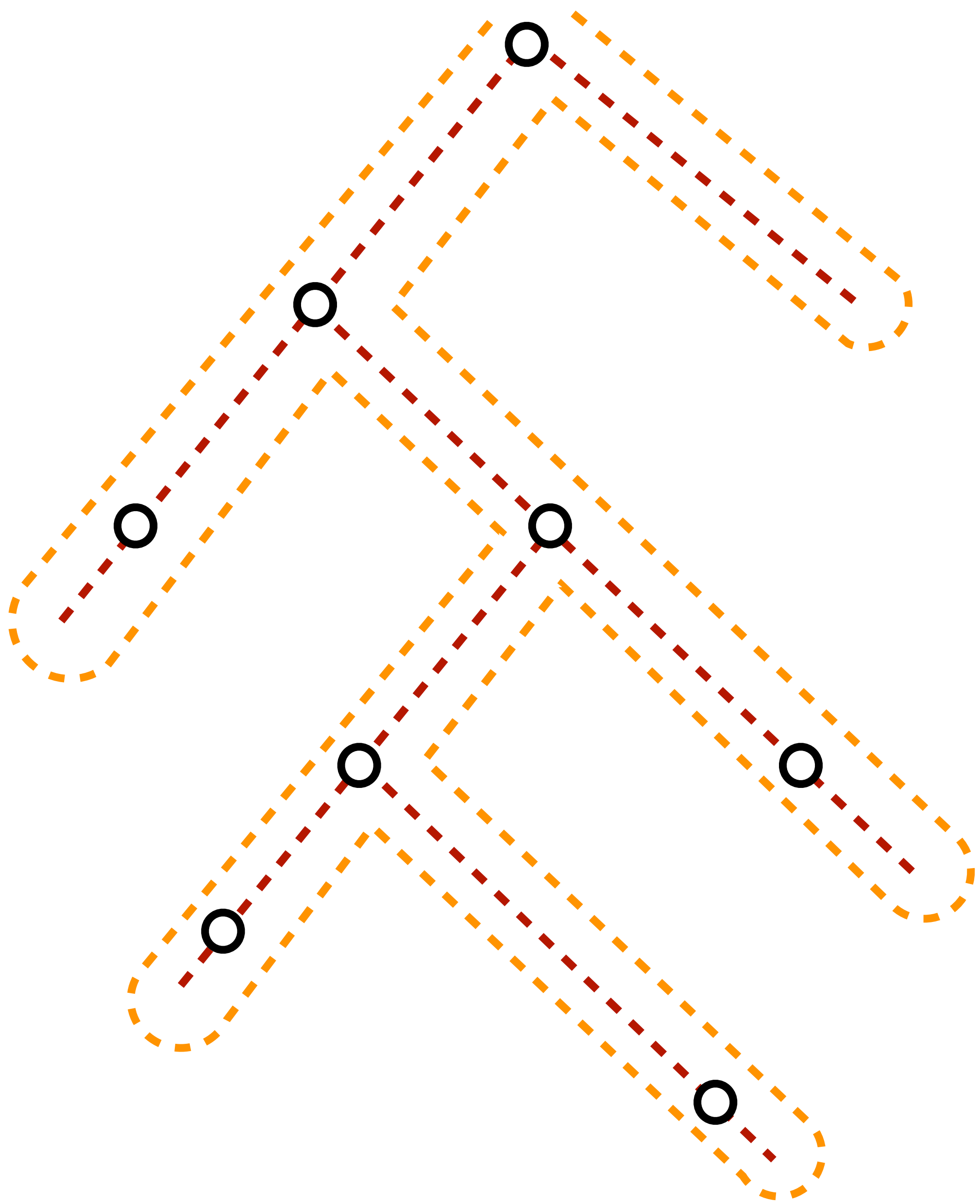

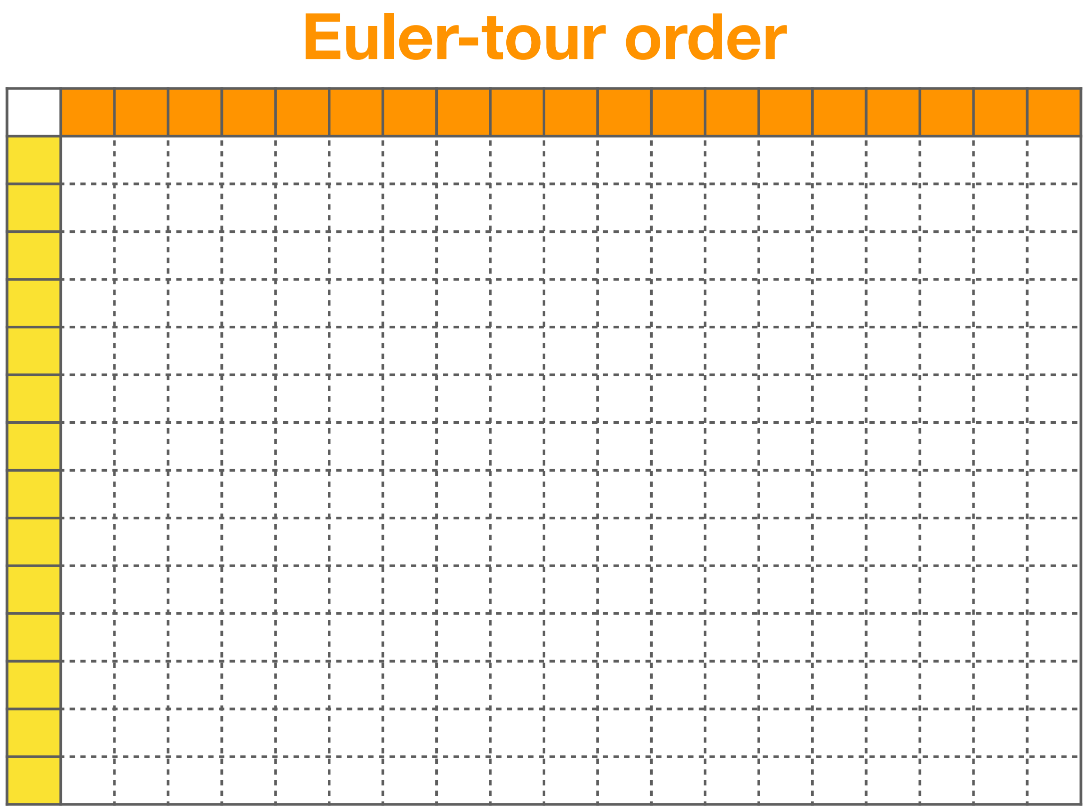

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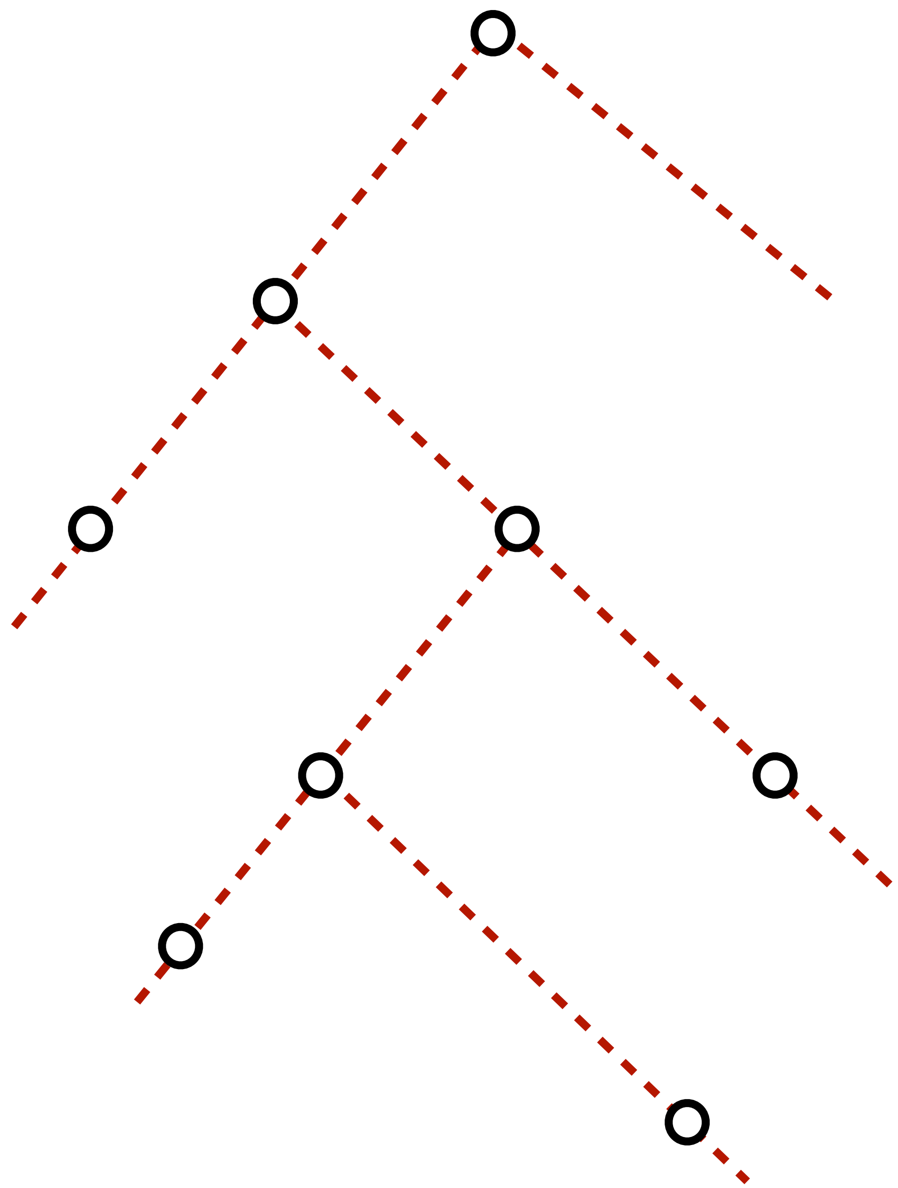

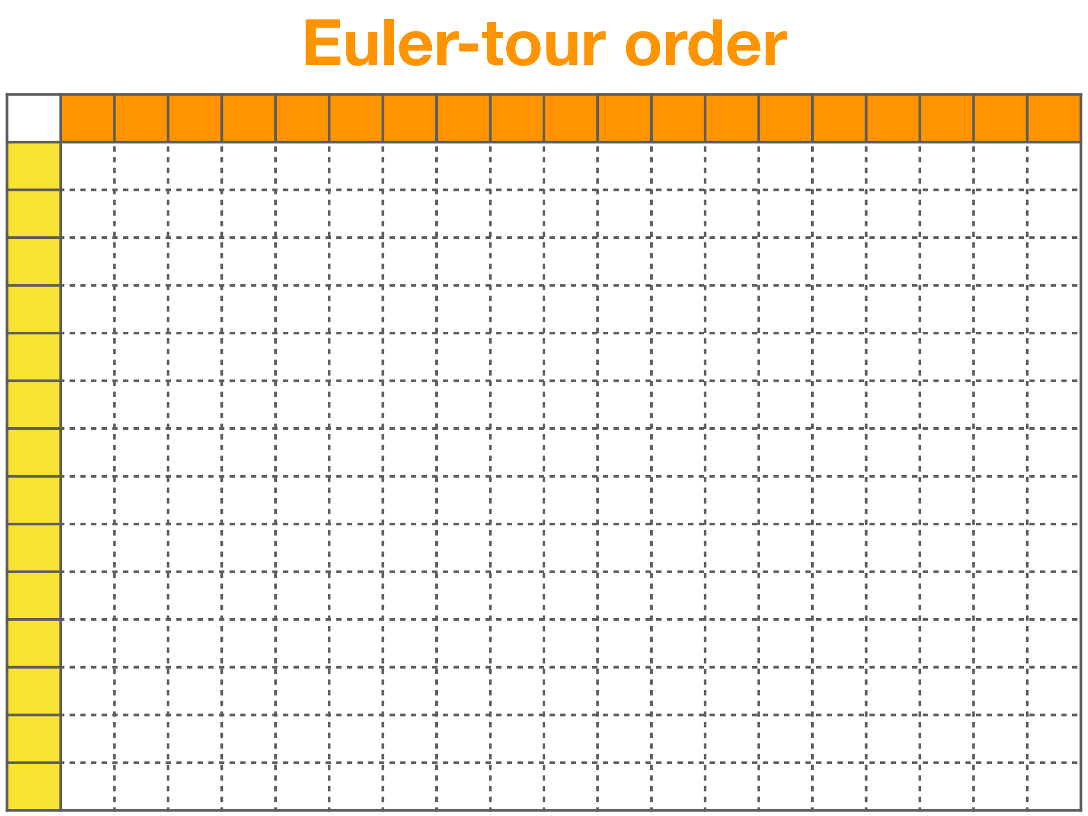

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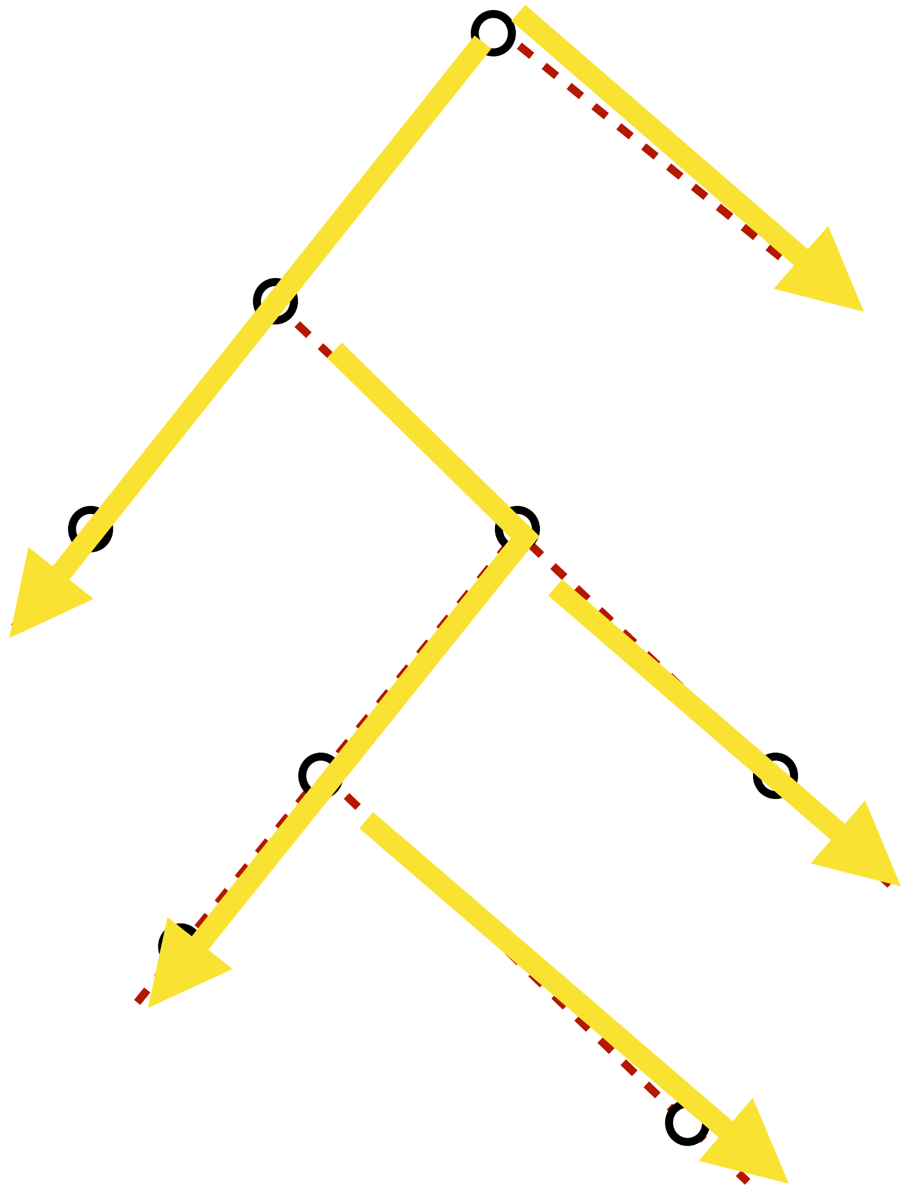

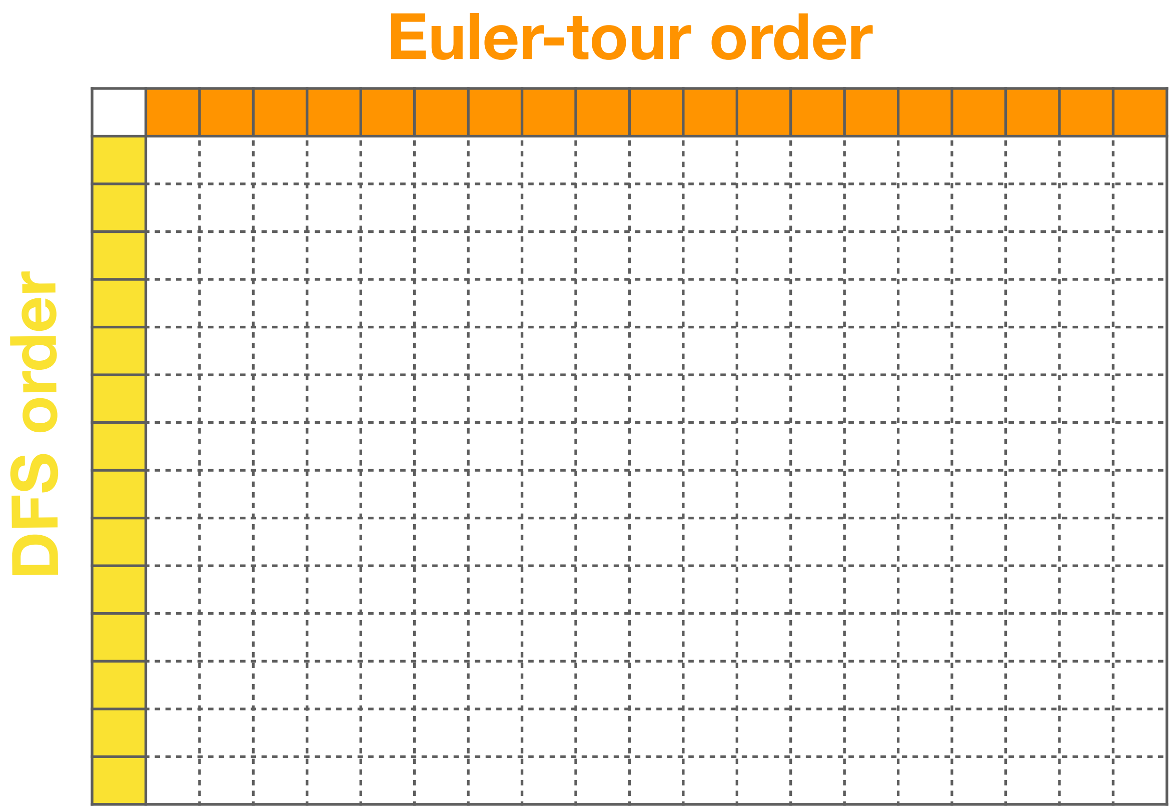

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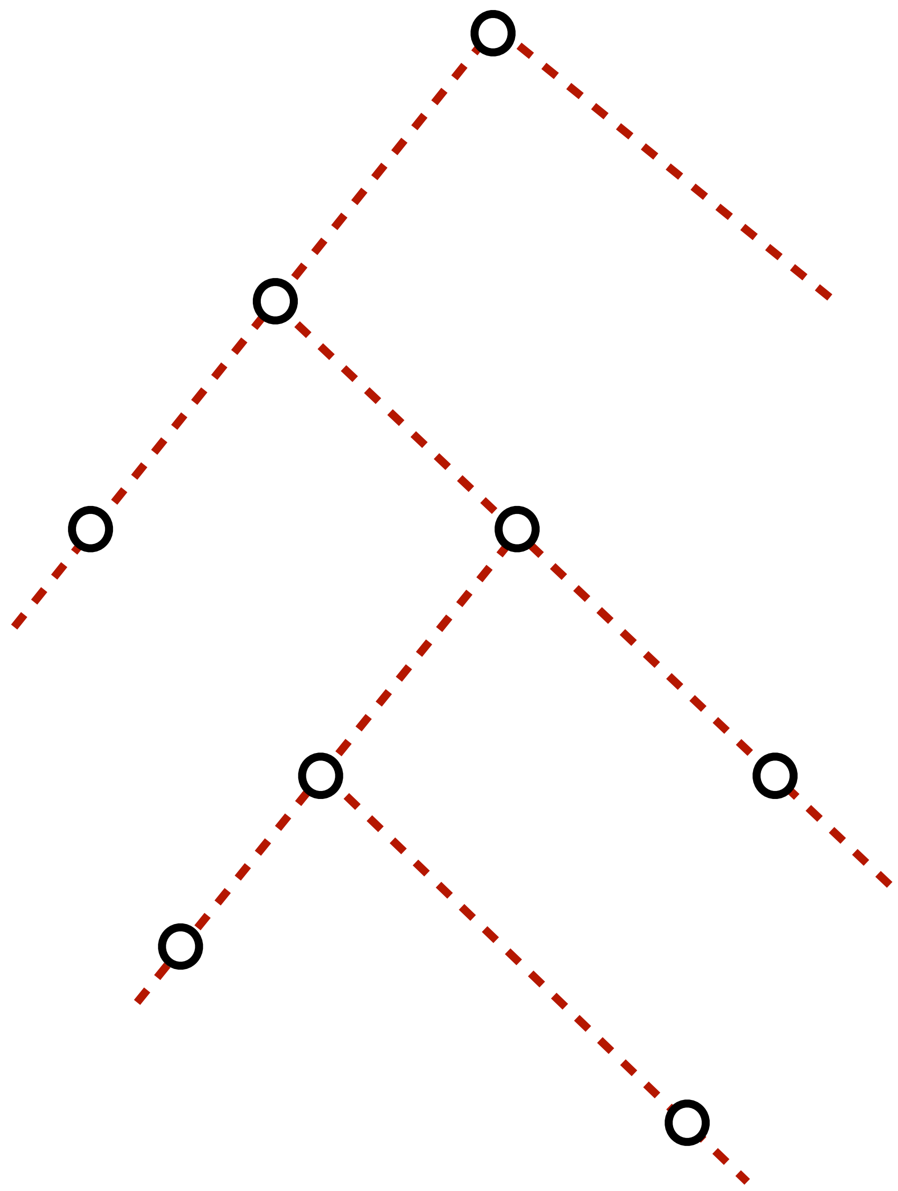

