Faster $(\Delta + 1)$-Edge Coloring: Breaking the $mn^{1/2}$ Time Barrier

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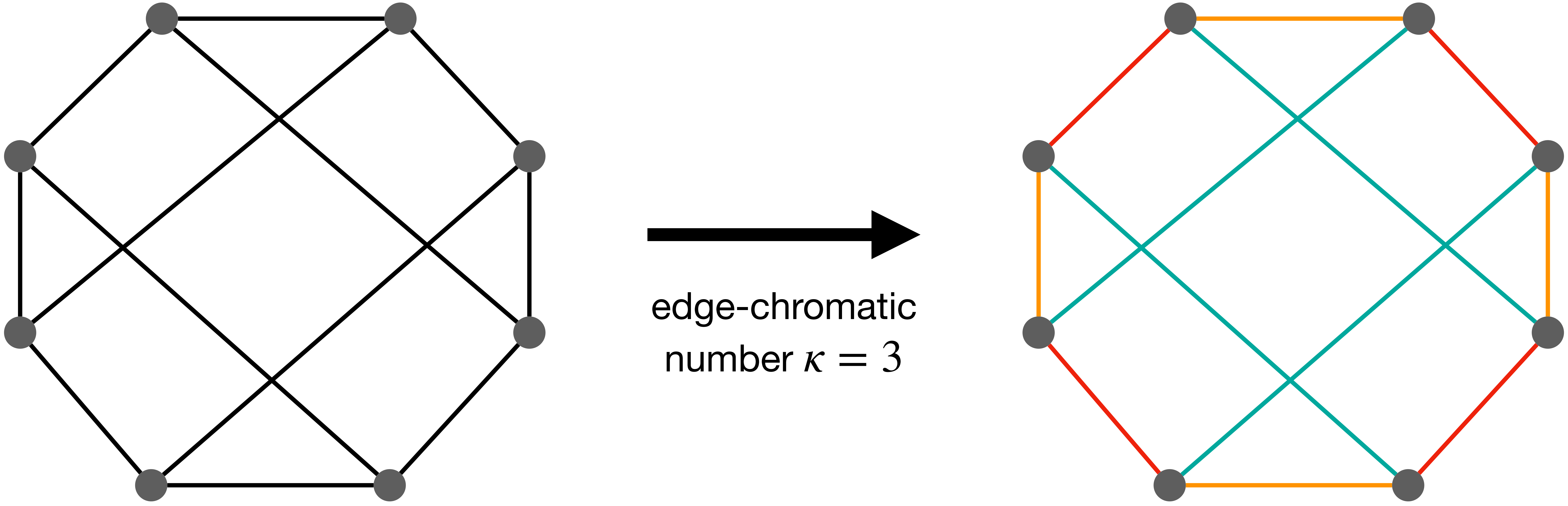
Edge Coloring

- Given an undirected simple graph $G = (V, E)$
- Compute a coloring $\chi : E \rightarrow \{1,2,\ldots,\kappa\}$ s.t. adjacent edges have different colors



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Why is edge coloring **relevant** to this workshop?

- Studies in the **distributed** setting:
  [EPS16] [BEPS16] [FGK17] [GHK18] [CHL+19] [Bern22] [Chr23] …
- Studies in the **dynamic** setting:
  [BM17] [BCHN18] [DHZ19] [Chr23] [BCPS24] [Chr24]

Edge Coloring

- Given an undirected simple graph $G = (V, E)$
- Compute a coloring $\chi : E \to \{1,2,\ldots,\kappa\}$ s.t. adjacent edges have different colors. What is the **smallest possible κ** ?
- **Folklore:** $\kappa \geq \Delta(G)$, here $\Delta = \max\{\deg(v)\}$
- **Upper bounds:**
  $\kappa \leq \Delta$ in bipartite graphs, $\kappa \leq \Delta + 1$ in general graphs [Vizing, 1964]
- **Hardness:**
  NP-hard to decide $\kappa = \Delta$ or $\Delta + 1$ [Holyer, 1981]

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Δ-edge coloring in bipartite graphs is in near-linear time e.g. in [COS, 2001]

Question: The exact runtime of $(\Delta + 1)$-edge coloring?

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A sparse history of runtime complexities: $n = |V|, m = |E|$

- $O(mn)$ [Vizing, Diskret. Analiz 1964]
- $\tilde{O}(\min\{mn^{1/2}, m\Delta\})$ [Arjomandi, INFOR 1982] [Gabow et al, 1985]
- $O(mn^{1/2})$ log-shaving [Sinnamon, 2019]