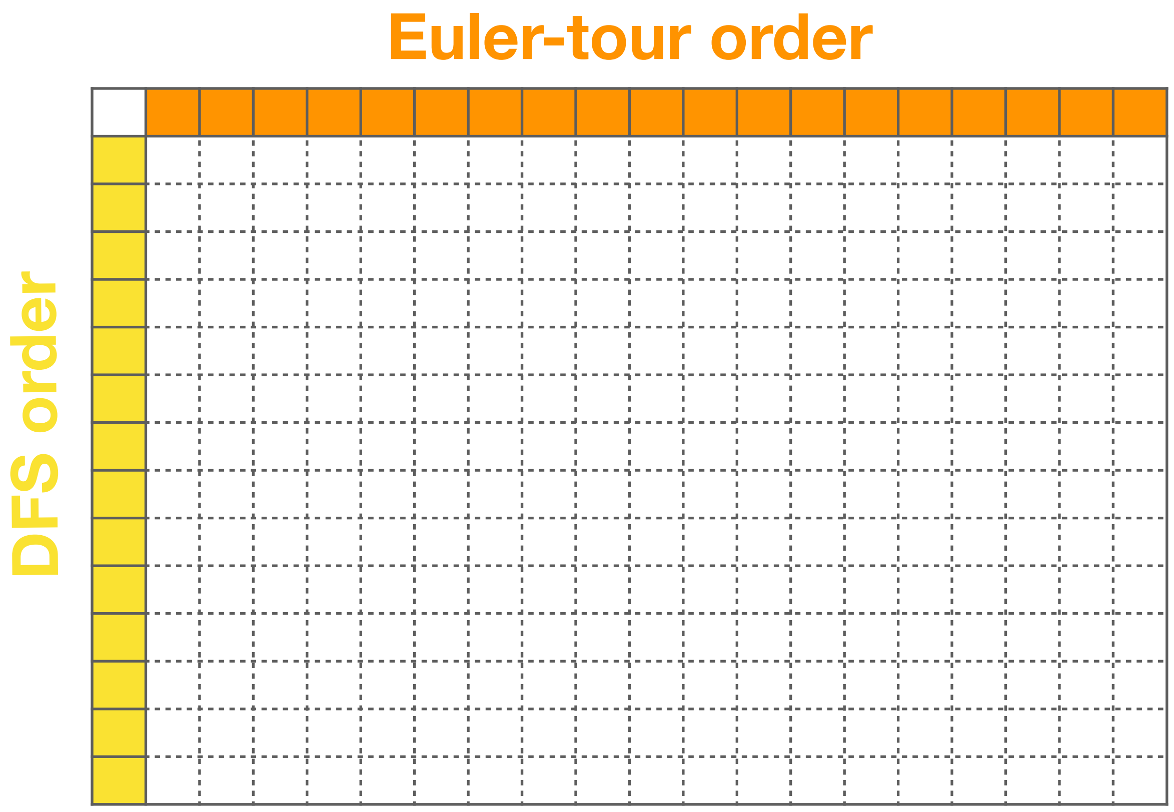

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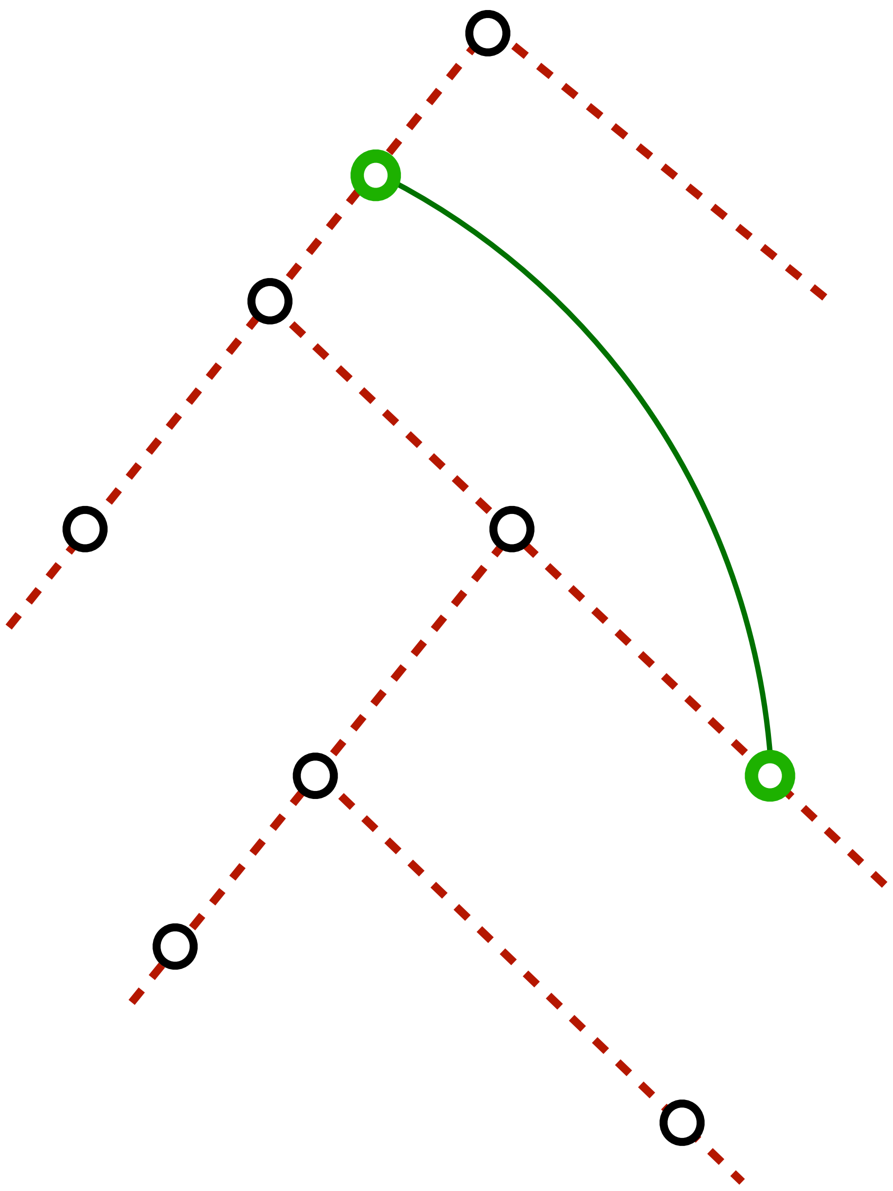

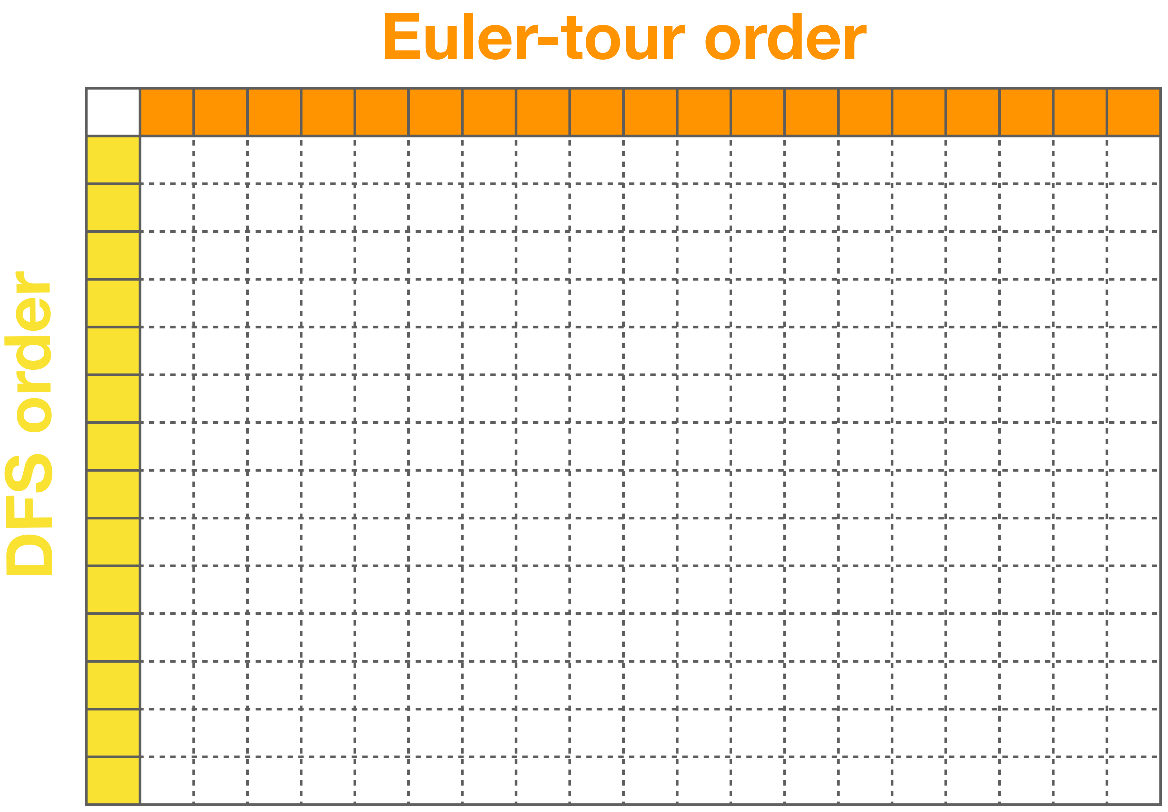

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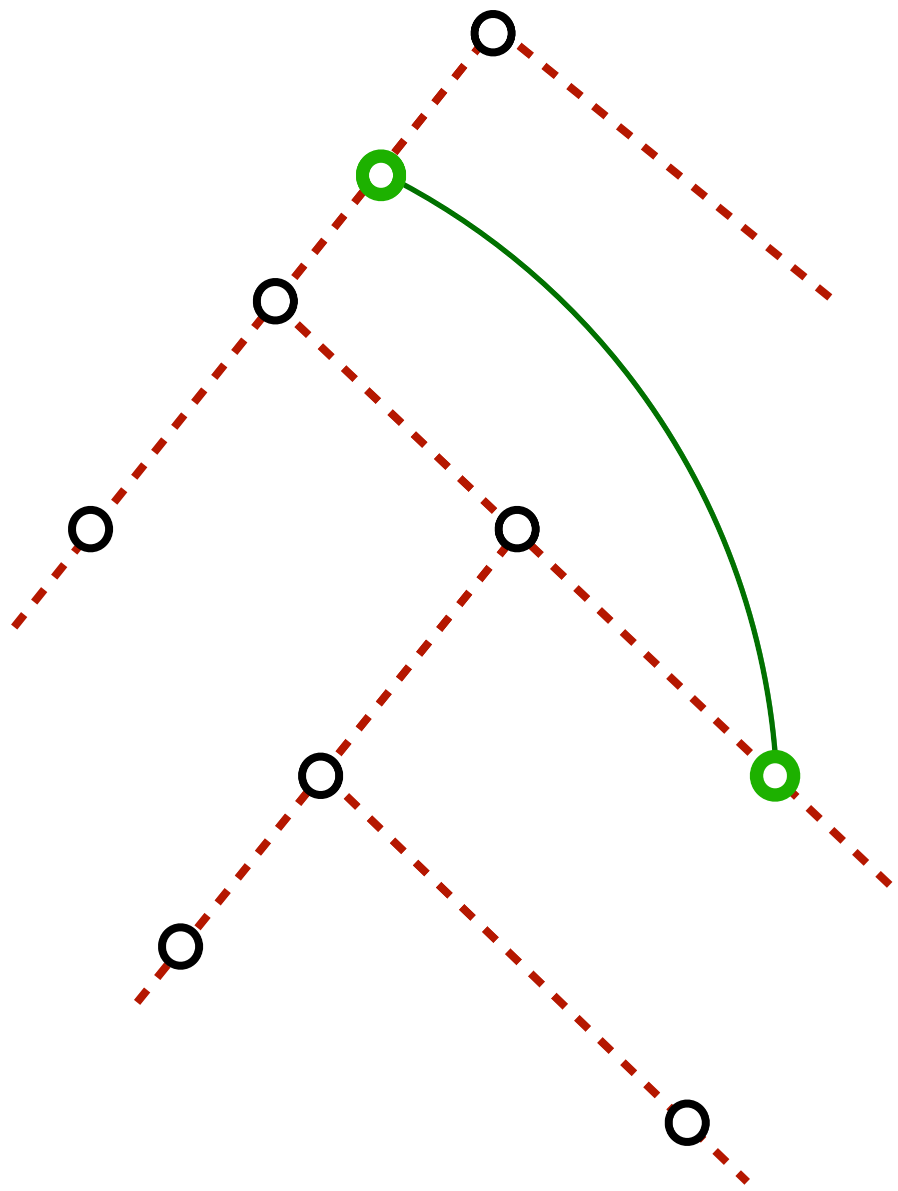

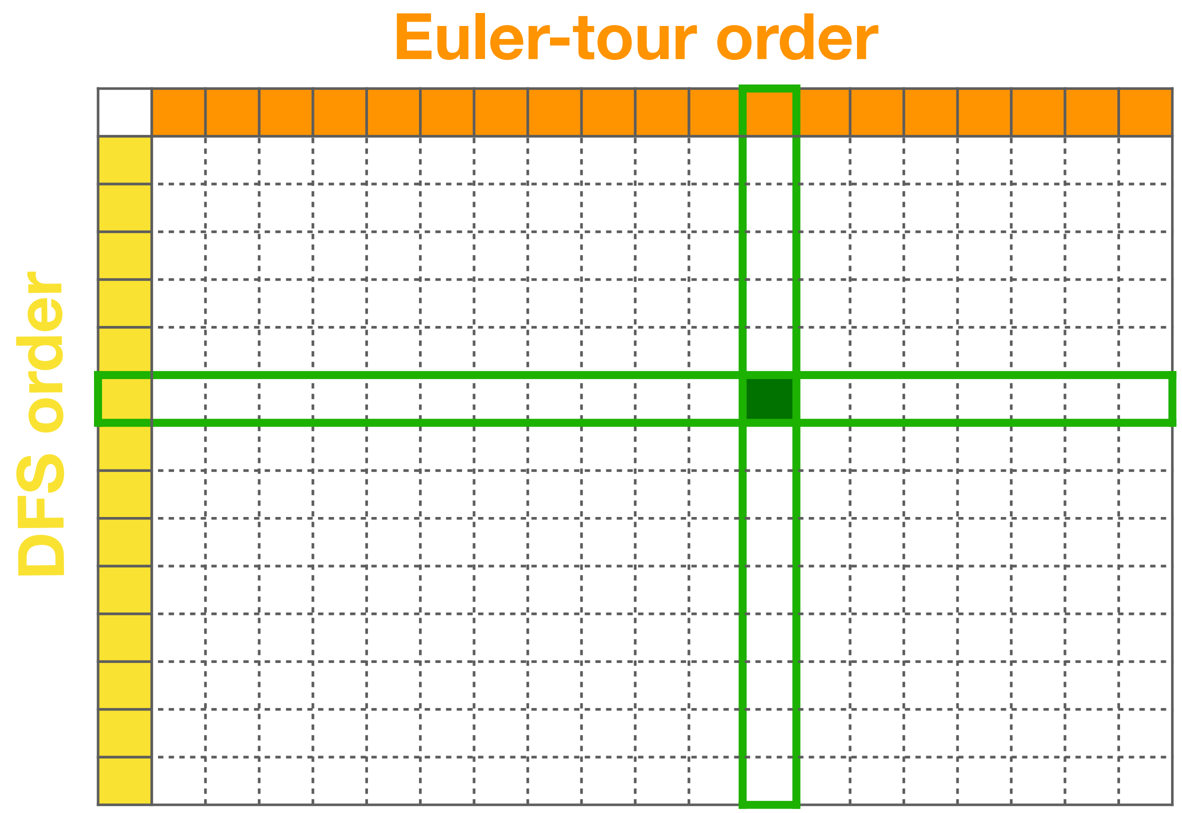

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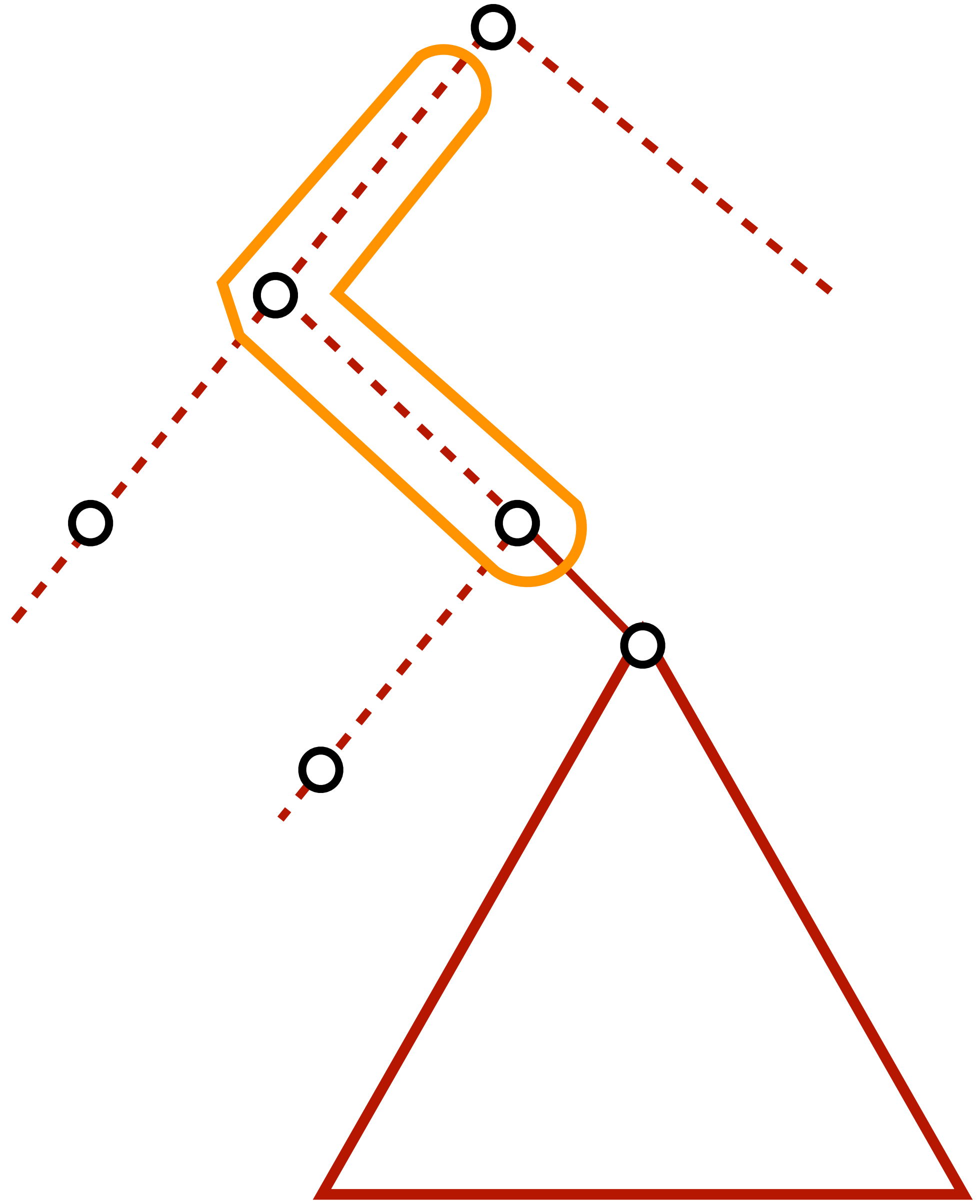
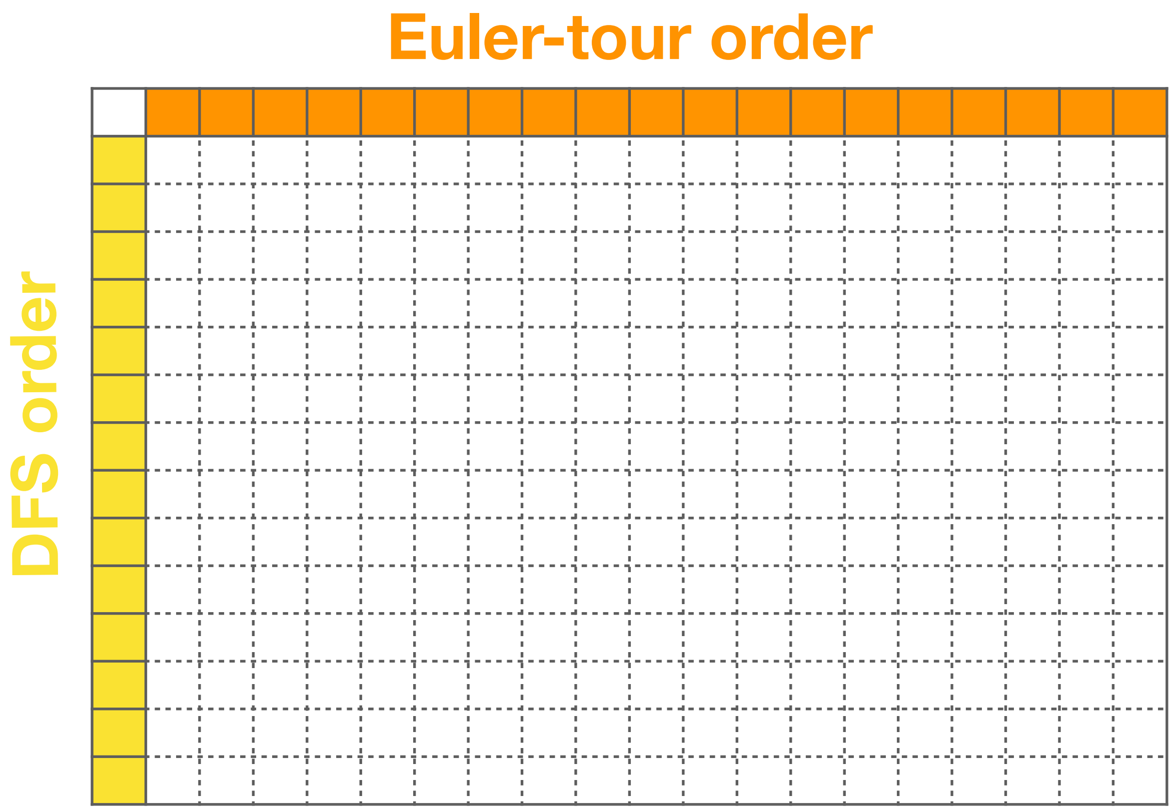
**Tool I:** 2D-range query





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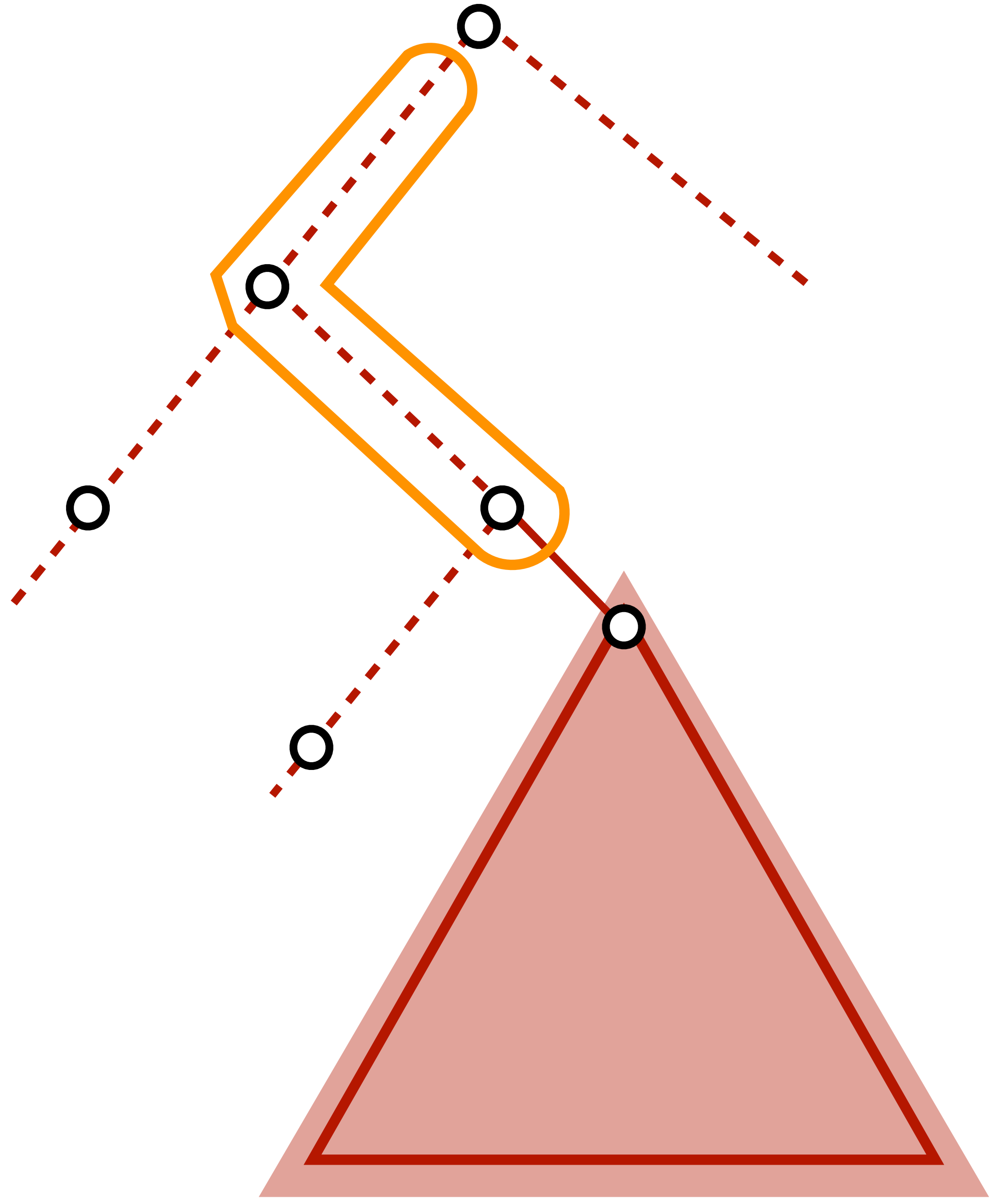

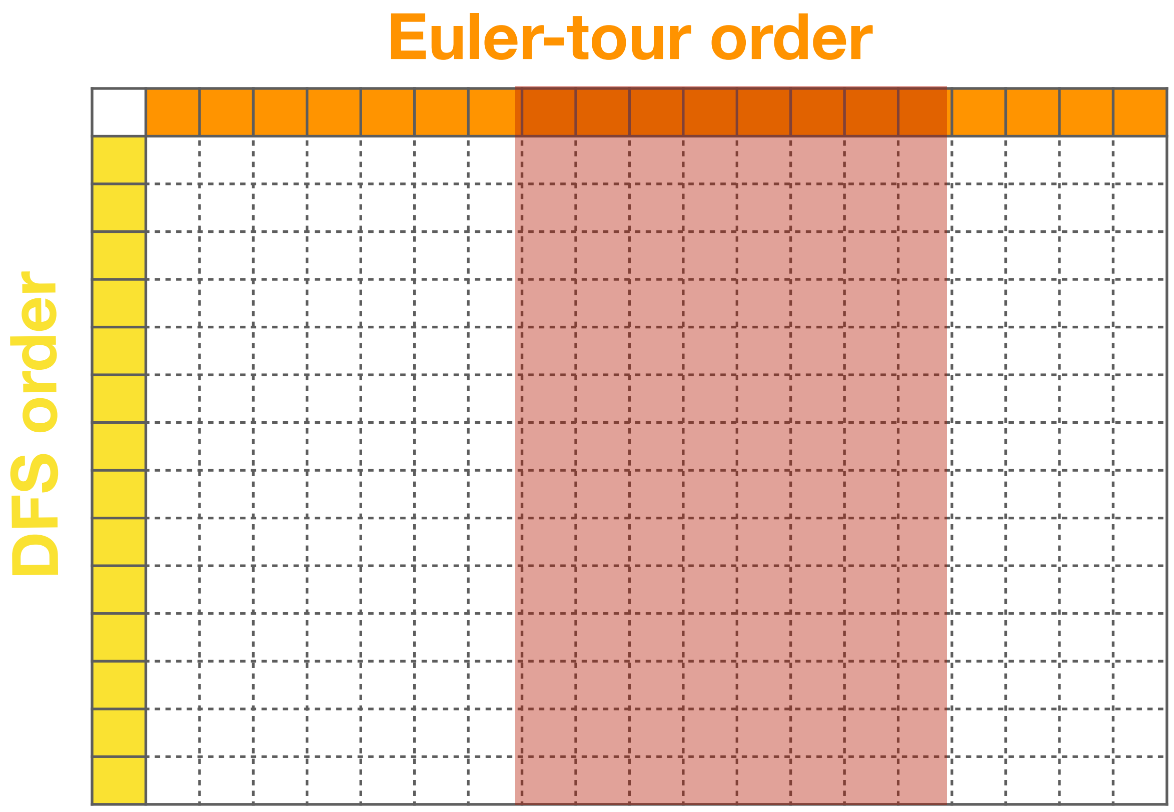

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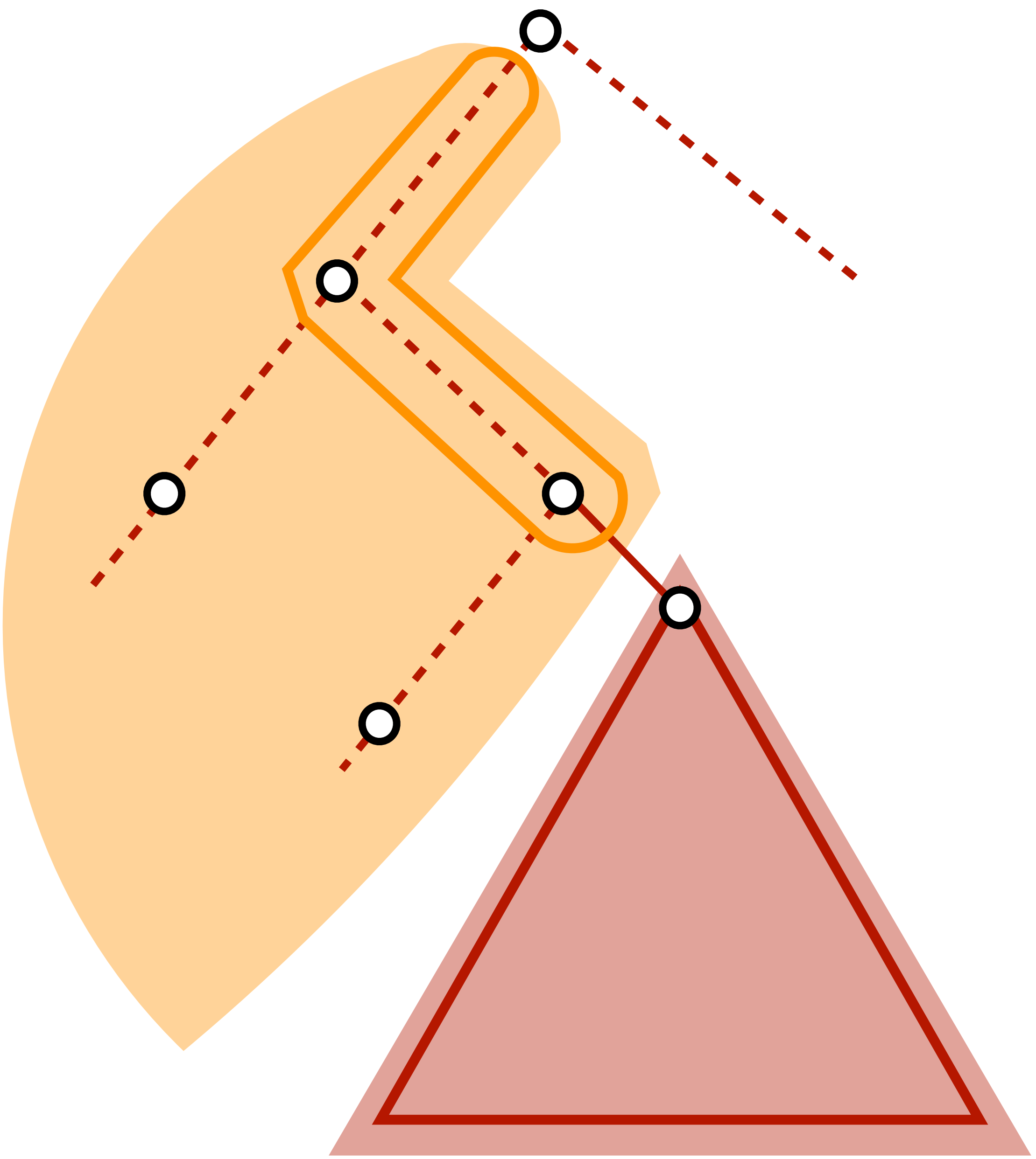

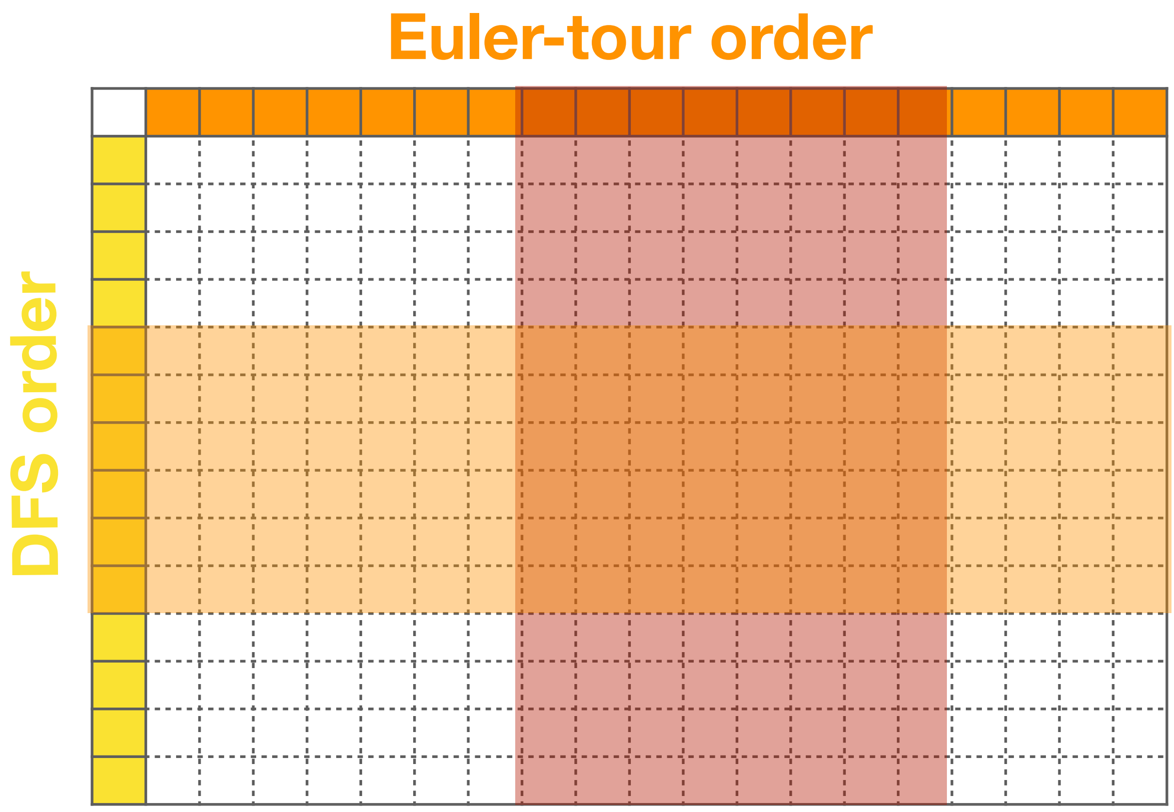

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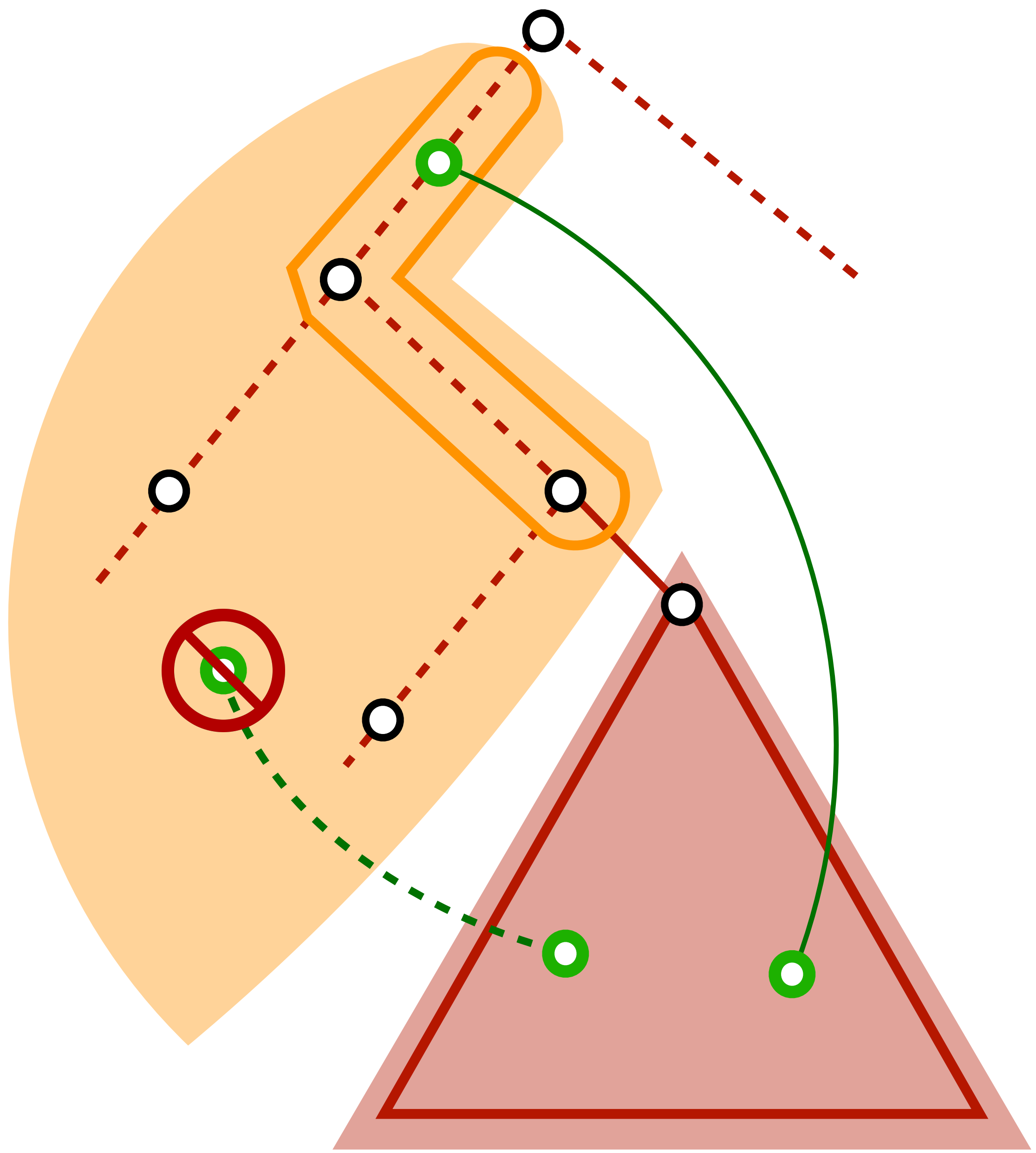

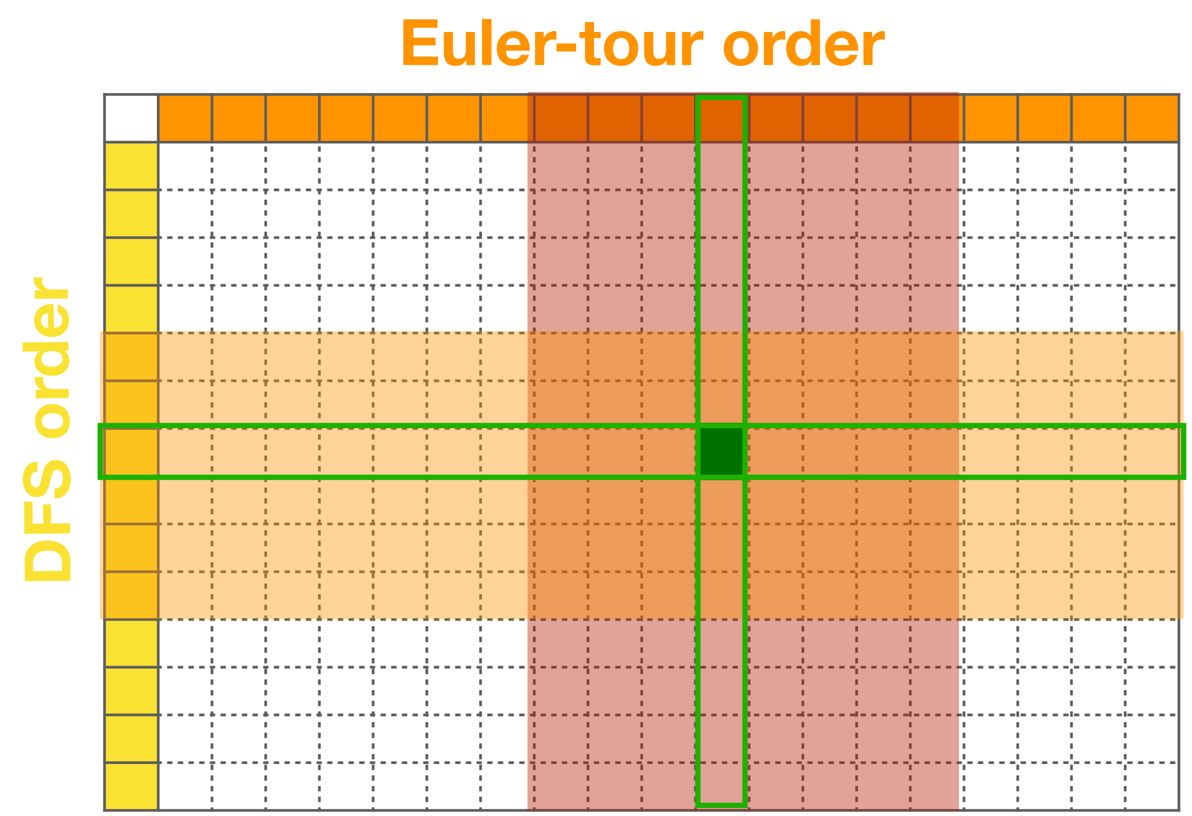


**2D-range minimum takes $O(\log n)$ time**

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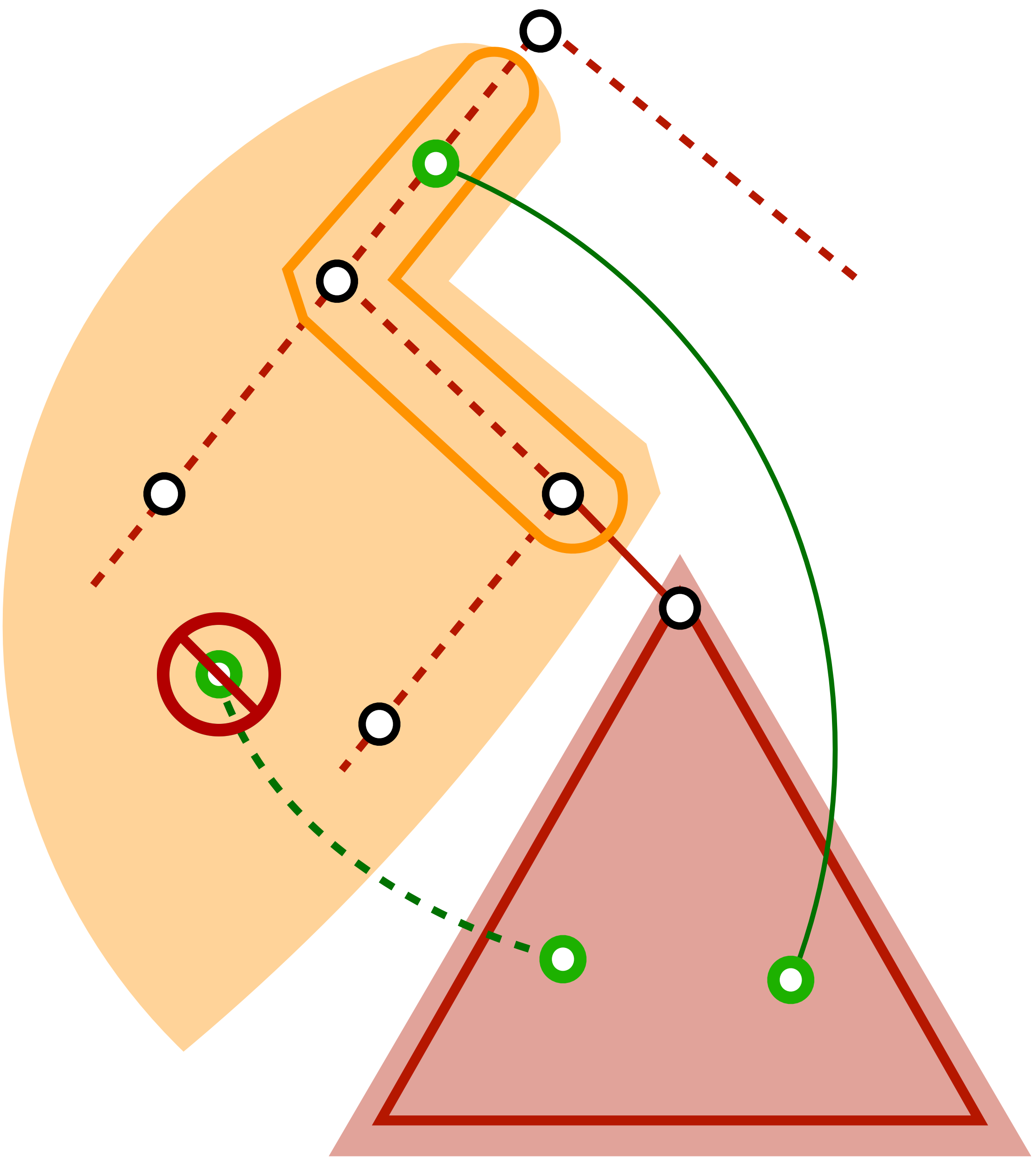