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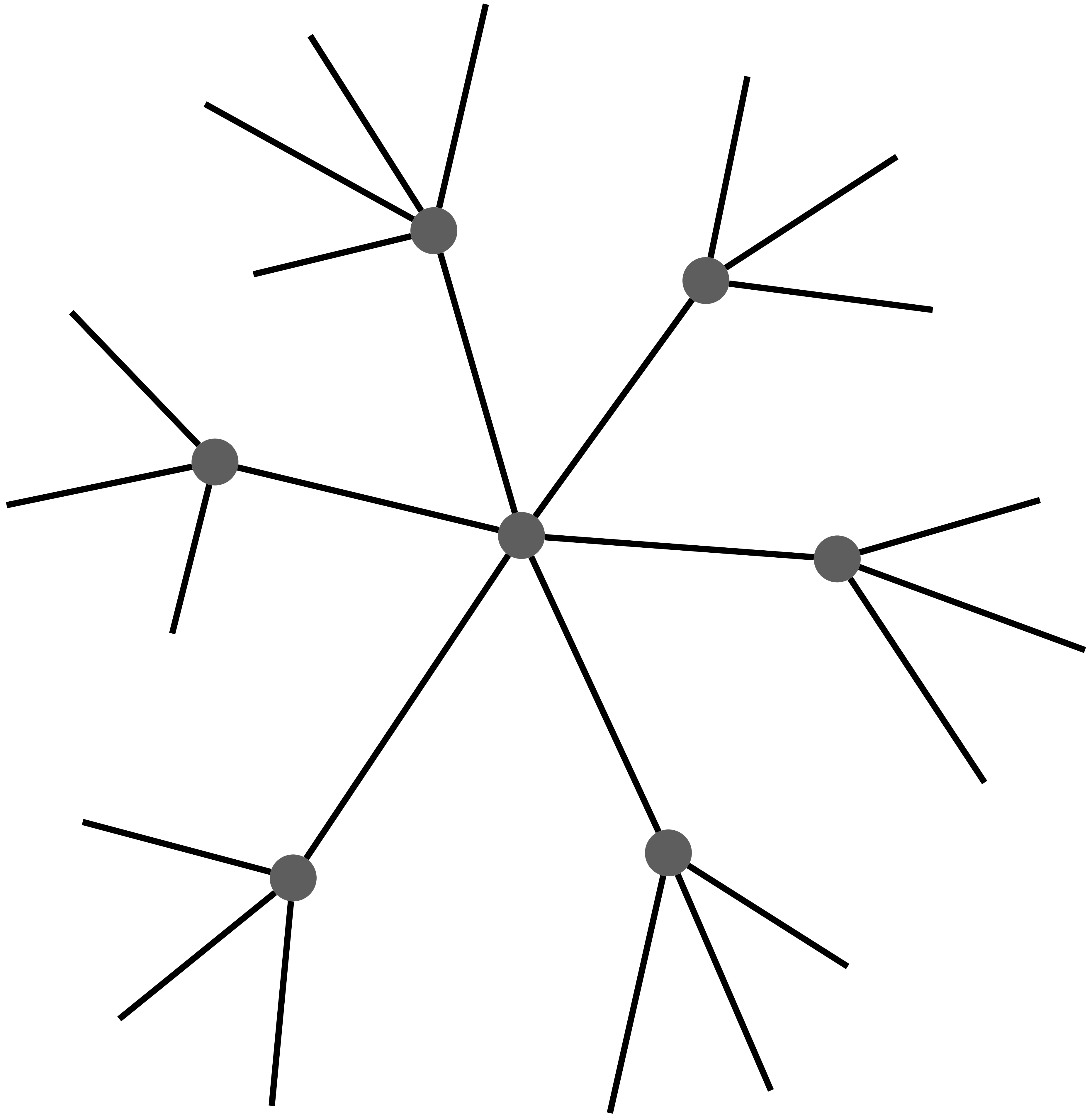
Reduce to $O(m/\Delta) = O(n)$ edges

1. Divide $G = G_1 \cup G_2$ such that $\Delta(G_1), \Delta(G_2) \le \Delta/2$

2. Color G_1, G_2 recursively using $2 \cdot (\Delta/2 + 1) = \Delta + 2$ colors

3. Delete the least-popular color, $O(m/\Delta) = O(n)$ uncolored edges



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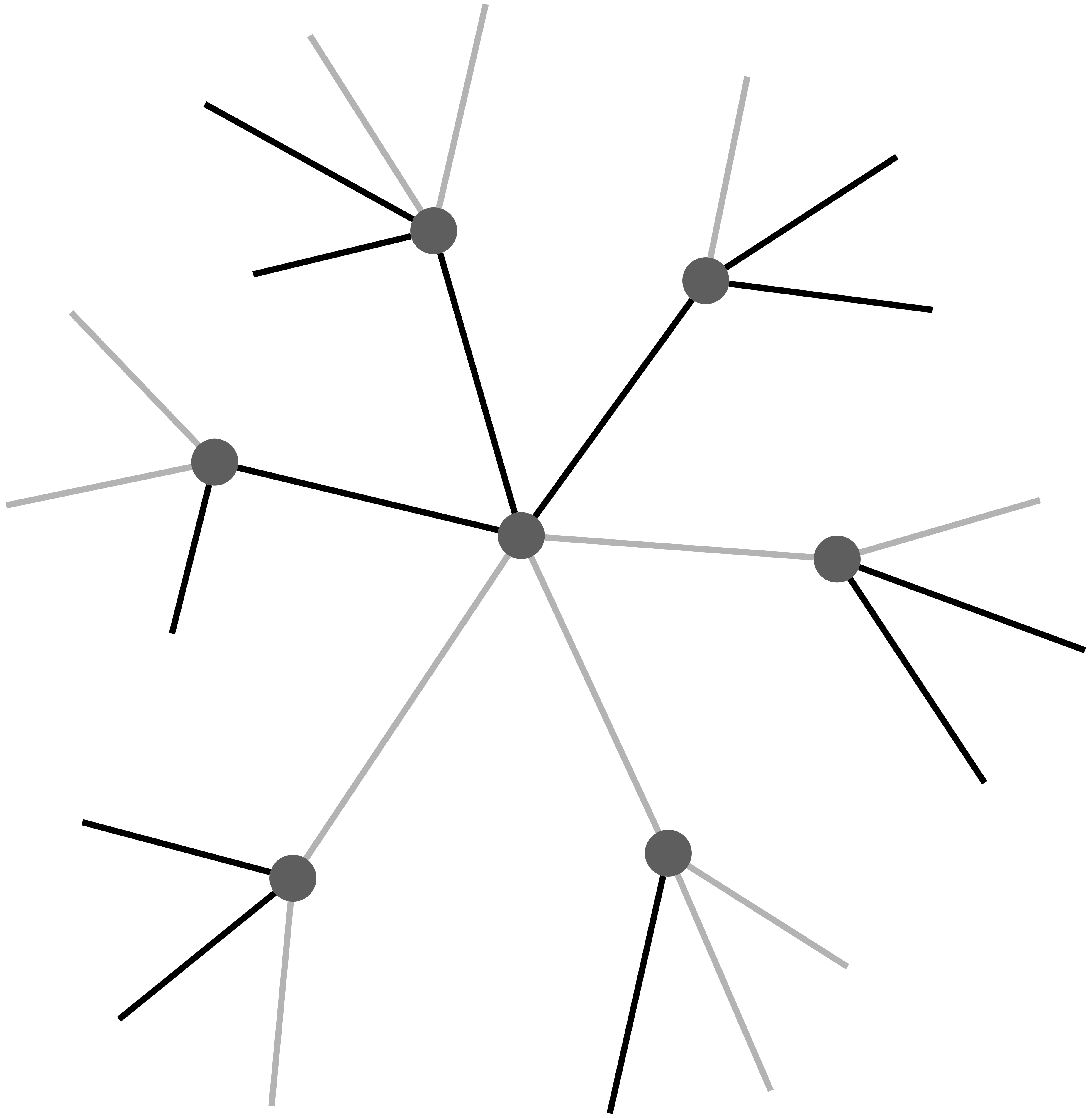