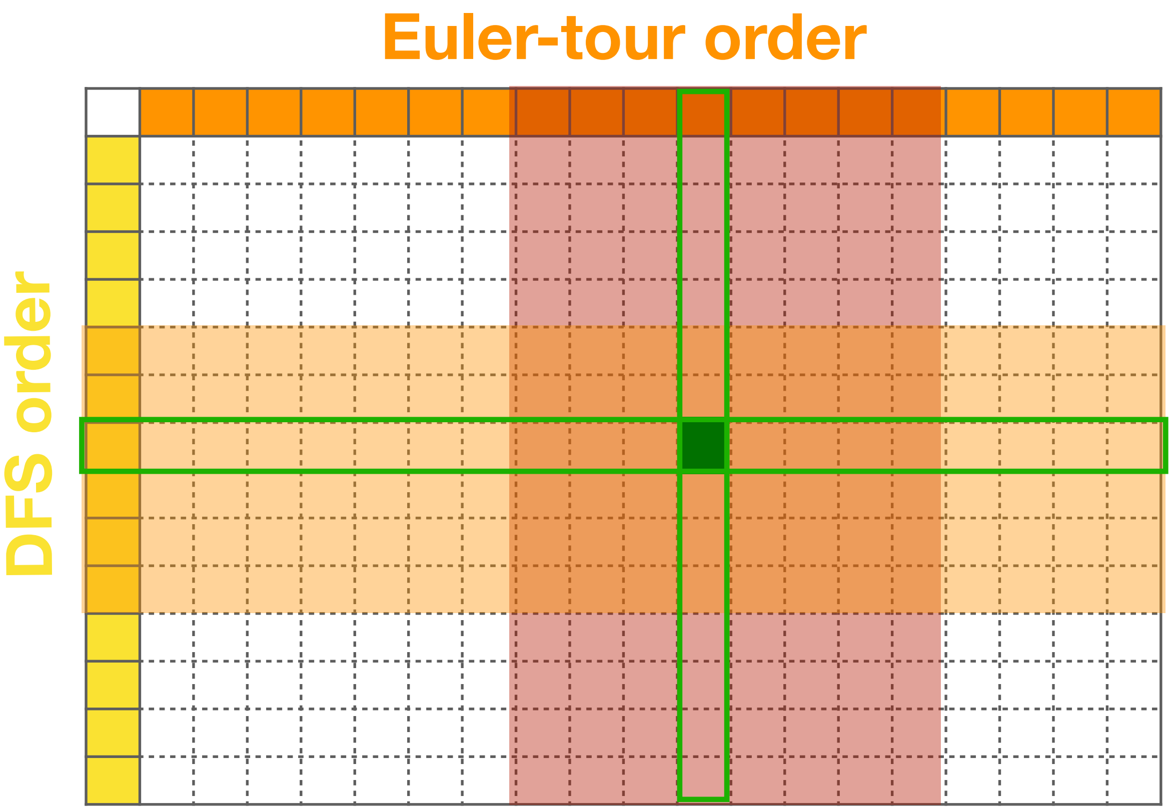


**2D-range minimum takes $O(\log n)$ time**

**Total time 'd be** $O(k + n \log n)$

# Finding highest ancestors

**Tool II:** Fractional cascading & tree partitioning

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**Lemma:** [CG'86]

Given *k* sorted integer arrays of total size *m*

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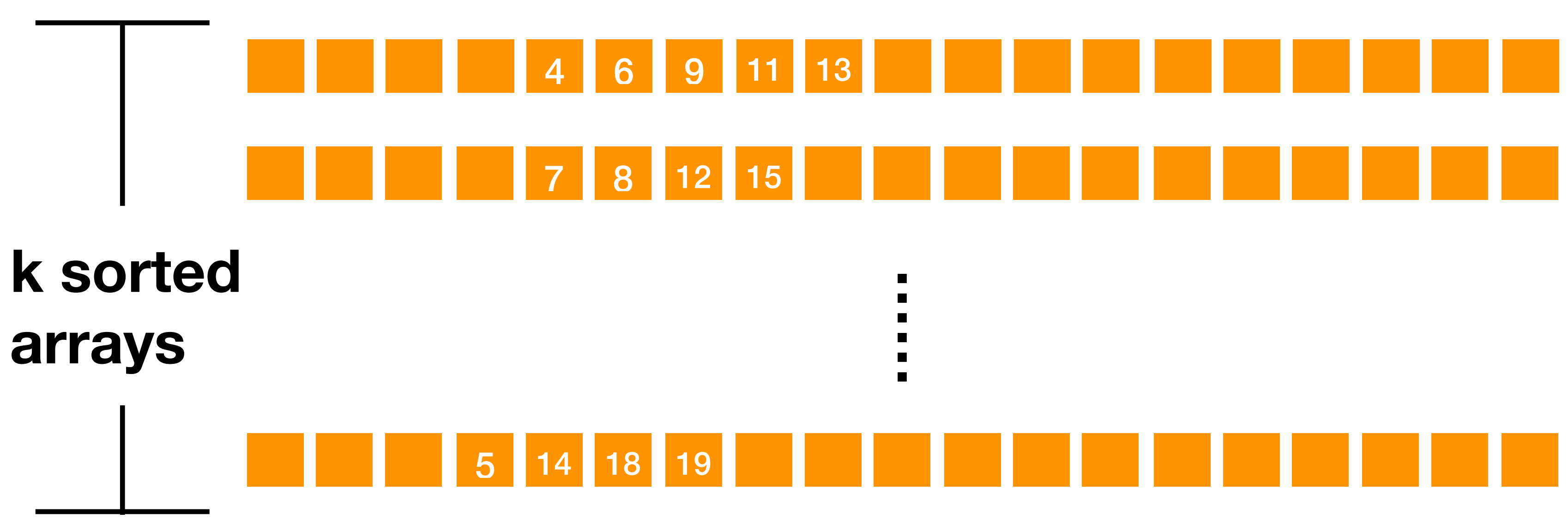

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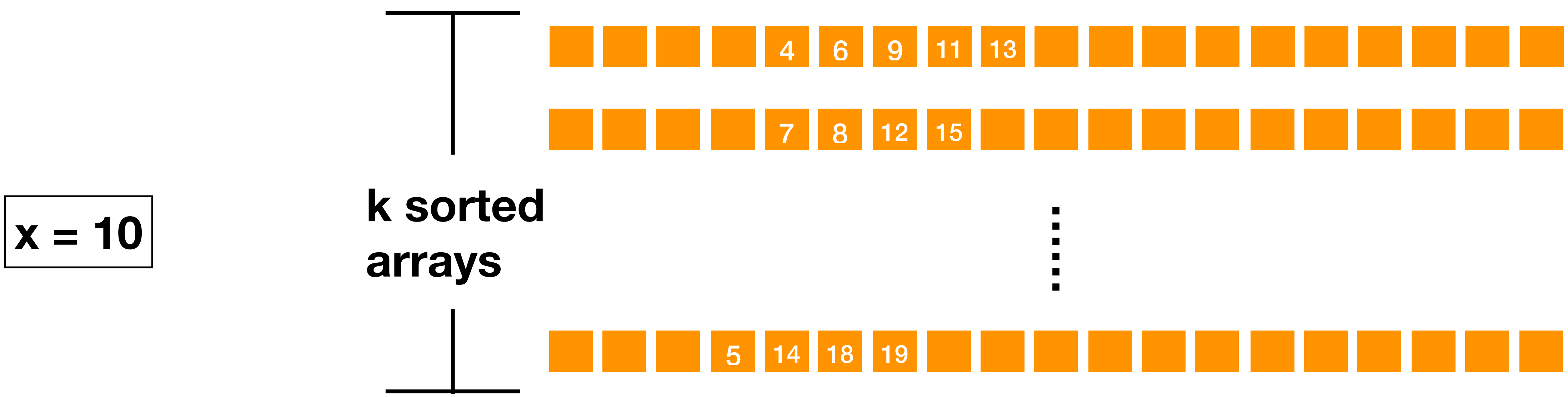

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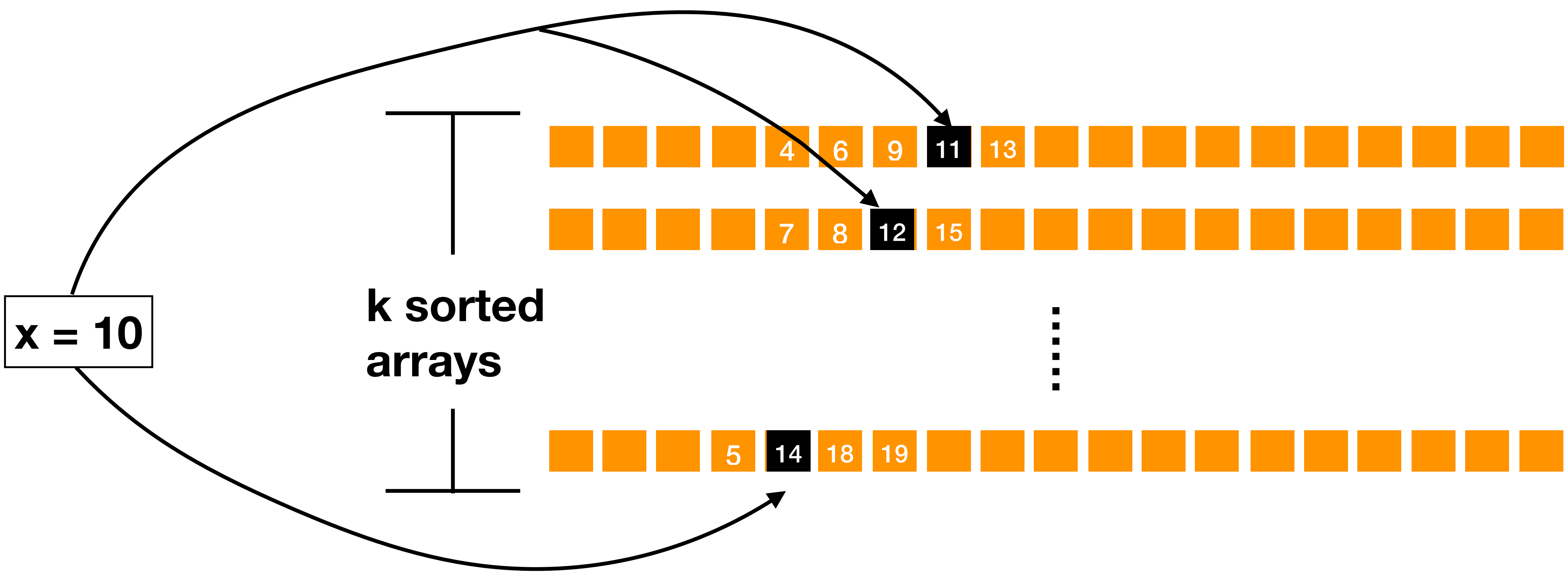

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<u>**Lemma:**</u> [DZ'17]

Given a tree *T* of size *n*,

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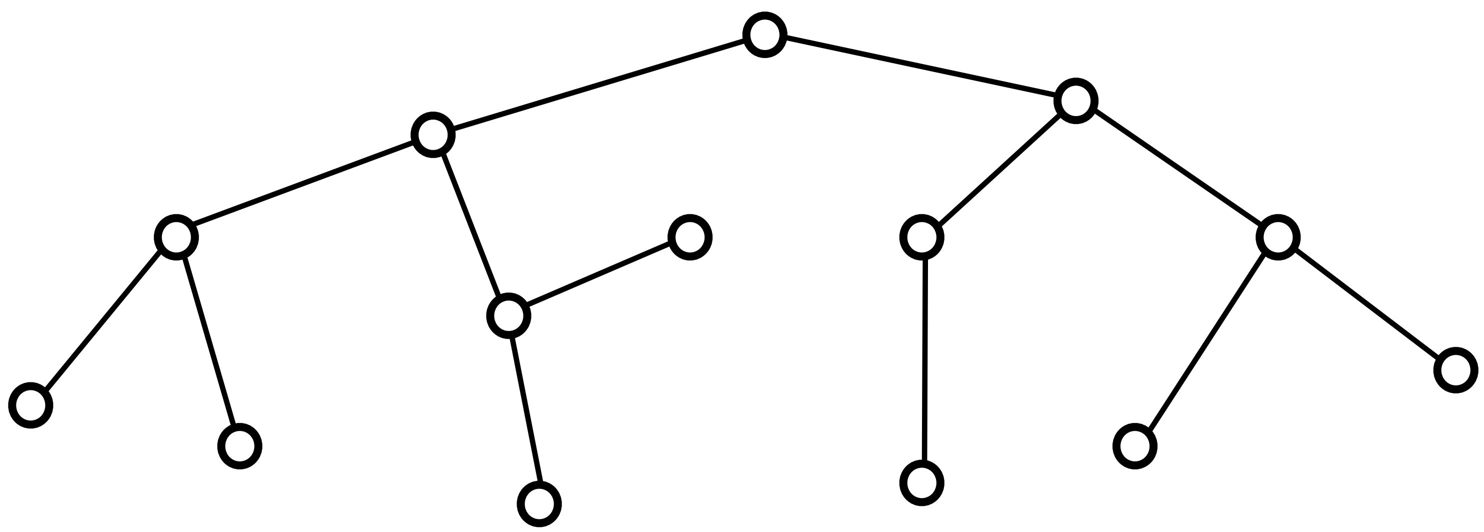

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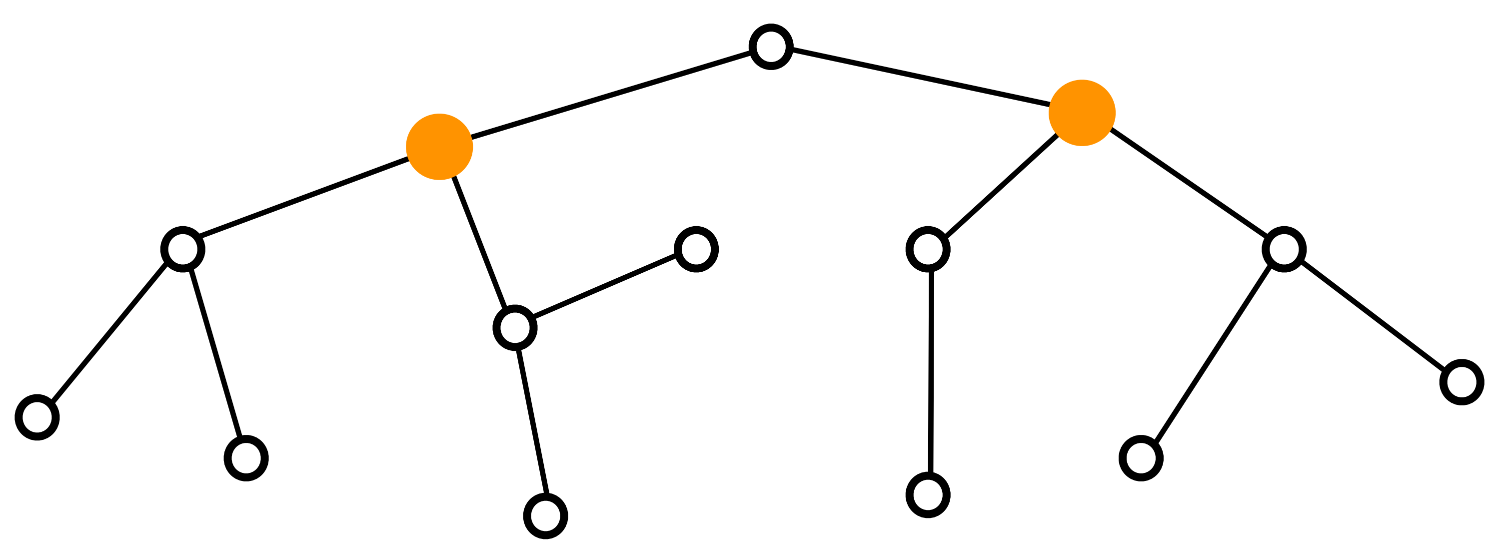

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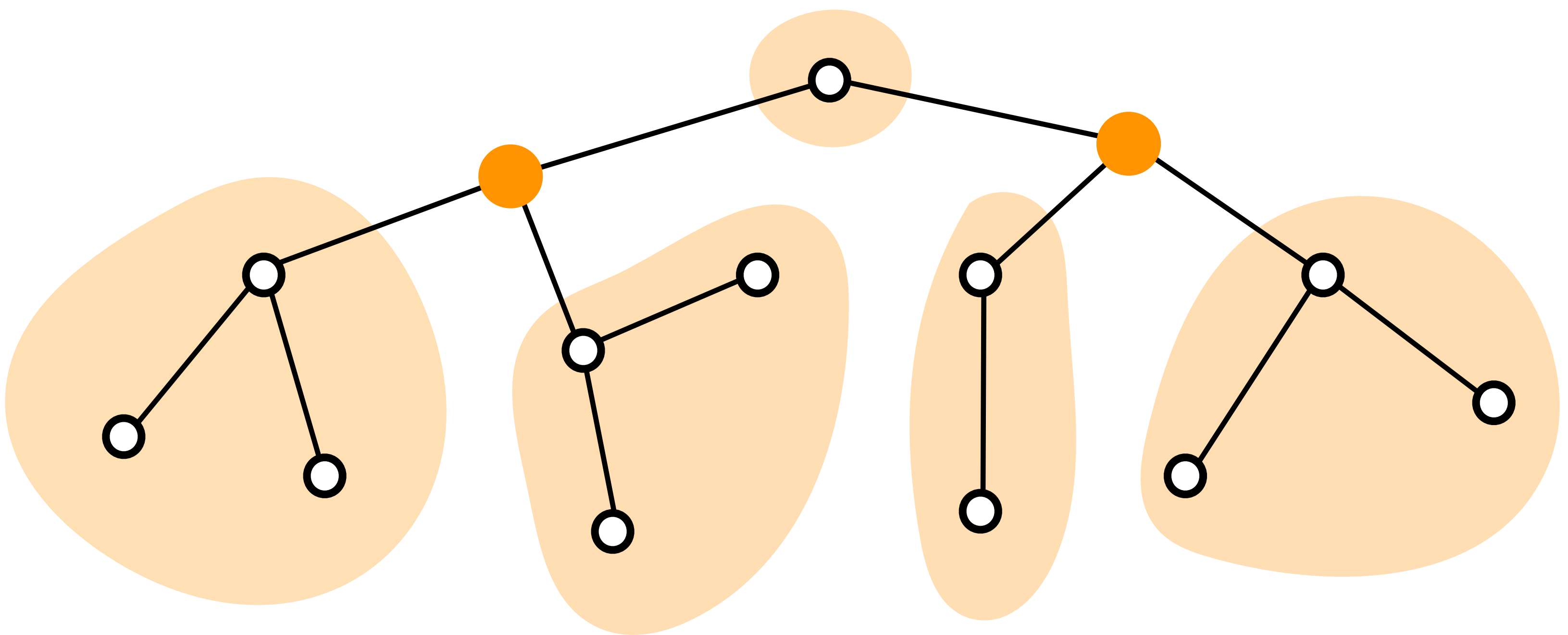
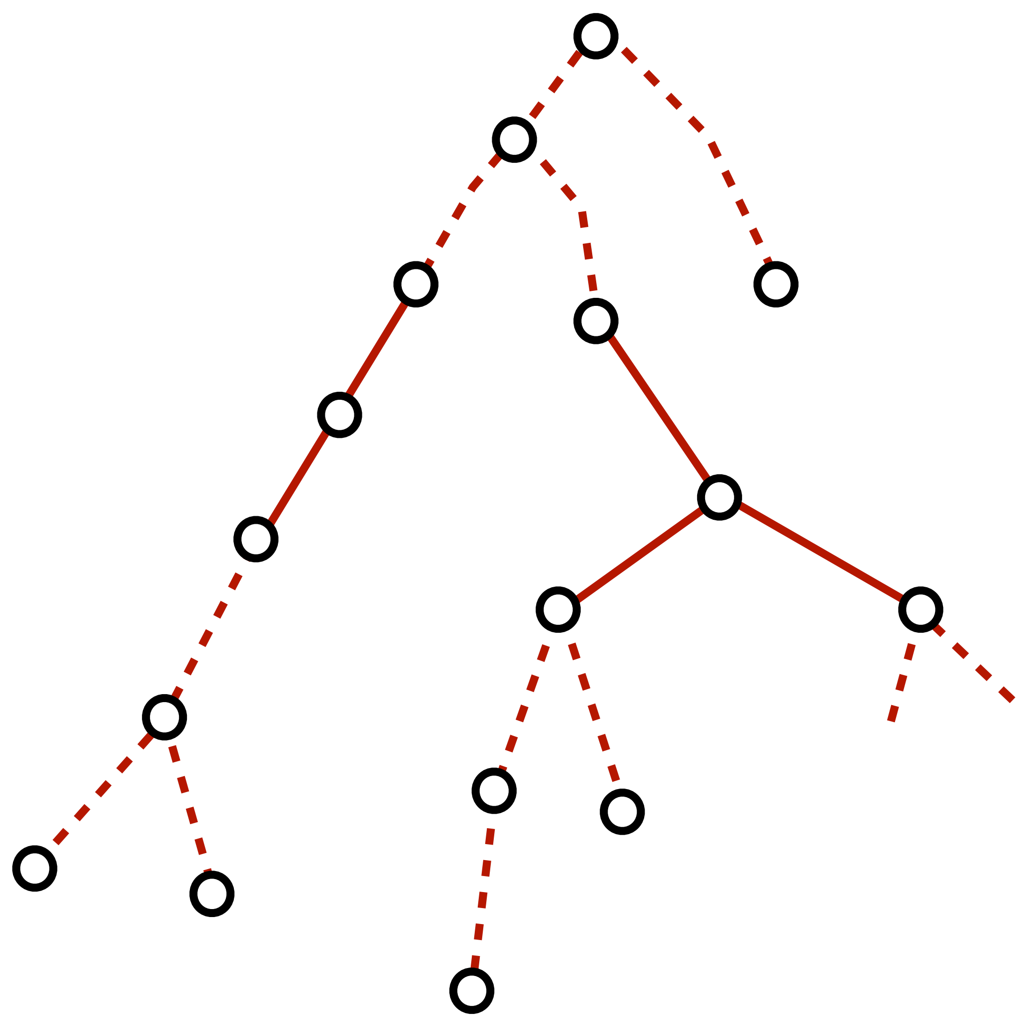