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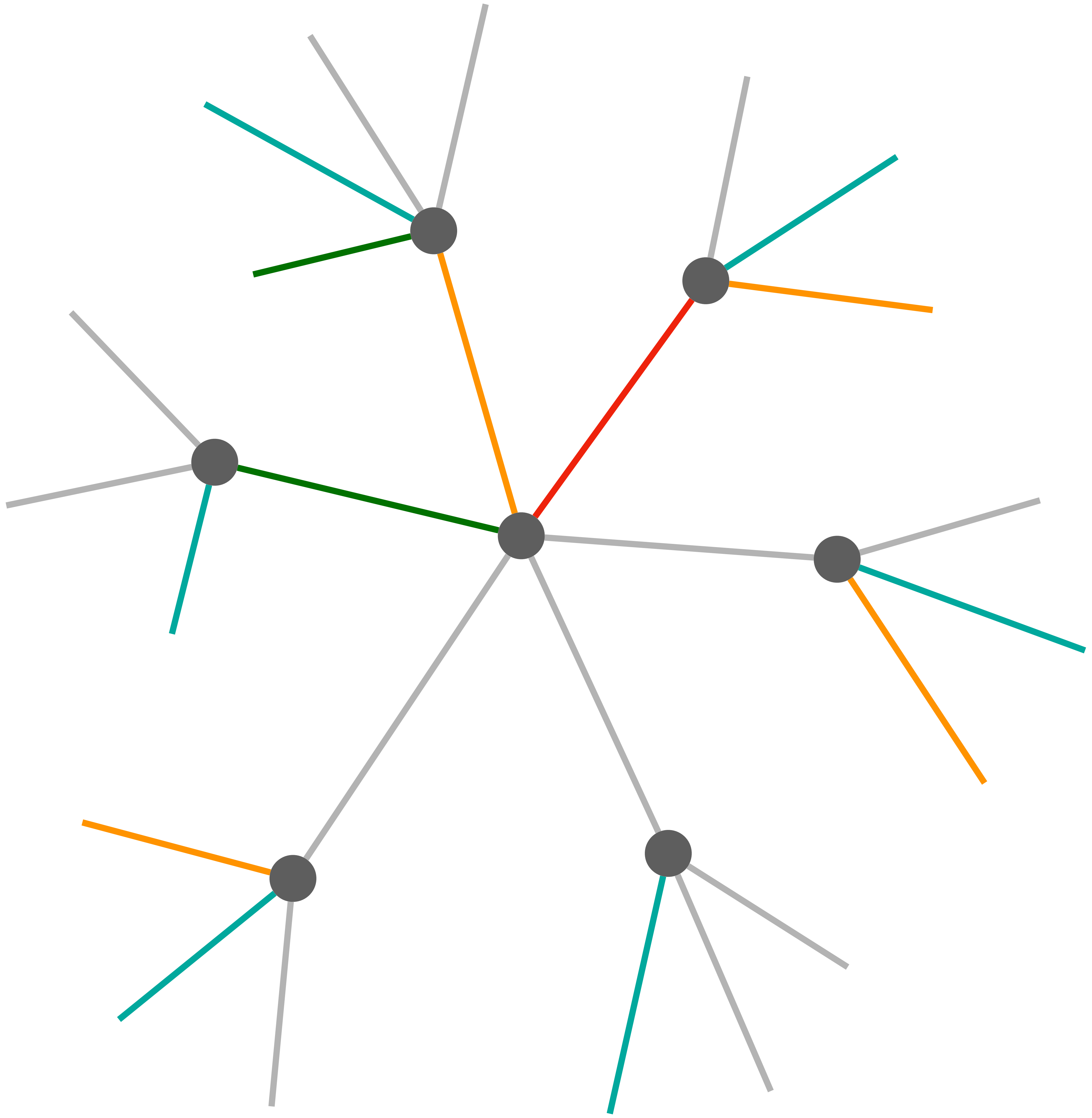

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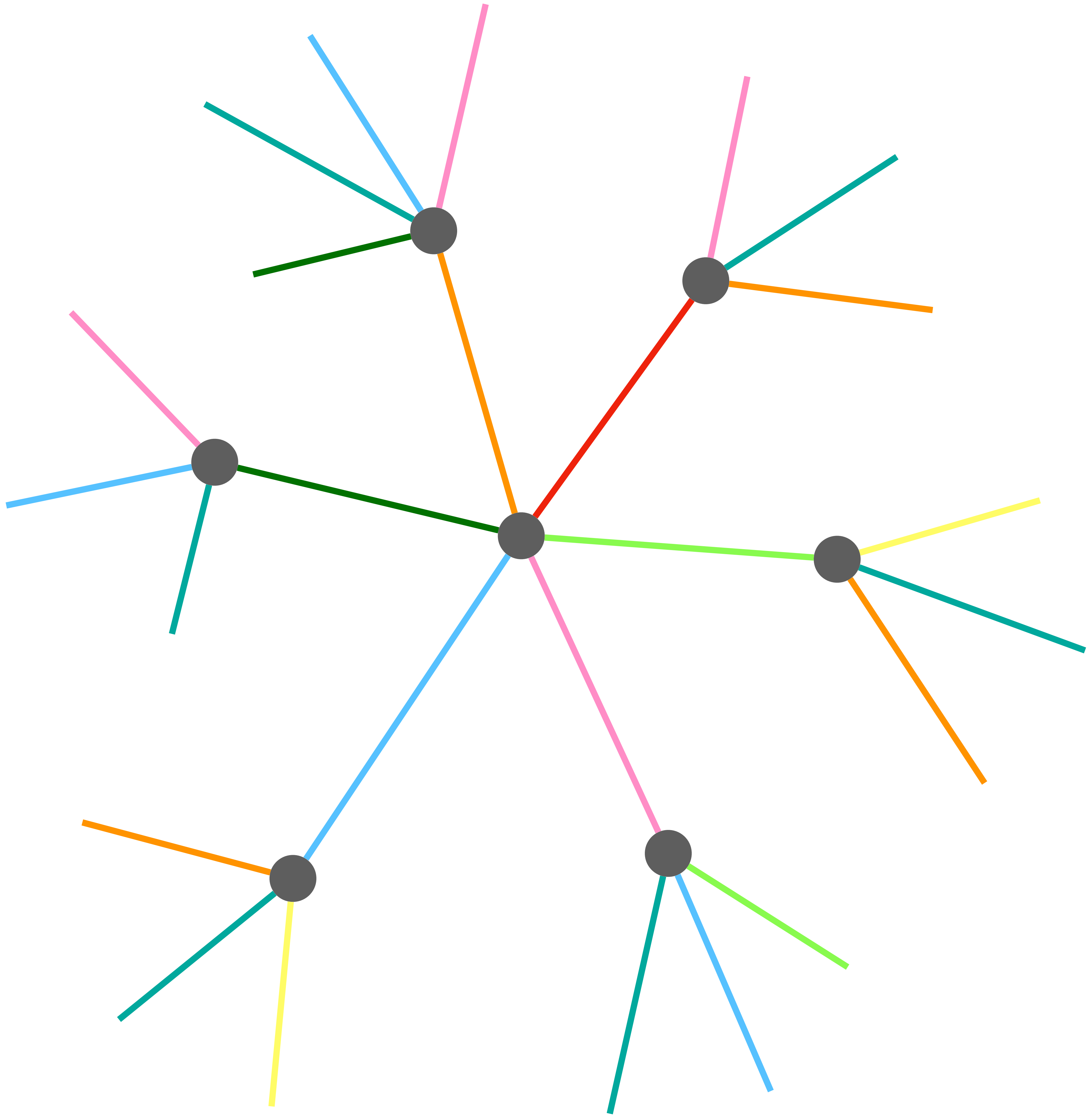

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G_1 uses {}

G_2 uses {}

remove color

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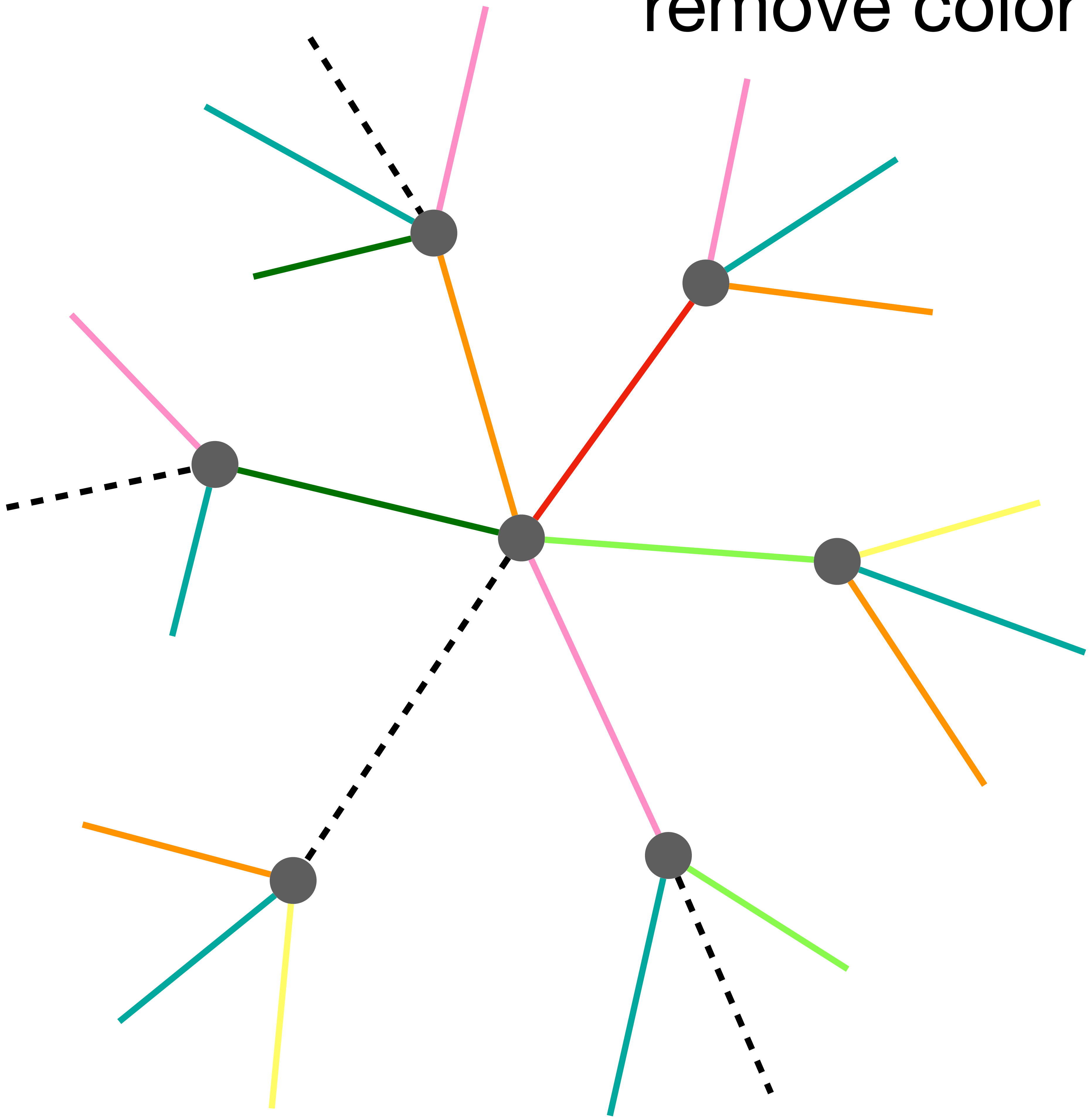

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Subtask: How to extend to a single uncolored edge?