# Finding highest ancestors

New data structure for finding highest ancestors



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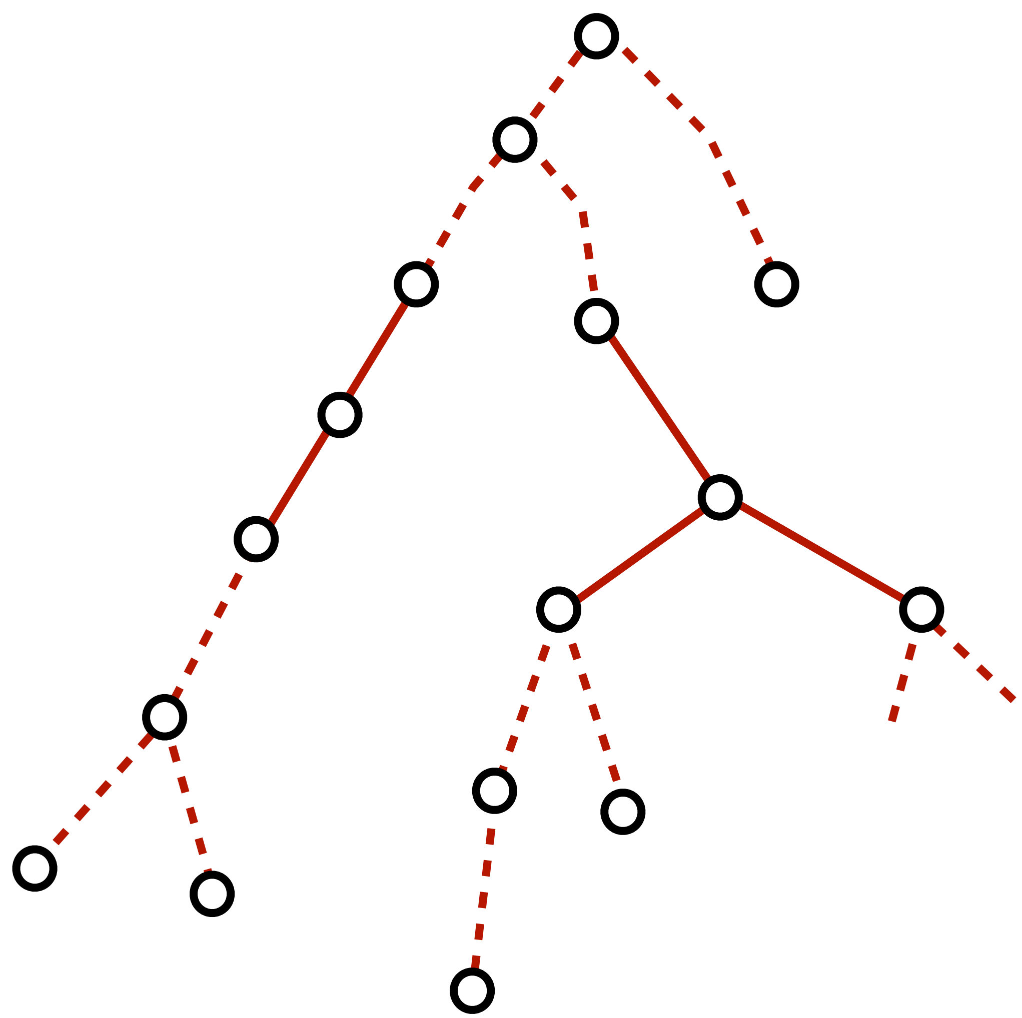
**Apply tree partition with** $k = \log n$
**So every subtree has size** $O(\log n)$



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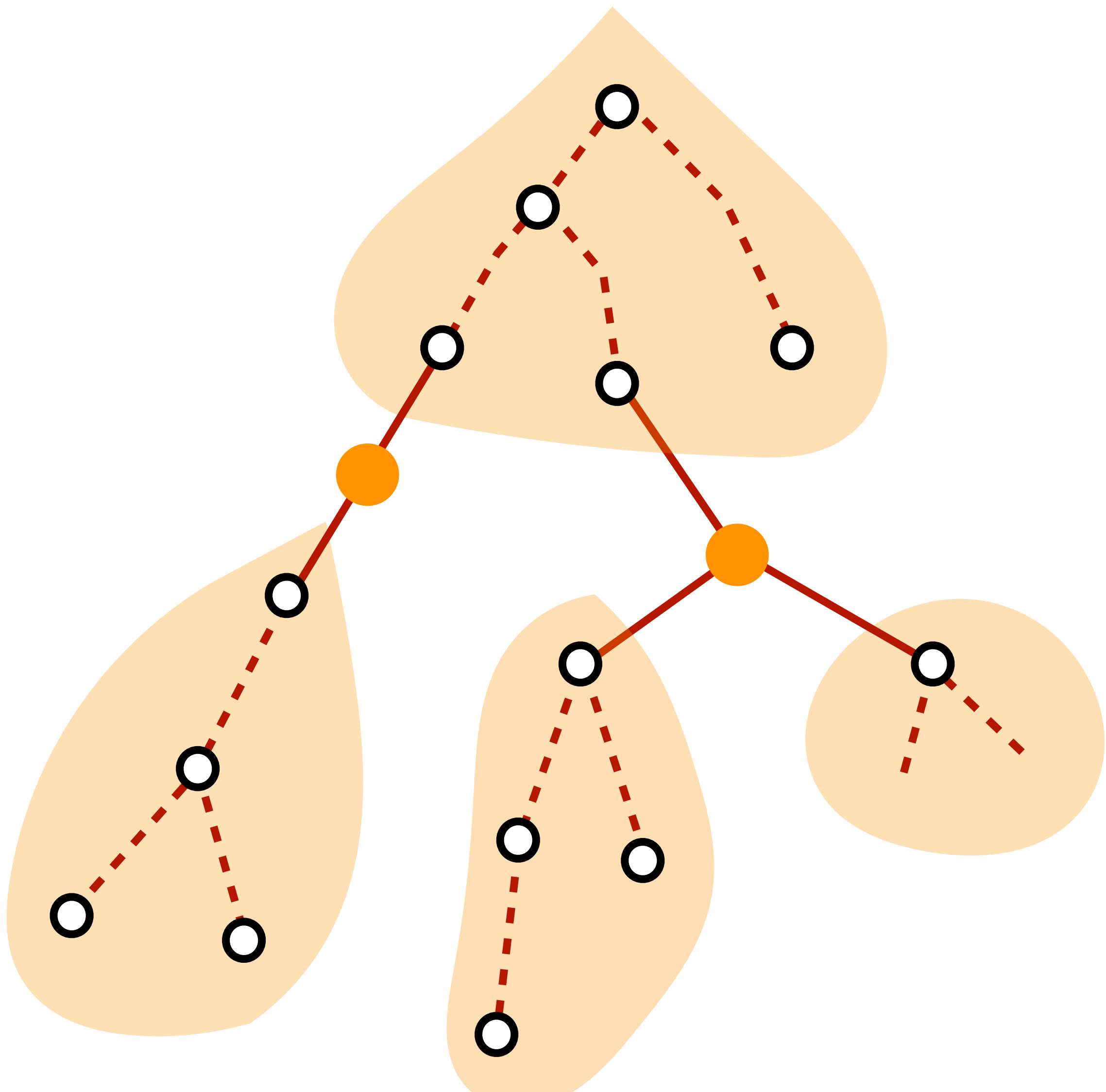

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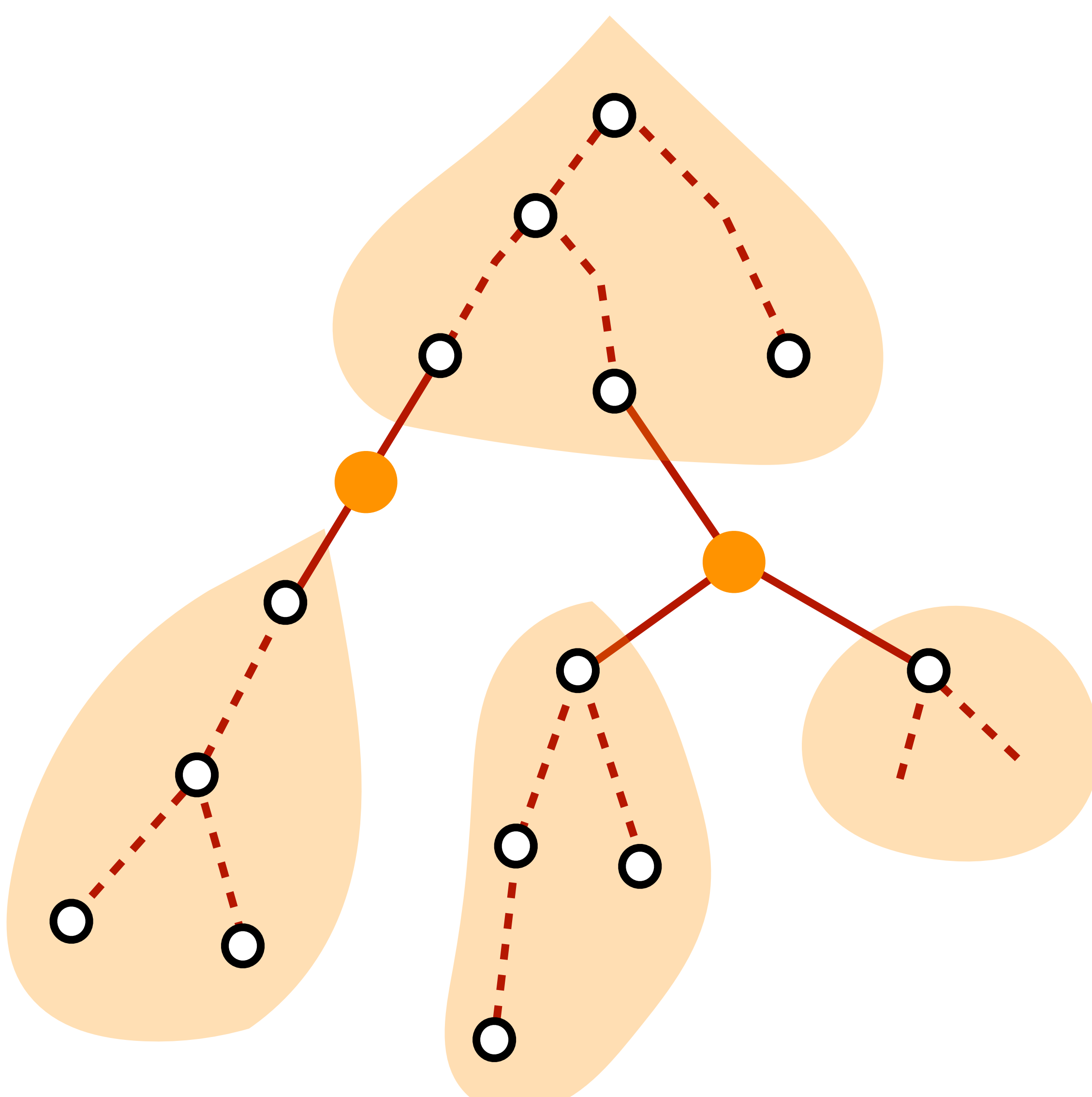
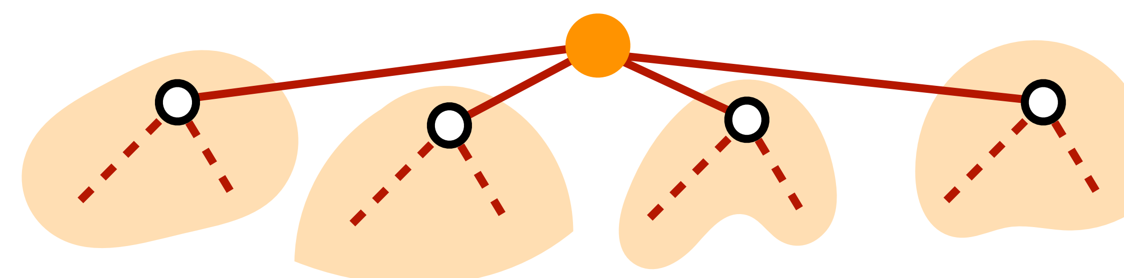


**Build** **fractional cascading** **on a bunch of** **subtrees**



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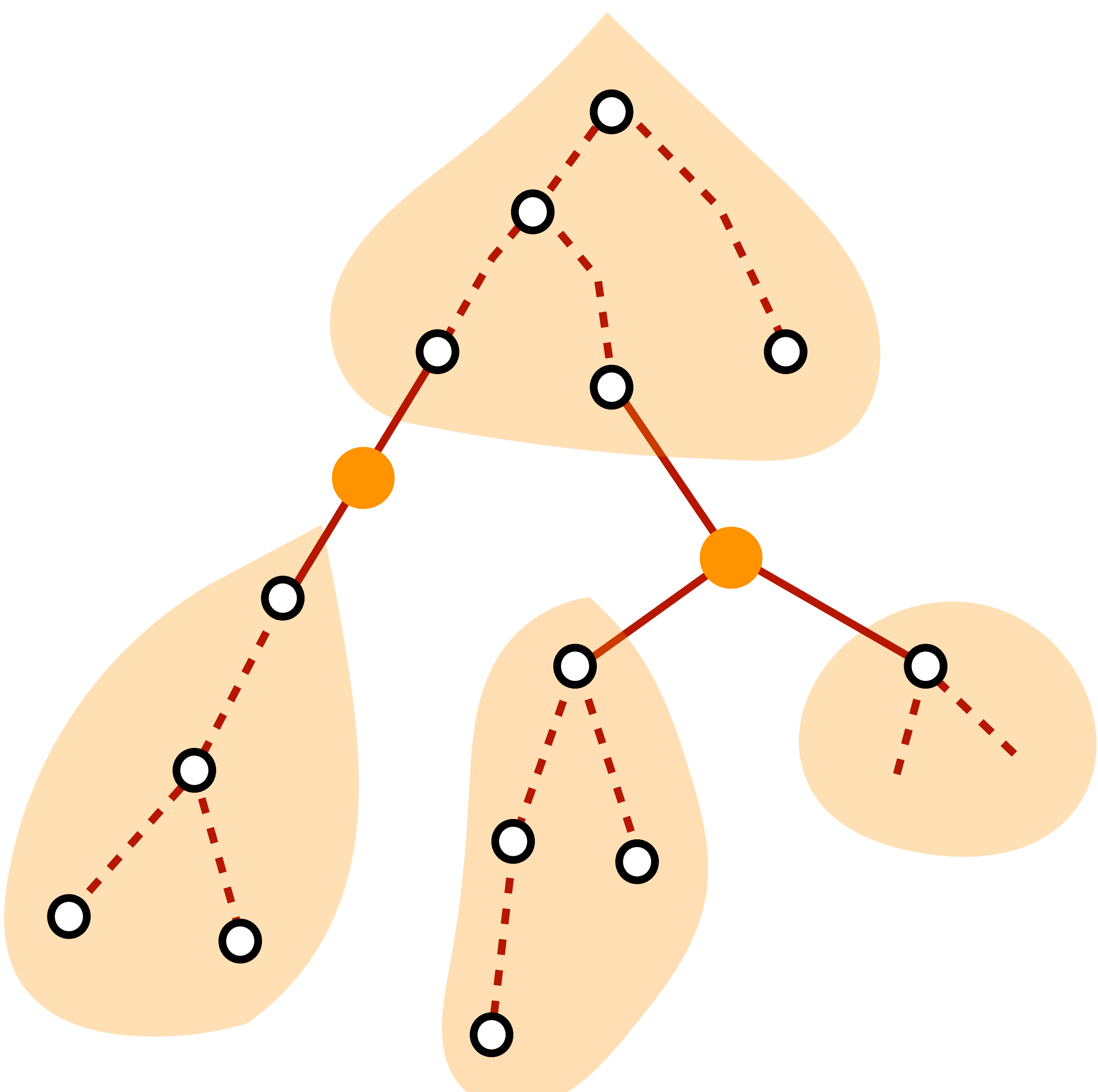

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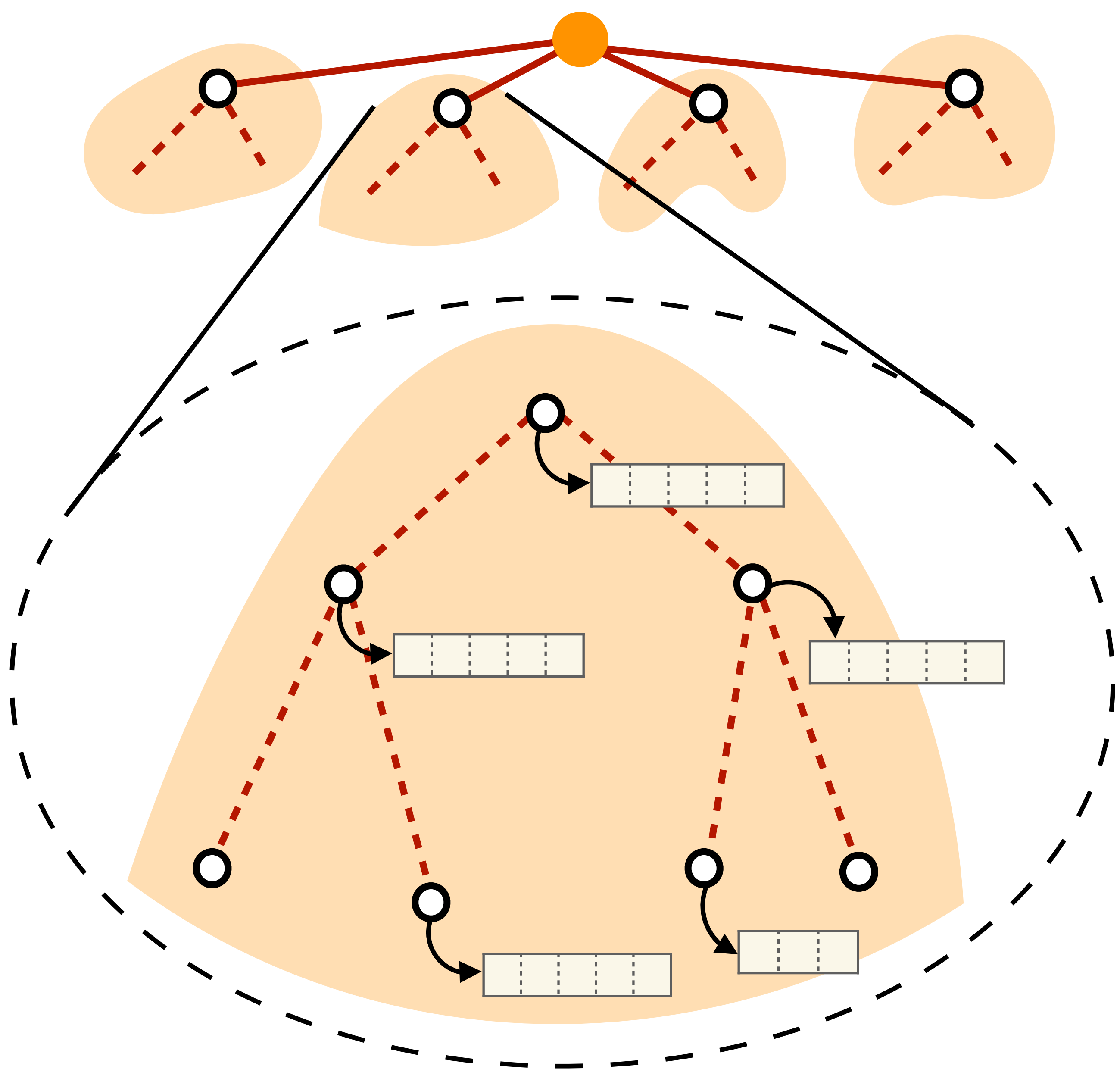

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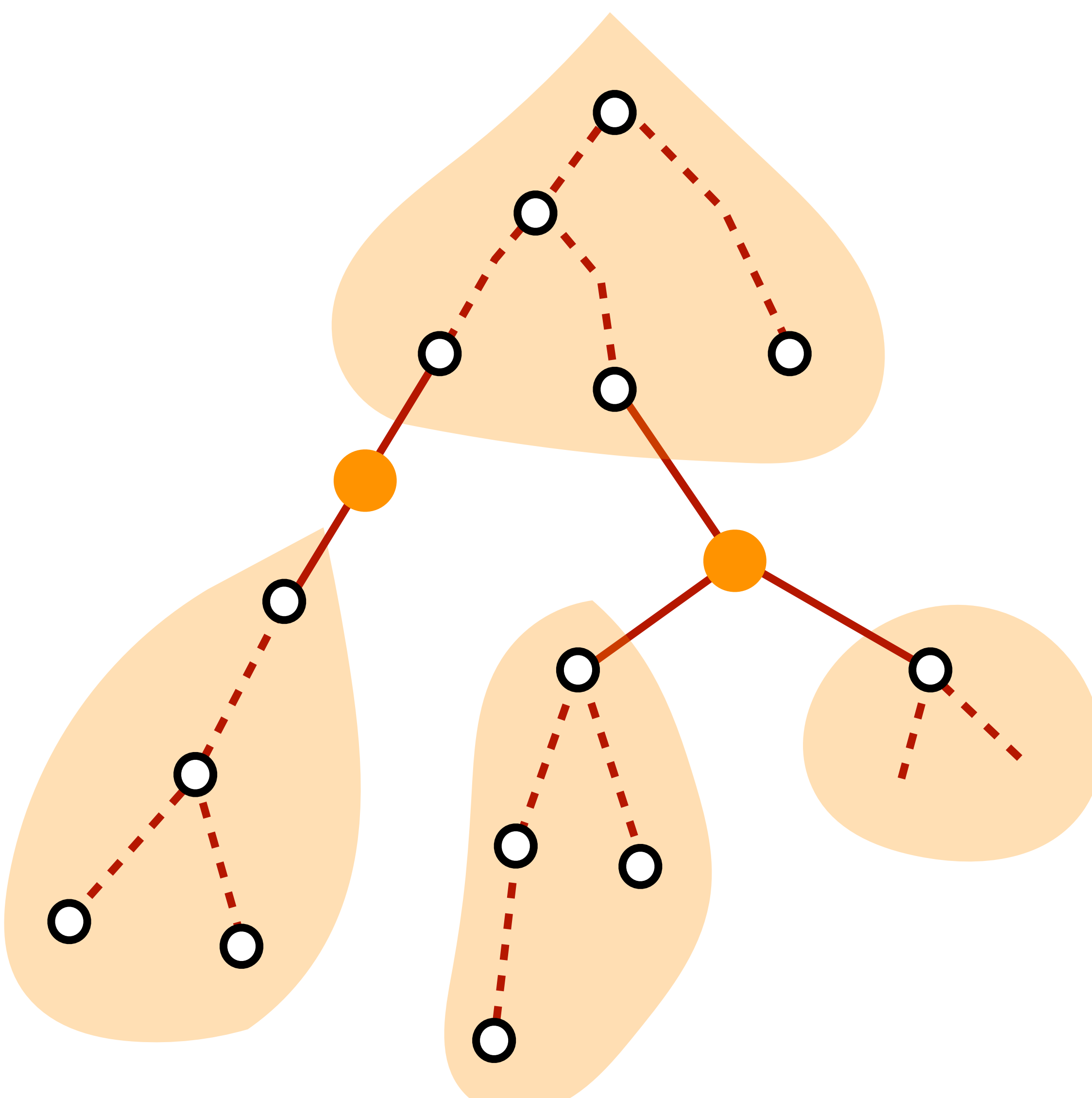
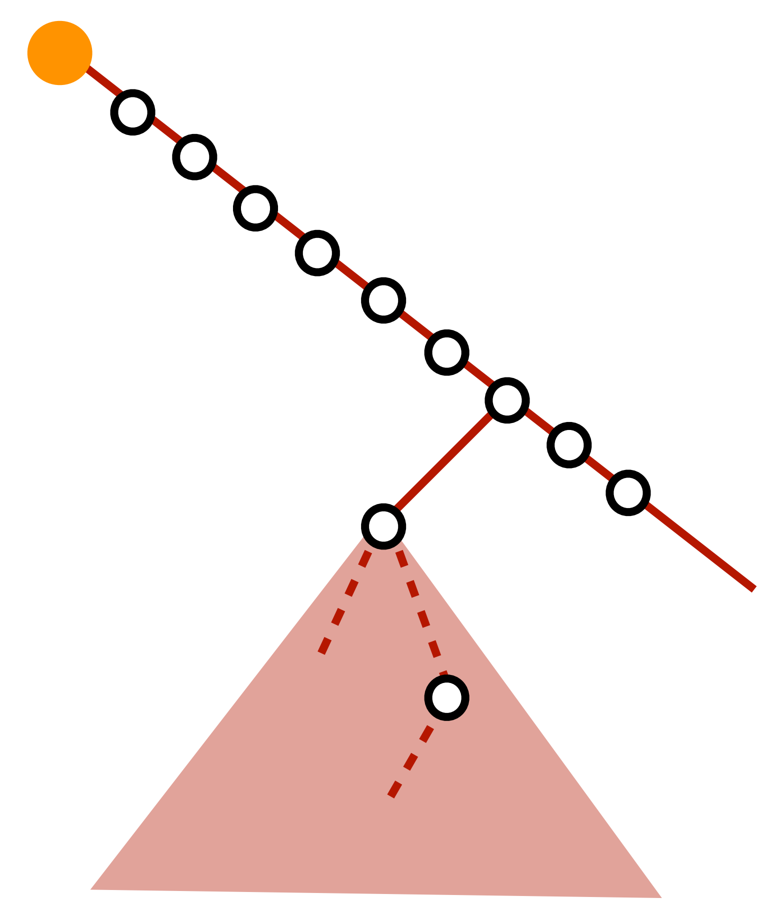


**Precompute for every choice of reverted tree path below the nearest special ancestor**



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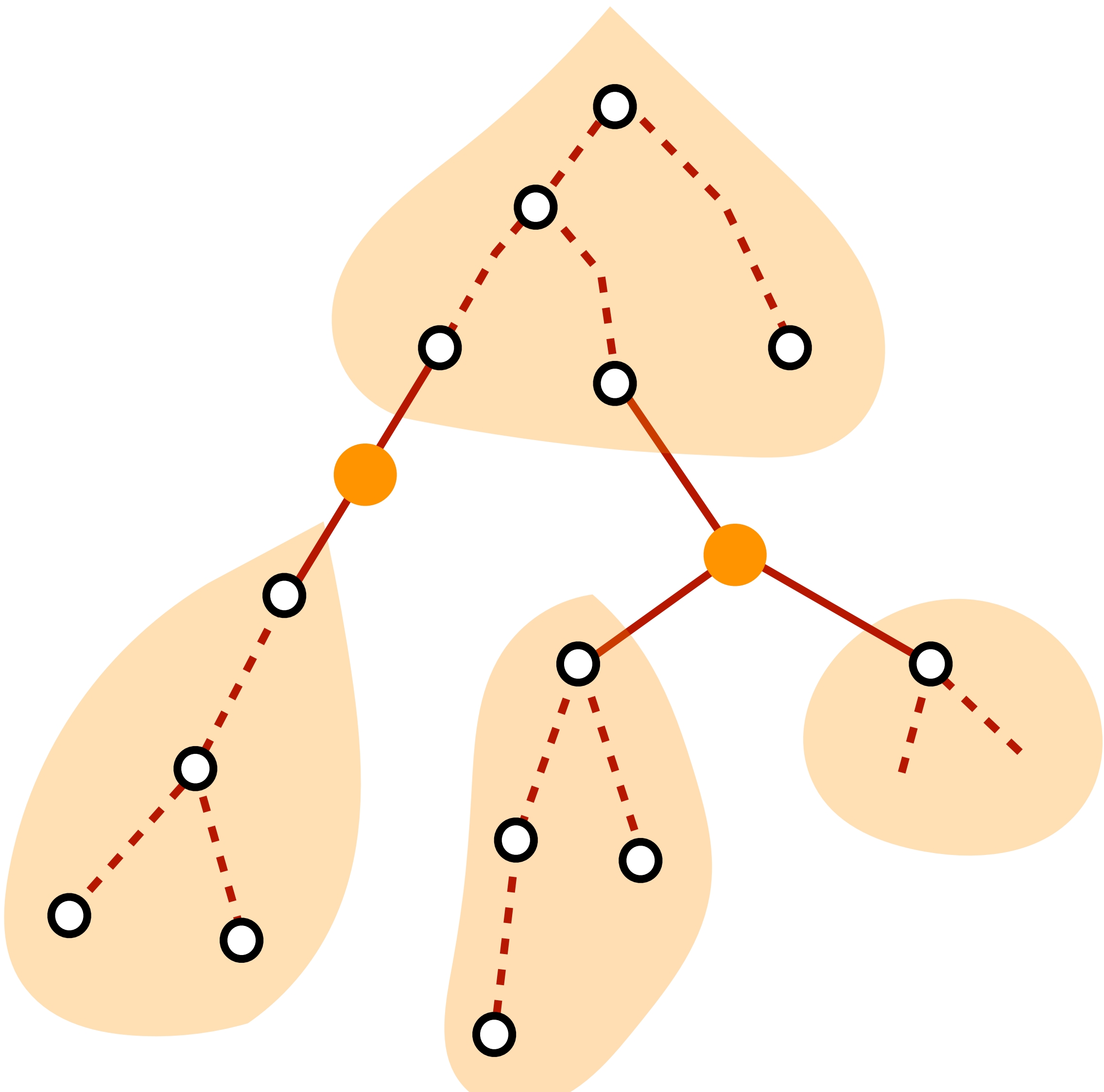

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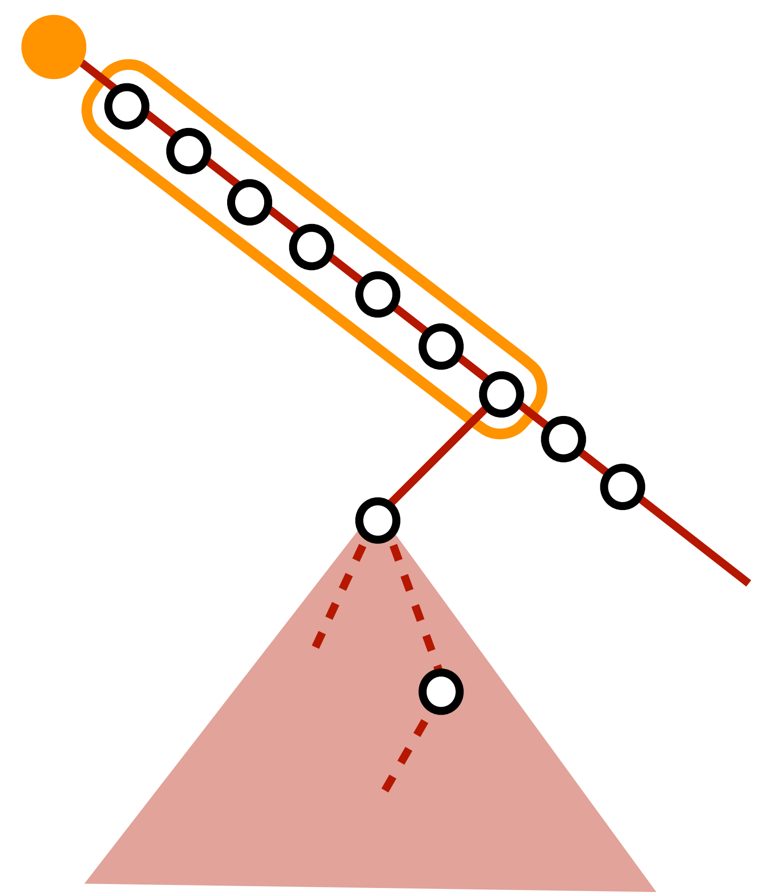

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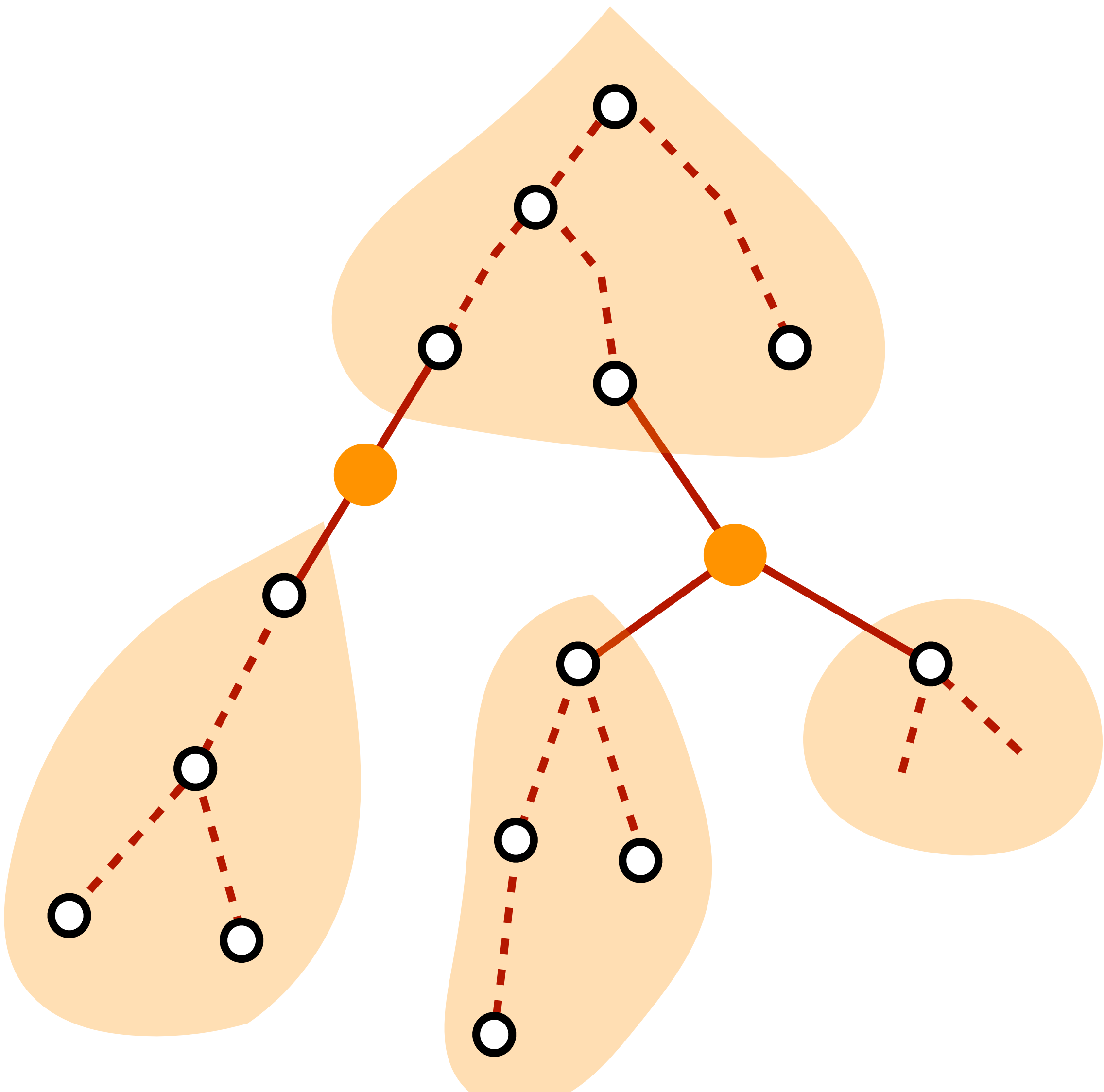

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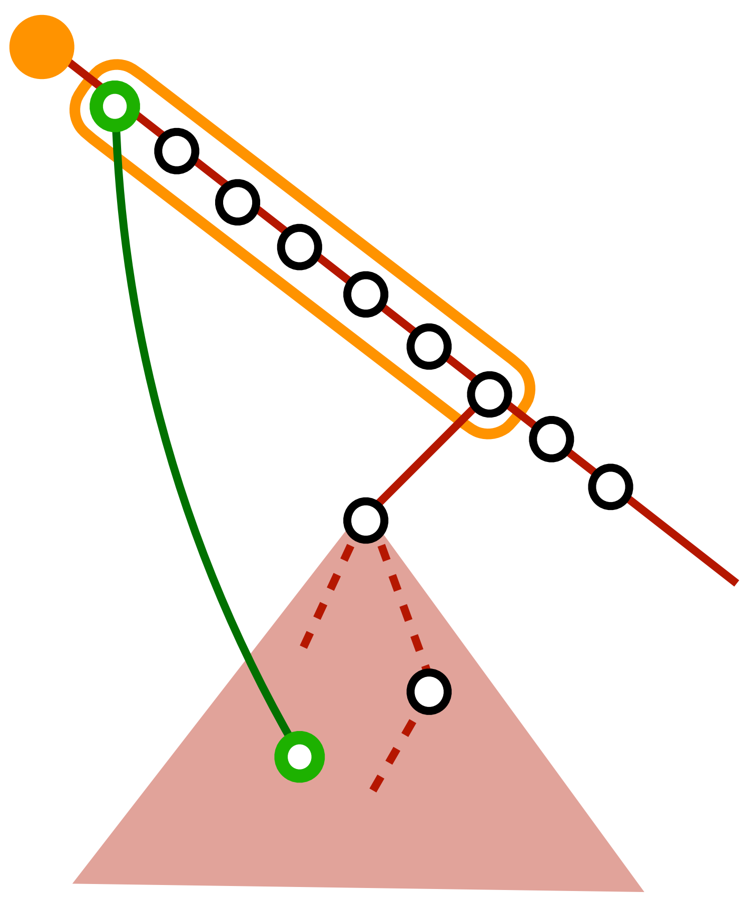

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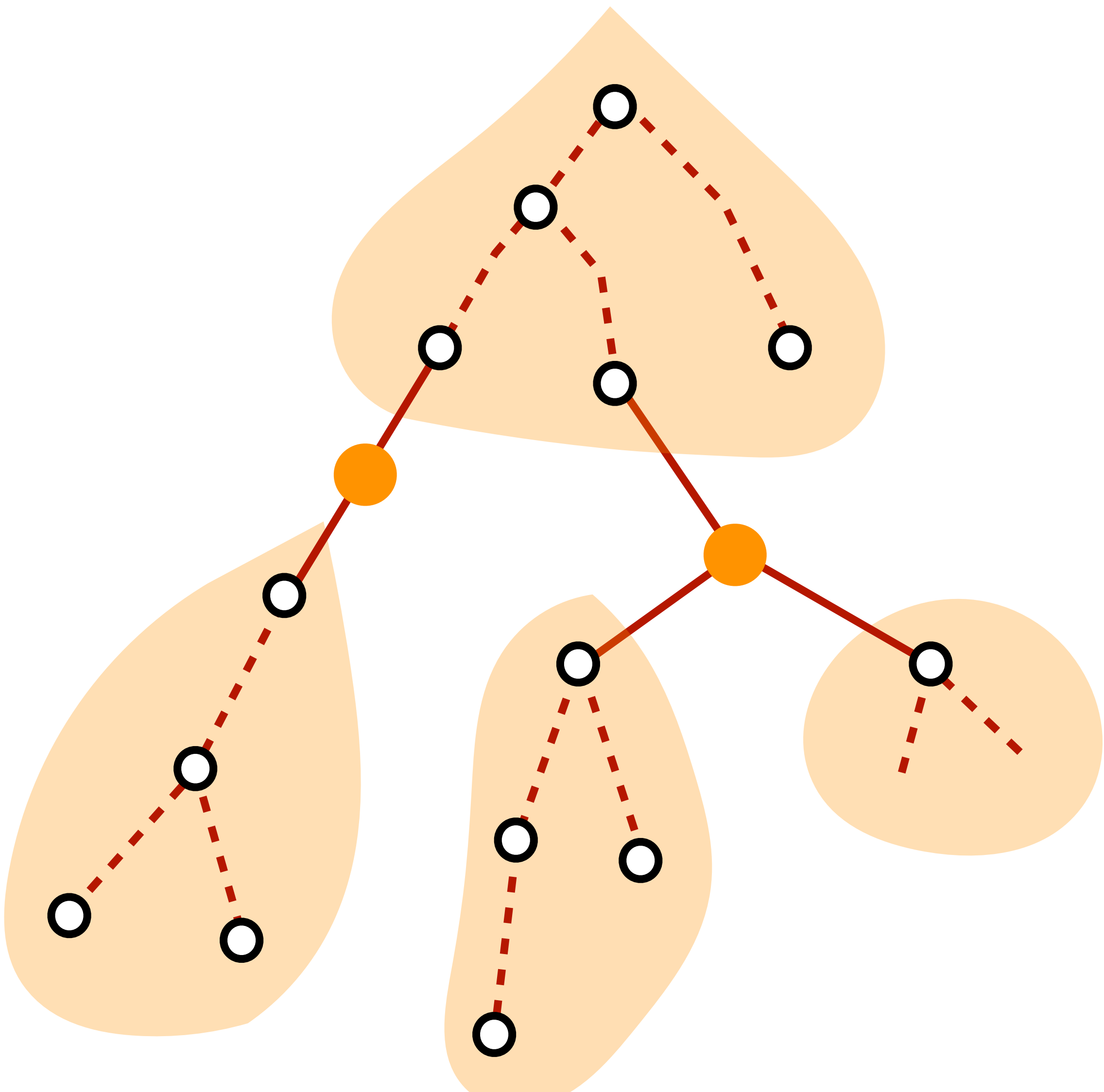

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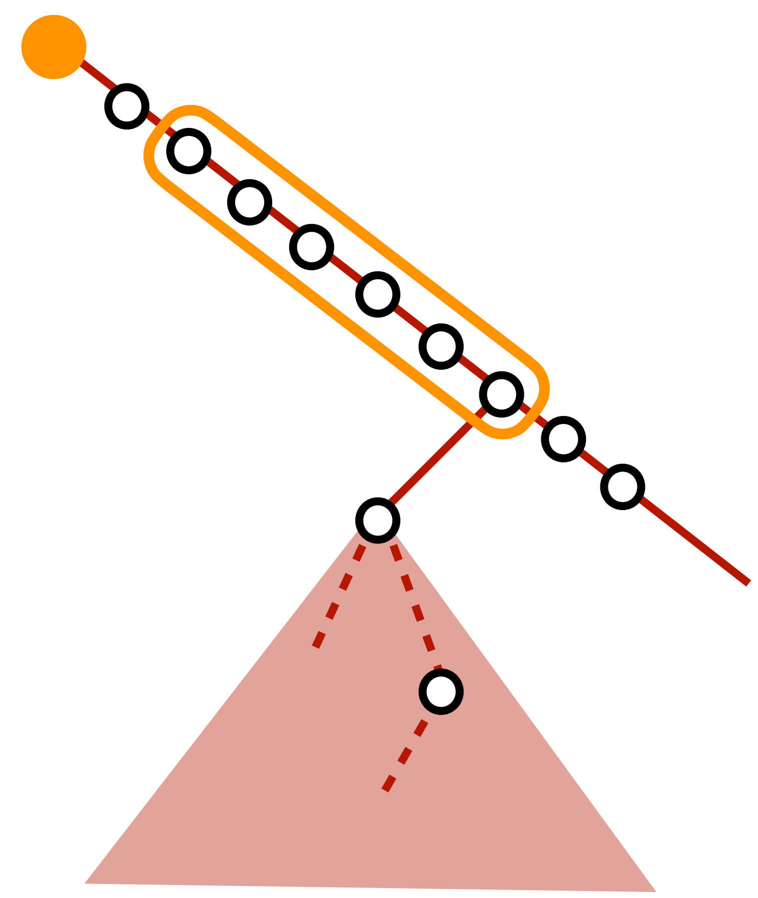