Answer: Flip alternating paths (bipartite), Vizing fans/chains (general)

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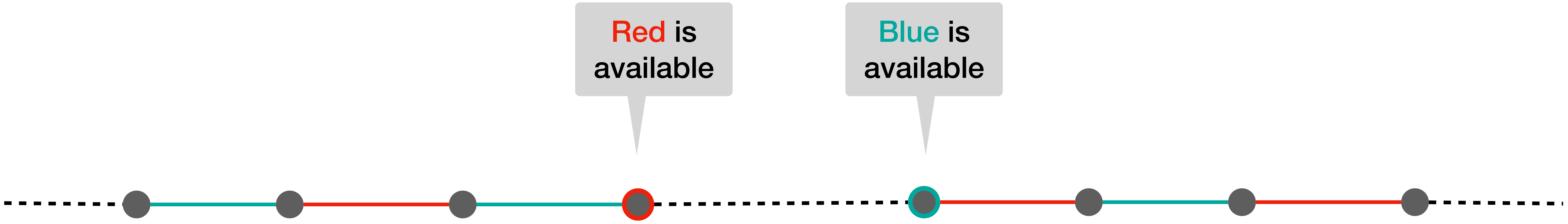

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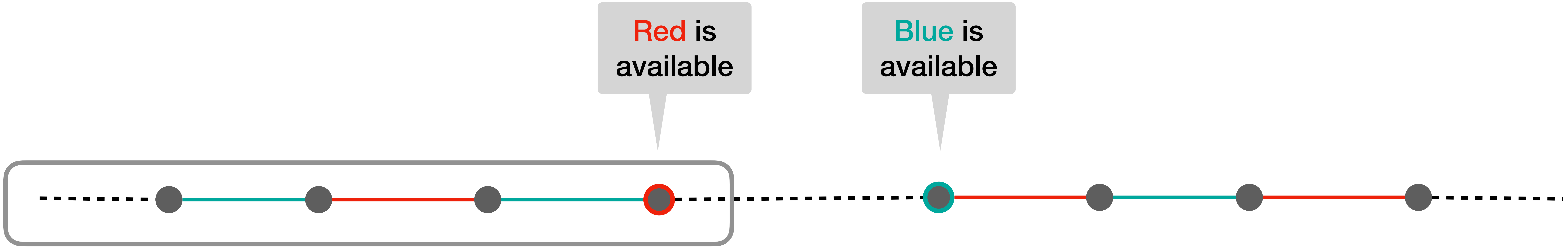

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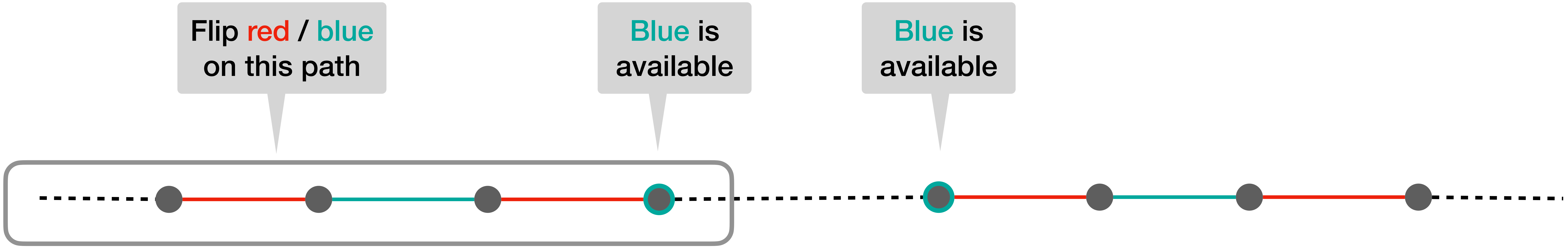

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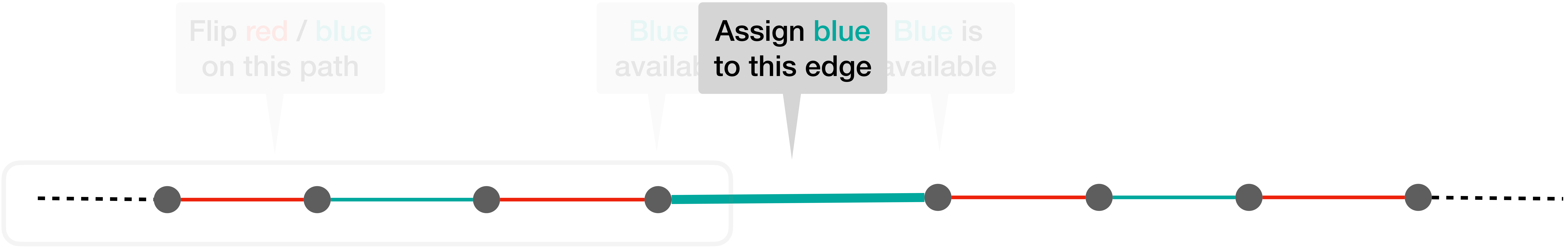
Exercise: Where did we use bipartite-ness?



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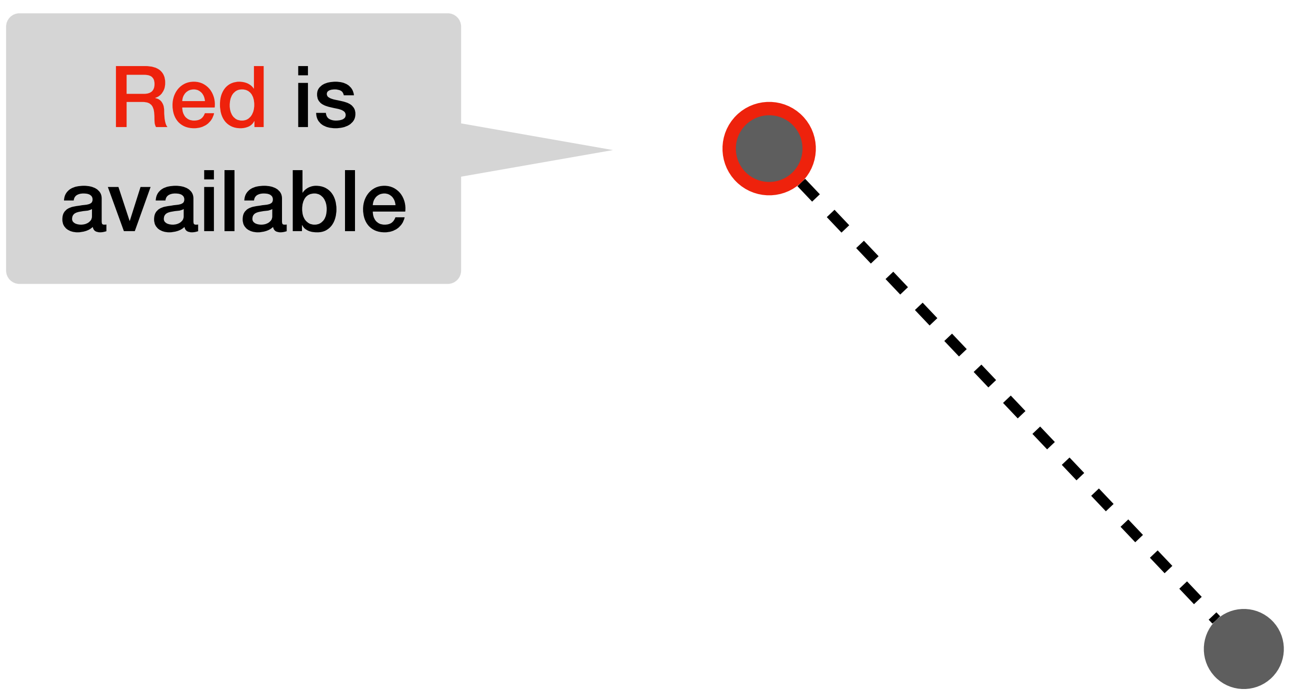

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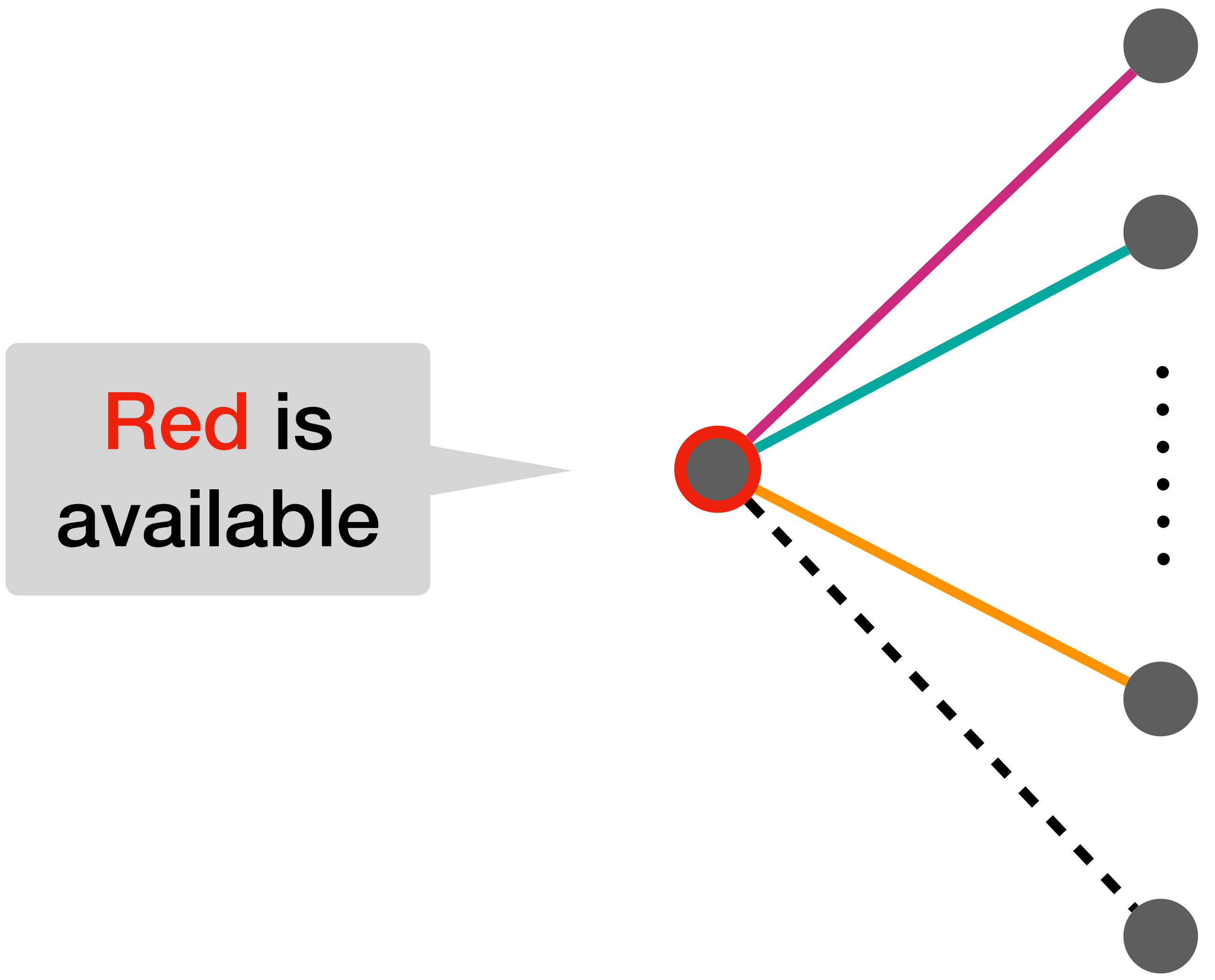