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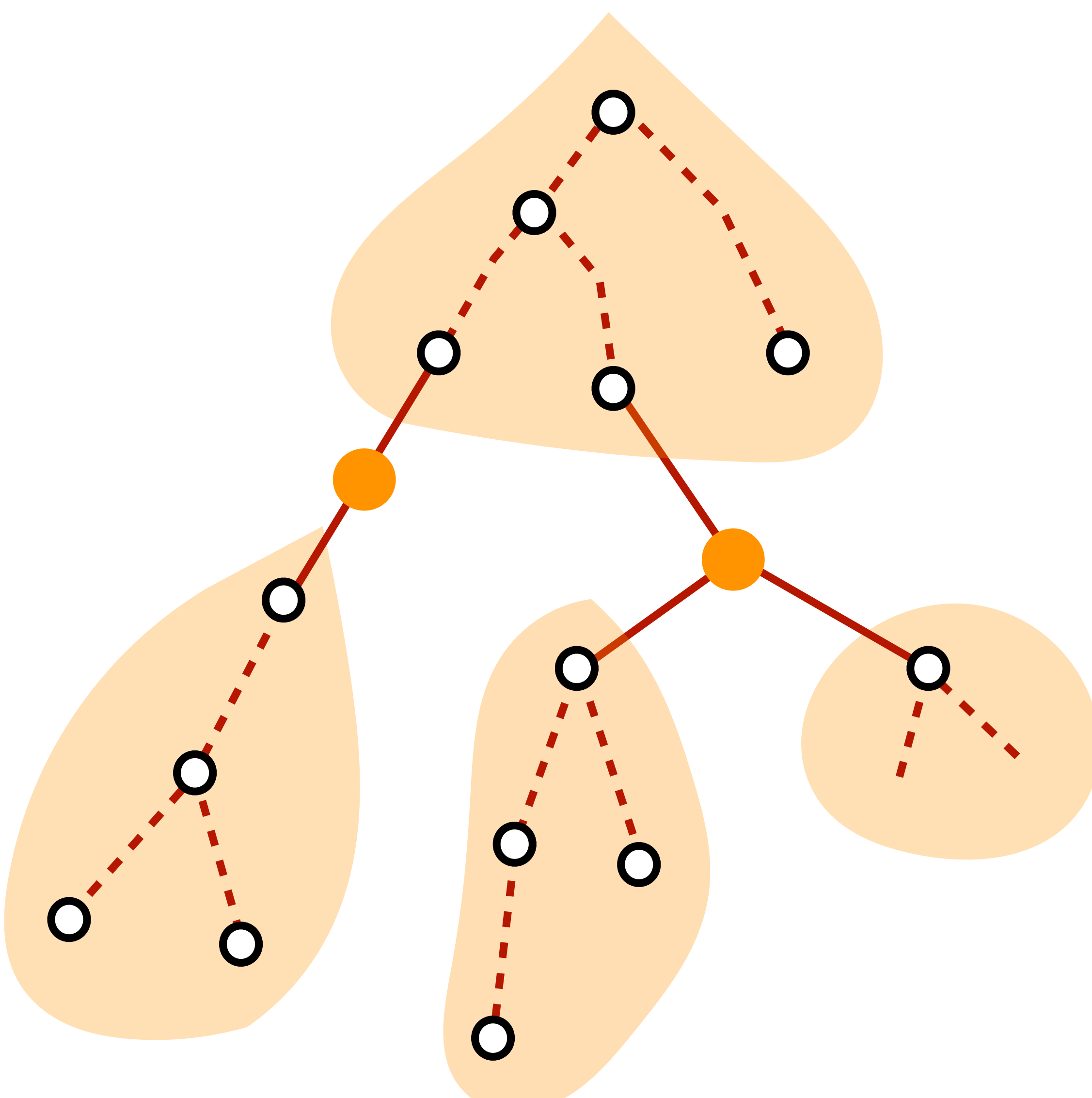

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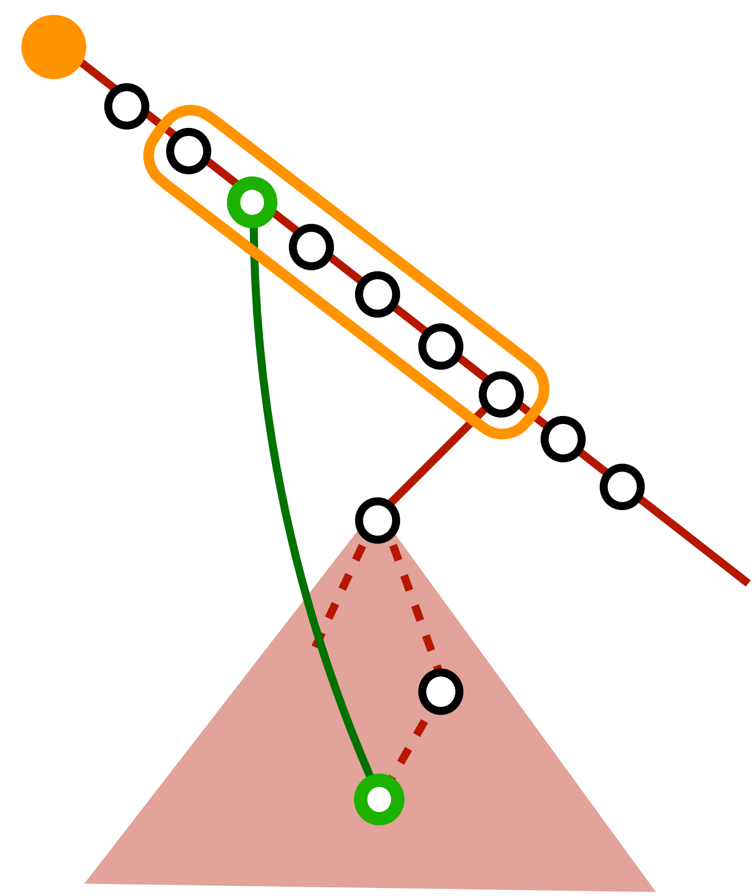

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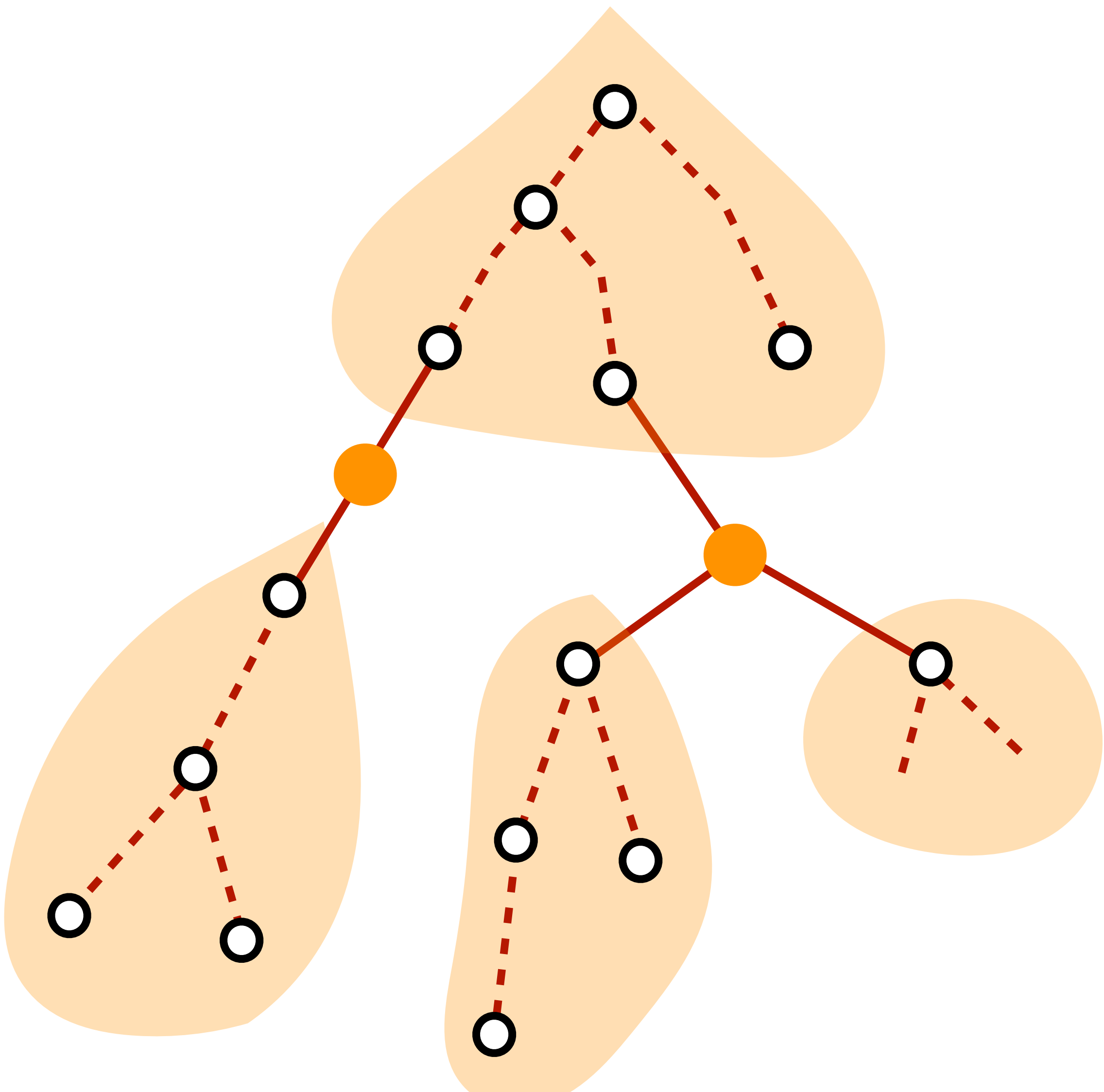

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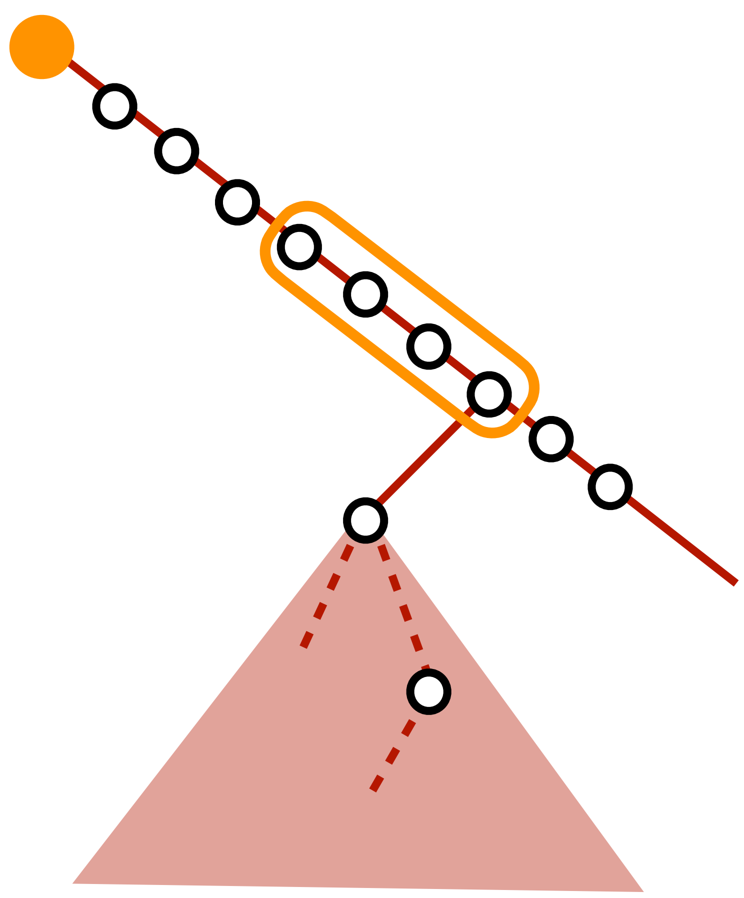

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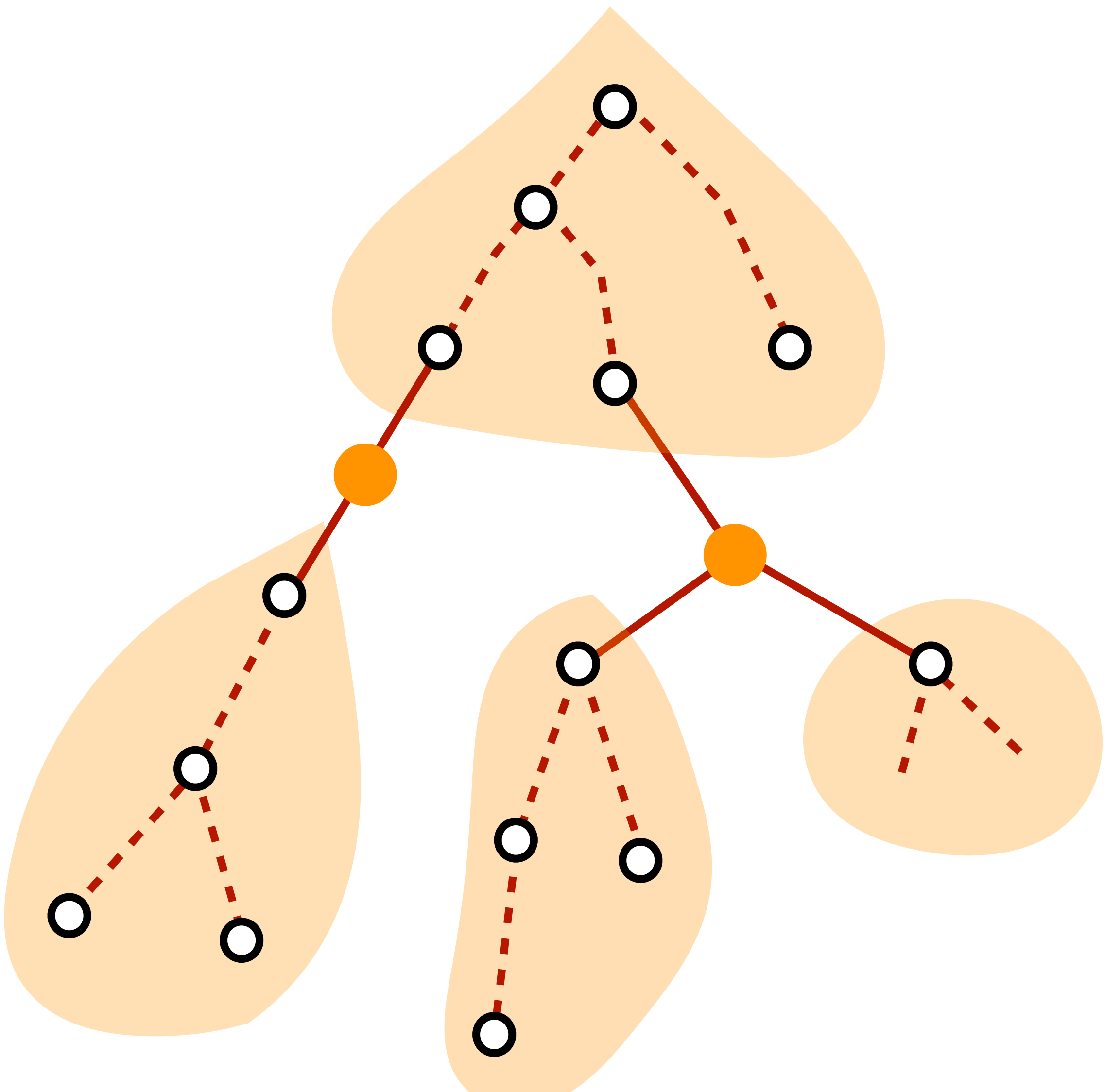

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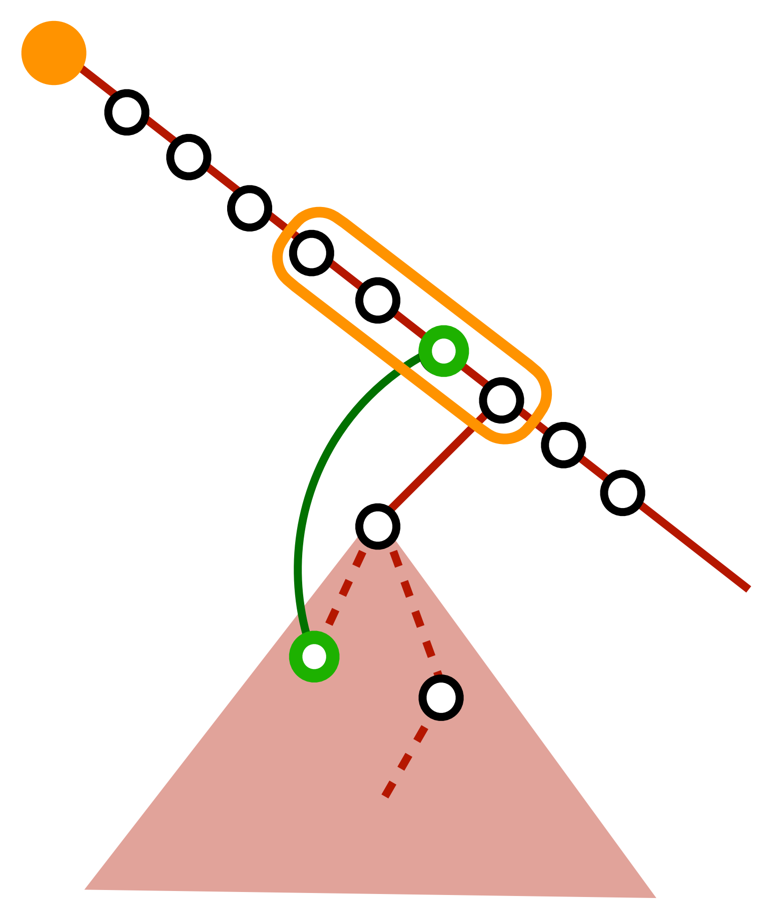

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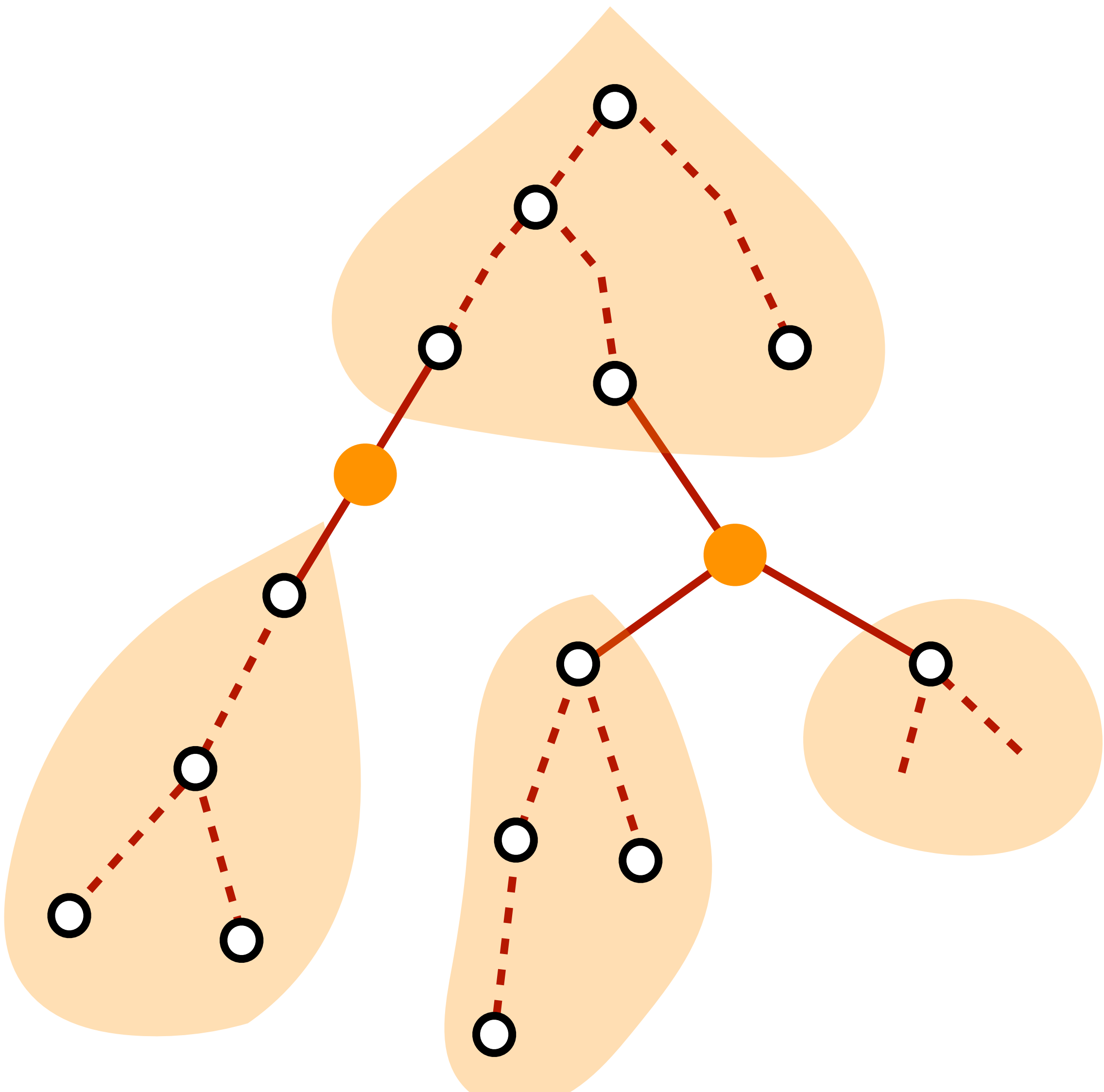

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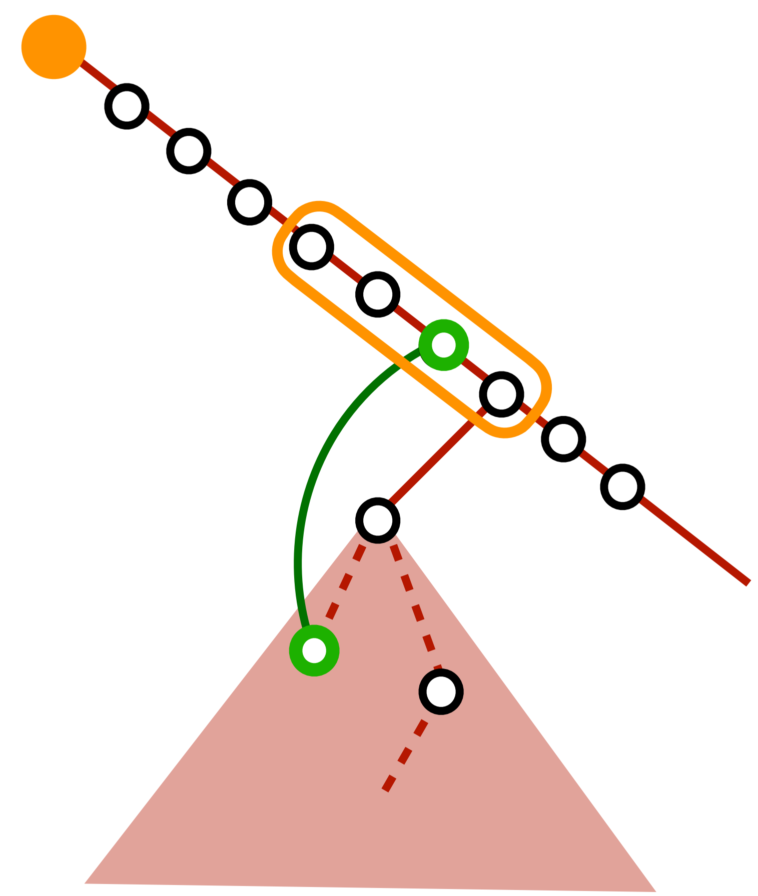


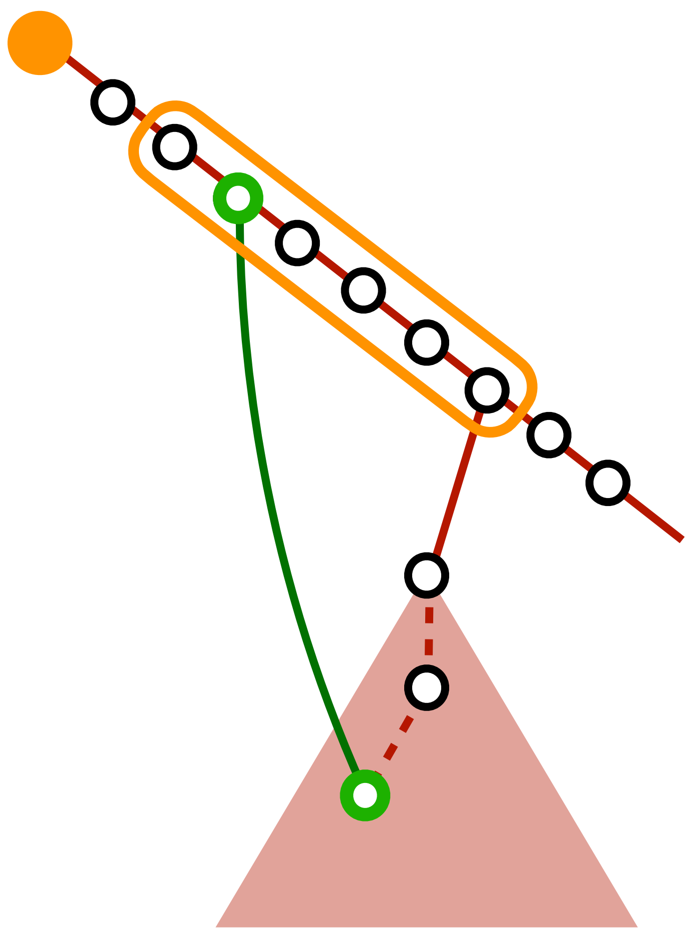
$O(n \log n)$ **entries in total**

# Highest ancestors in $O(n)$ total time

Consider three cases below

# Highest ancestors in *O(n)* total time

Consider three cases below



**The reverted tree path contains no special vertices**

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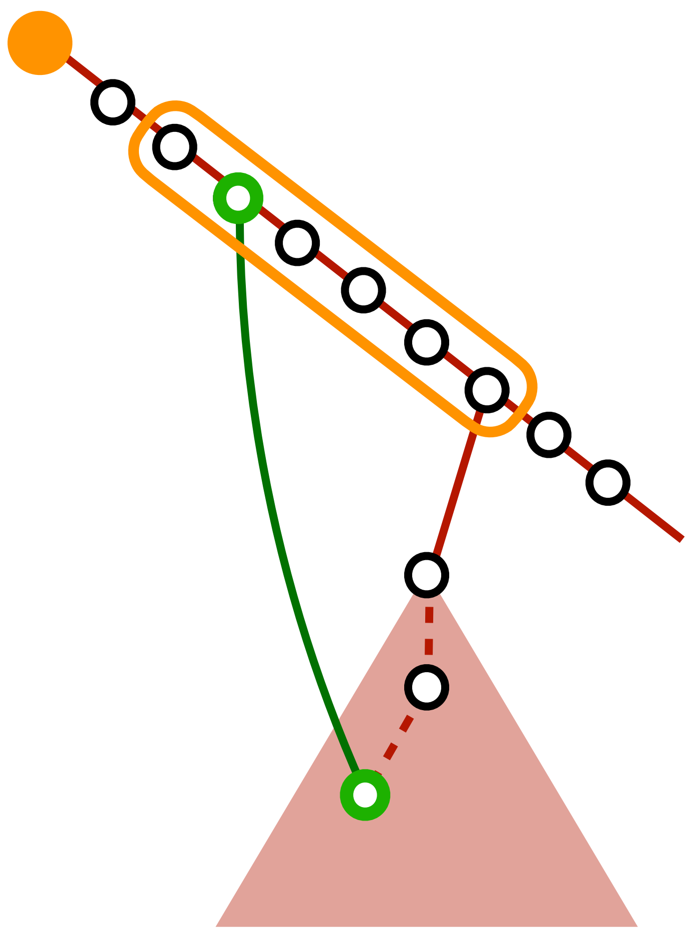

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**Use precomputed entries**
***O(1)* time**

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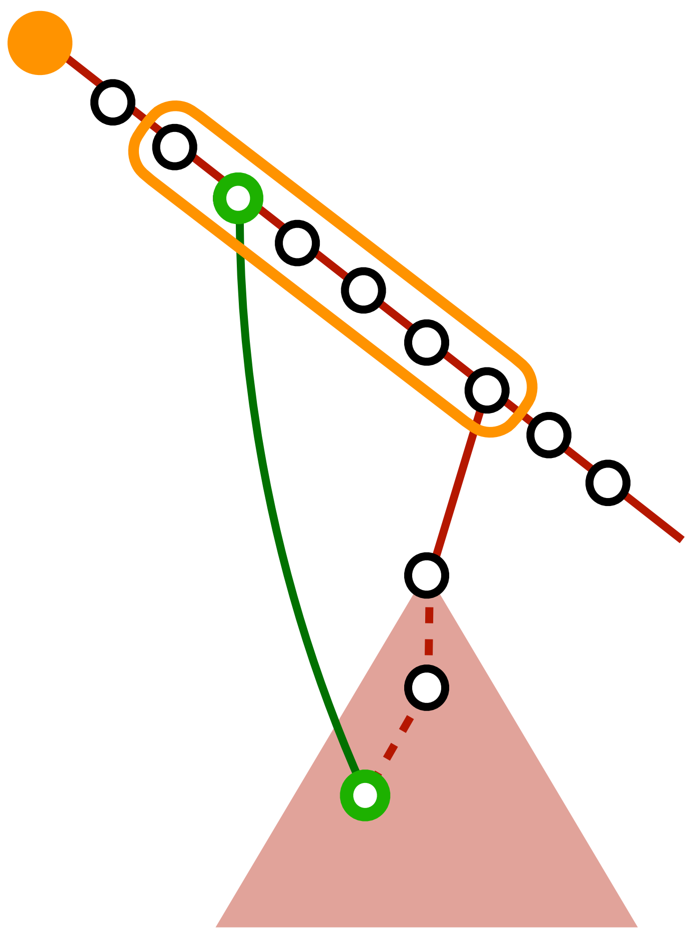

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**Use precomputed entries *O(1)* time**

**Total time = *O(n)***

# Highest ancestors in *O(n)* total time

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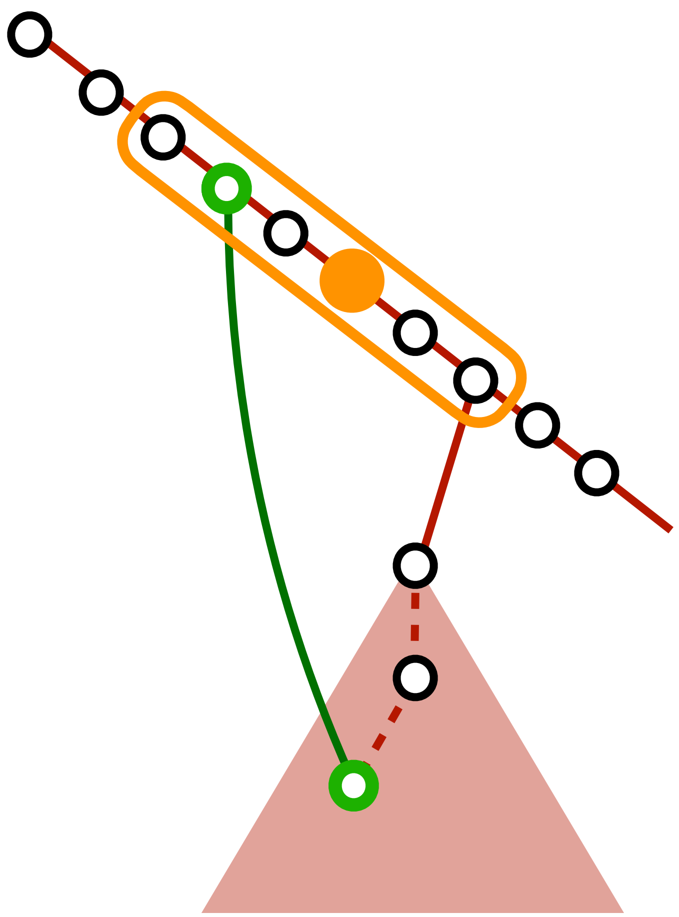

**The reverted tree path contains a special vertex, and subtree-size > *log n***

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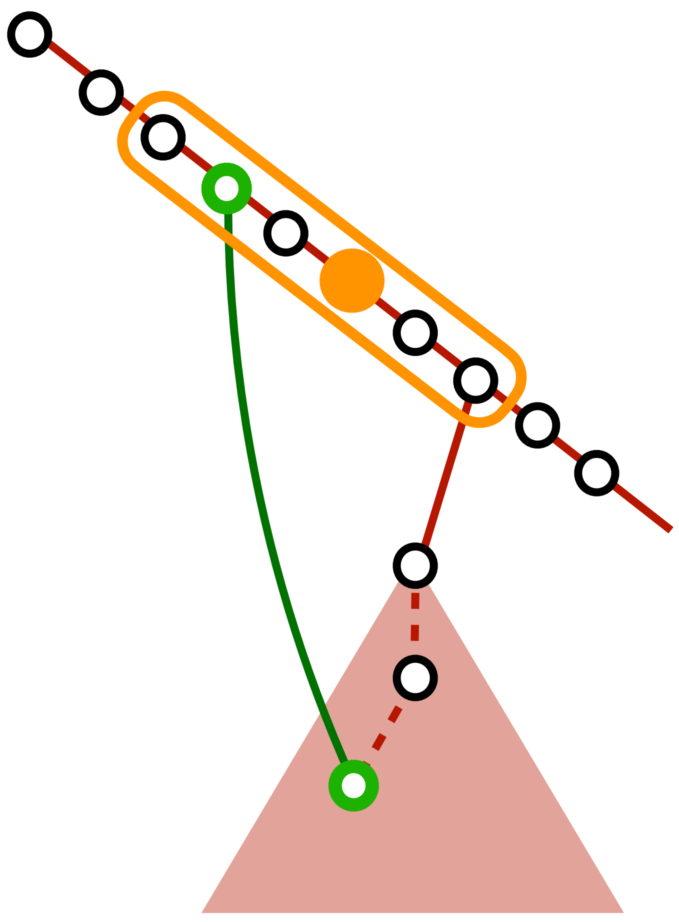

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**Apply 2D-range minimum *O(log n)* time**

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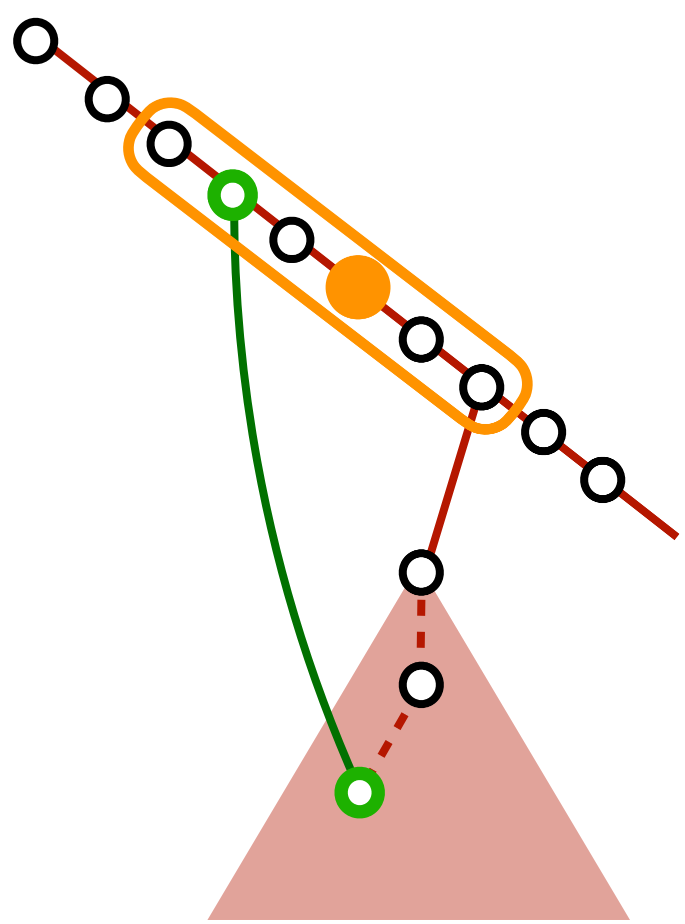

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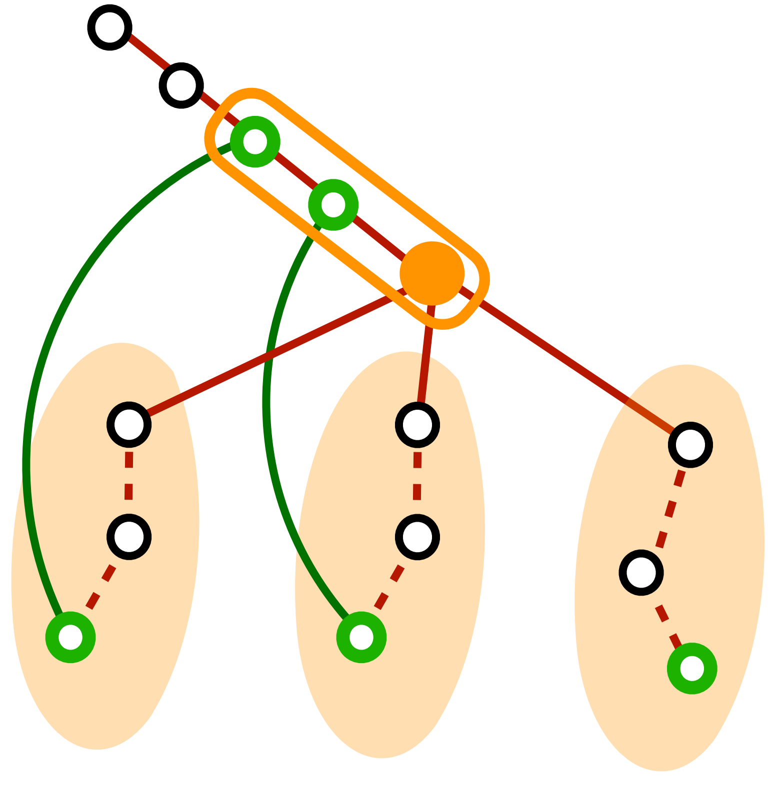

**A bunch of subtrees containing no special vertices**

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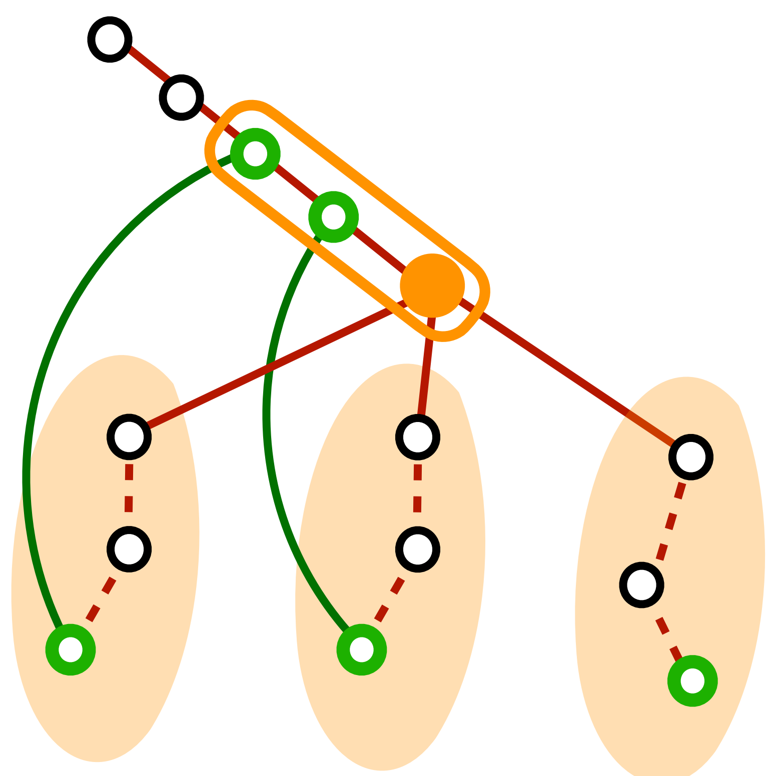

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**Apply fractional-cascade $O$(subtree-sizes + $\log n$) time**

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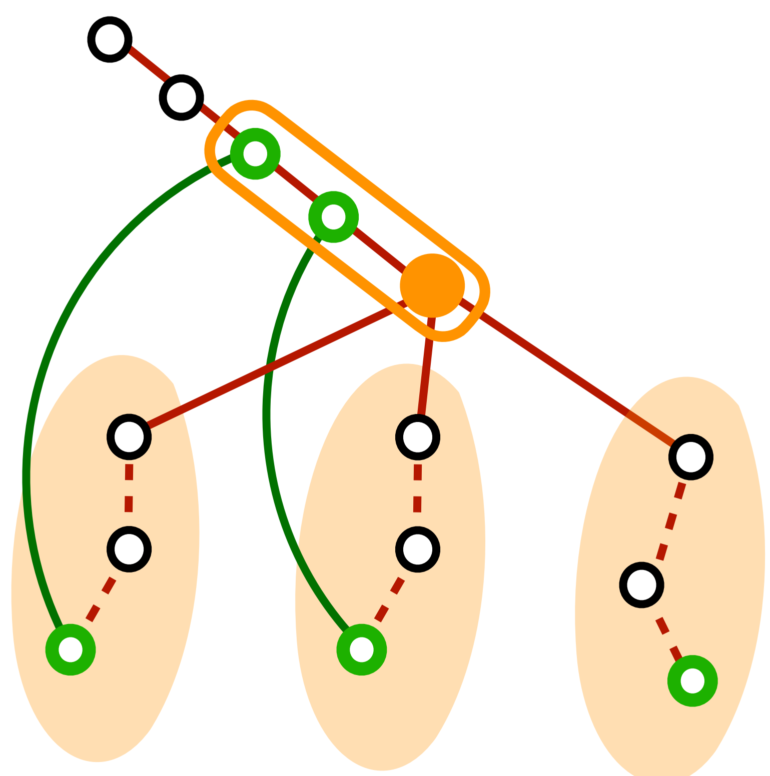

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**Apply fractional-cascade**
***O*(subtree-sizes + *log n*) time**

**Sum of subtree-sizes = *O(n)***

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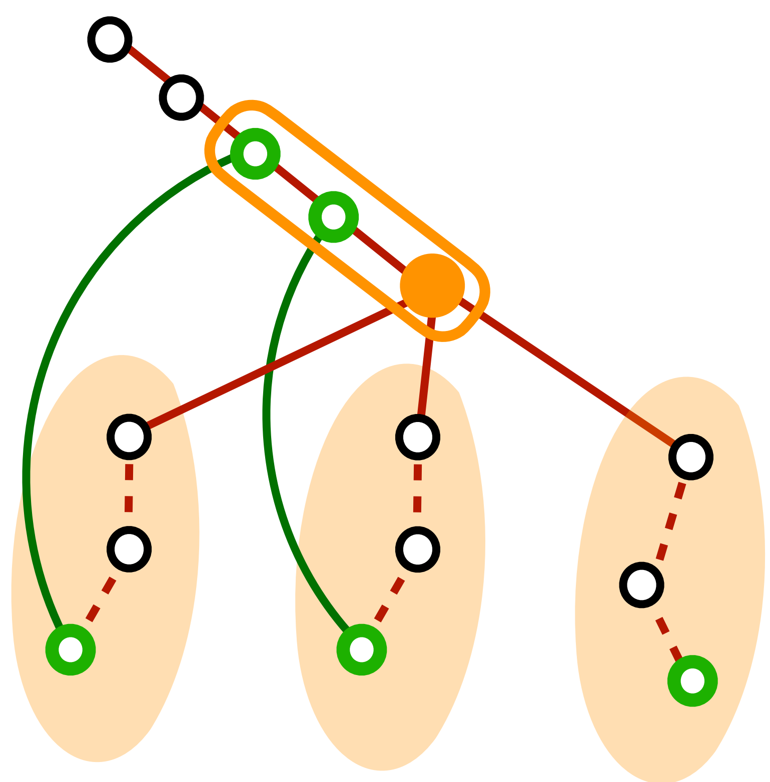

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***O*(subtree-sizes + *log n*) time**

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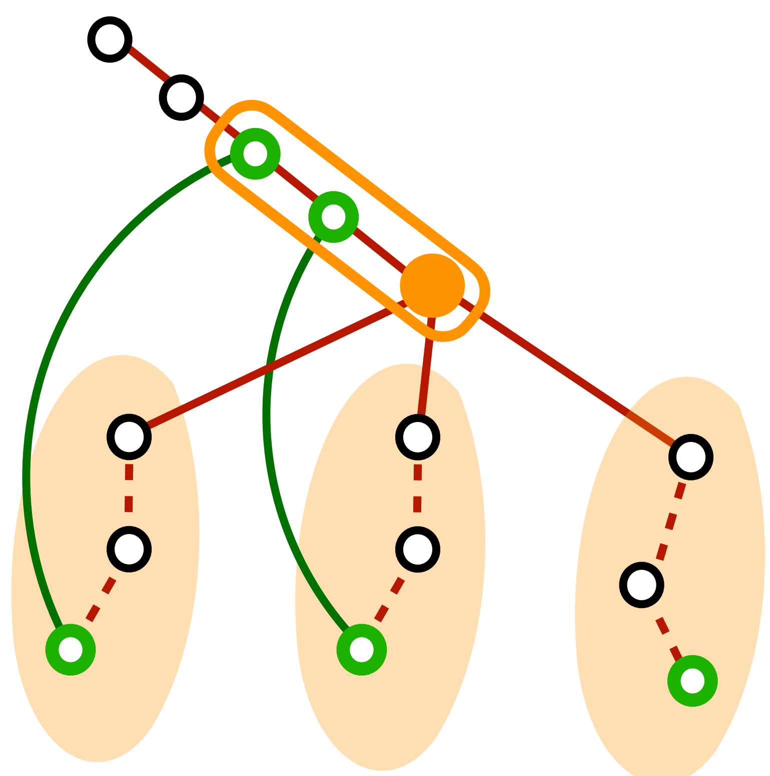

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**Summary: total time is *O(n)***

# Thanks