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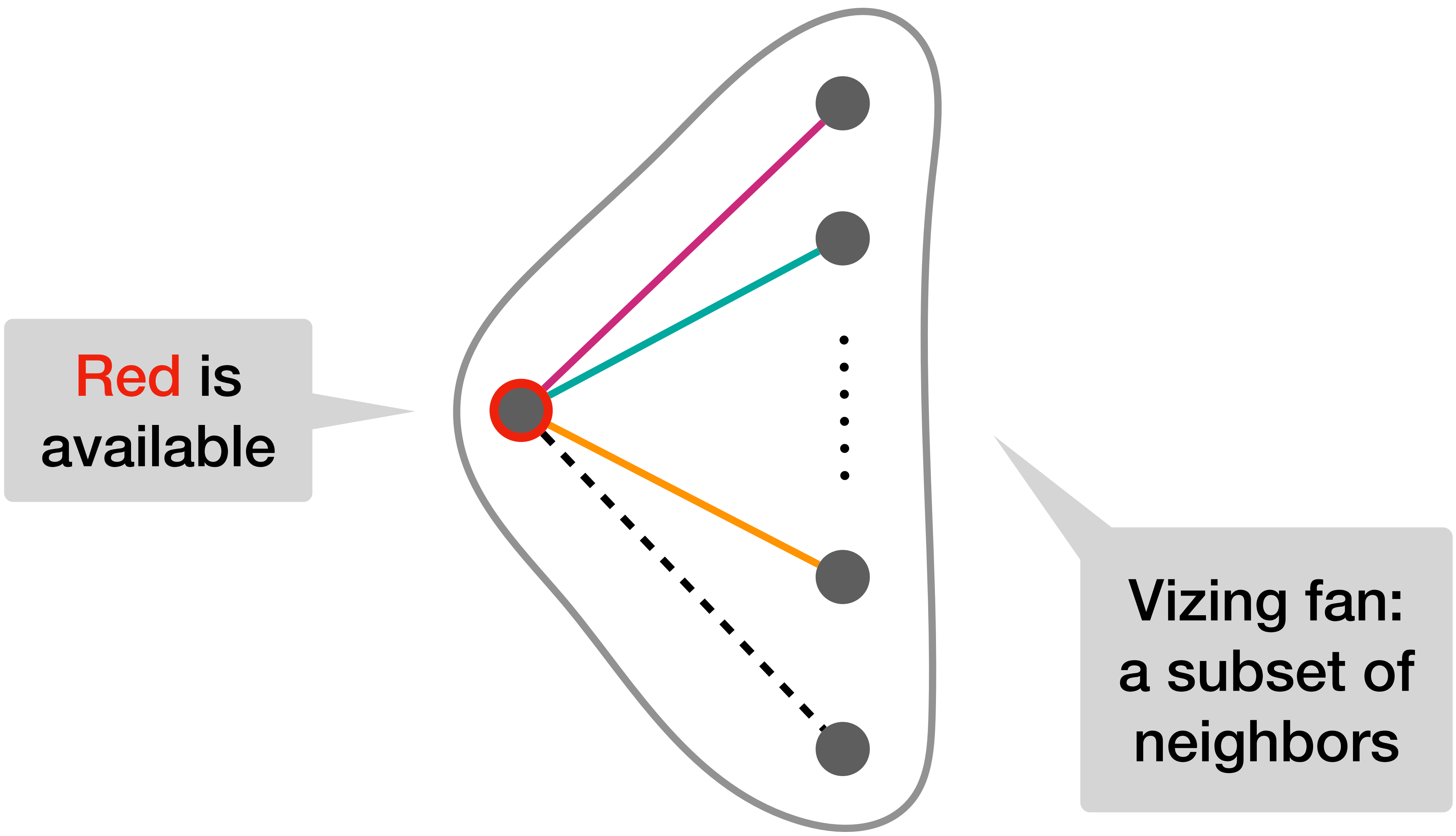

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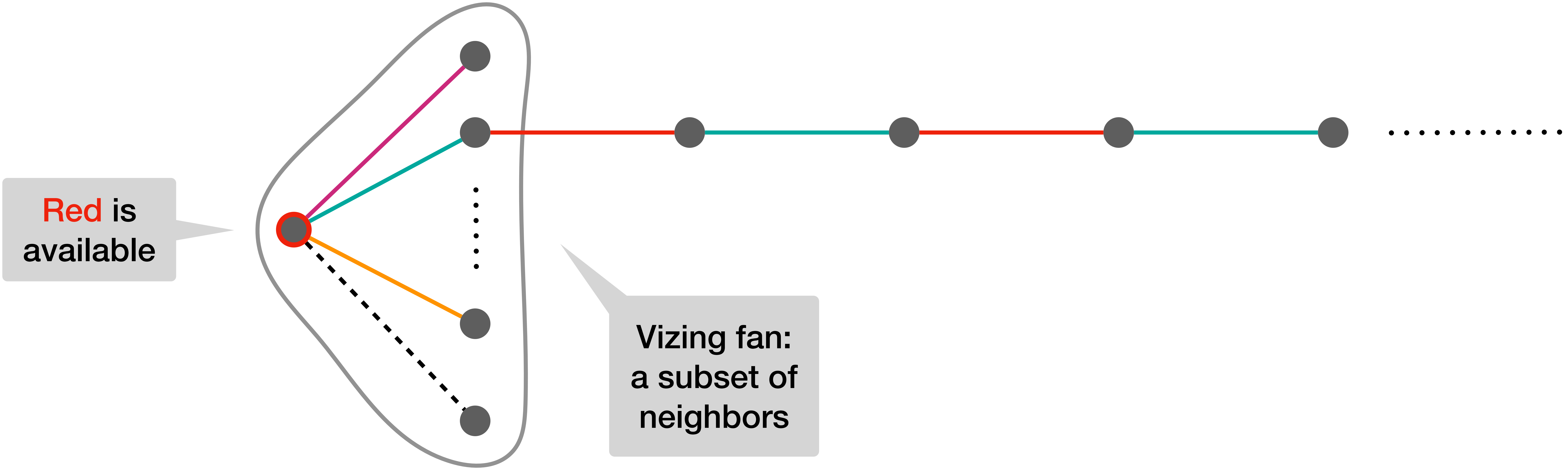

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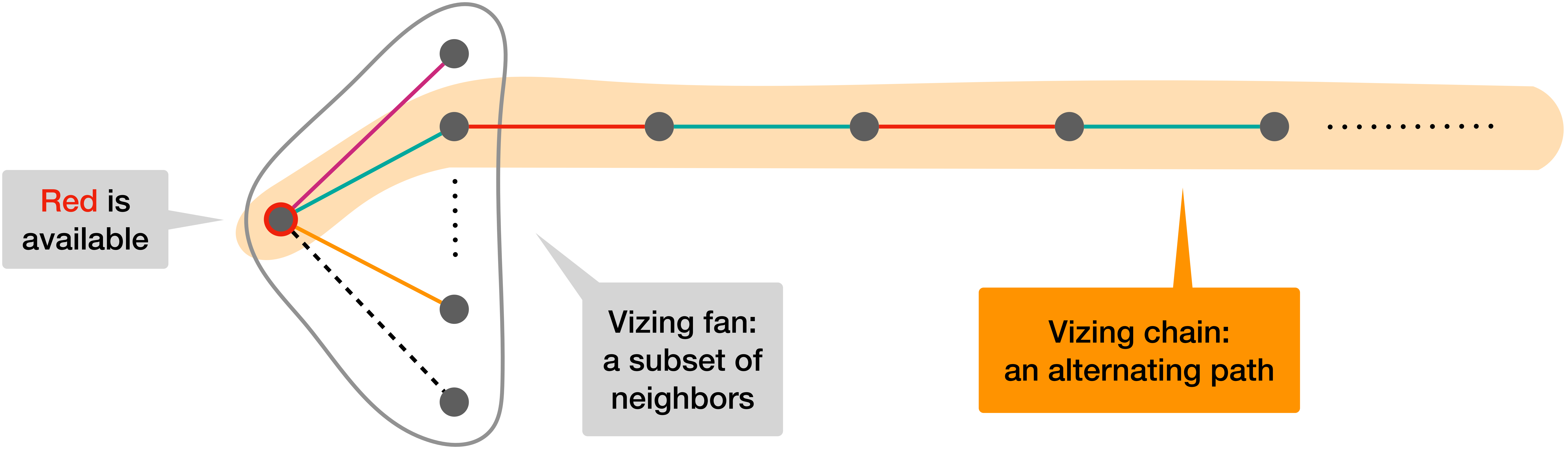

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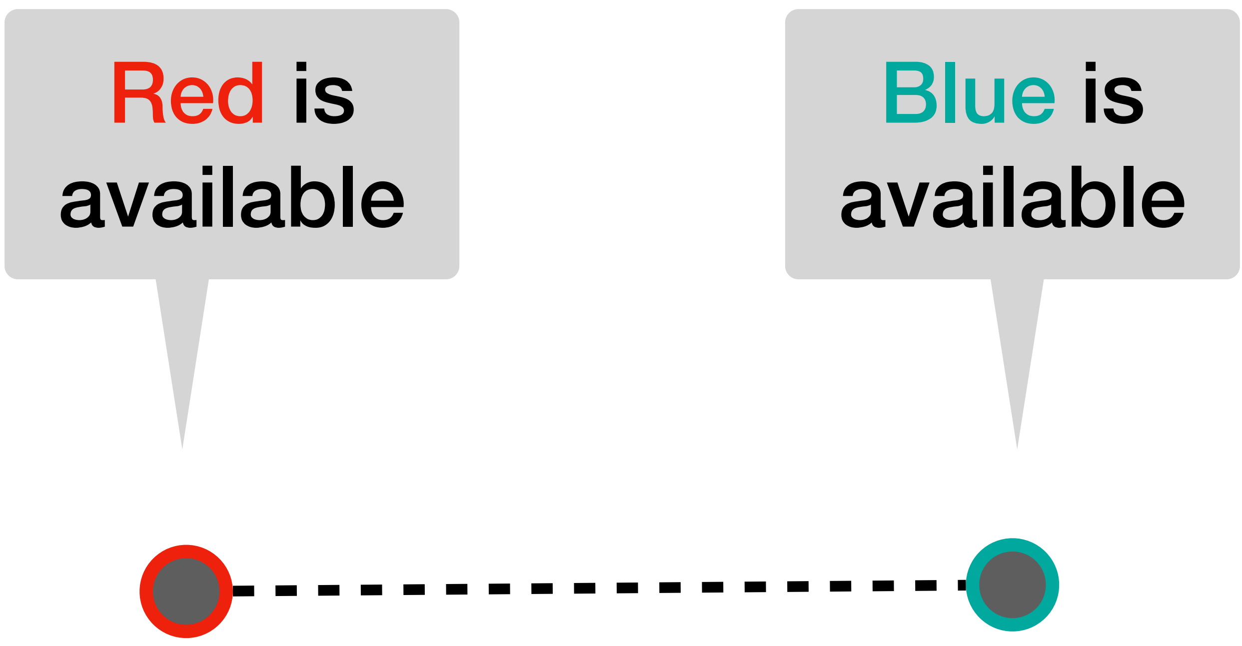
Runtime of color extension = **length** of alternating path

Technique: Analyze the average length of alternating paths



Call it a **type**-{**red**, **blue**} edge

All edges are sorted into Δ^2 **types**

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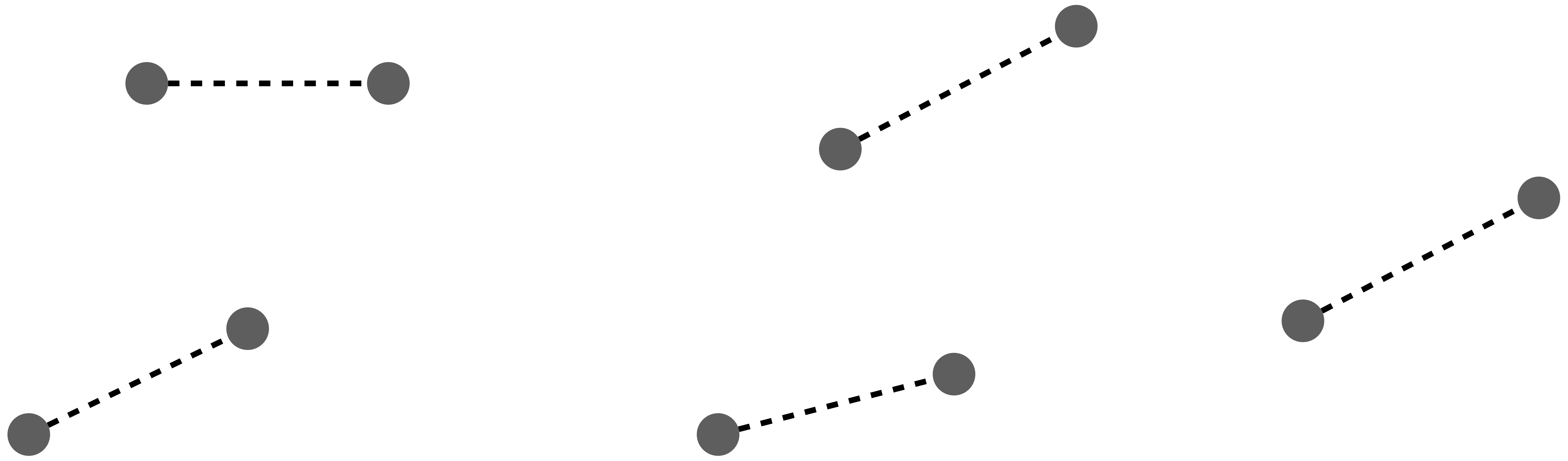

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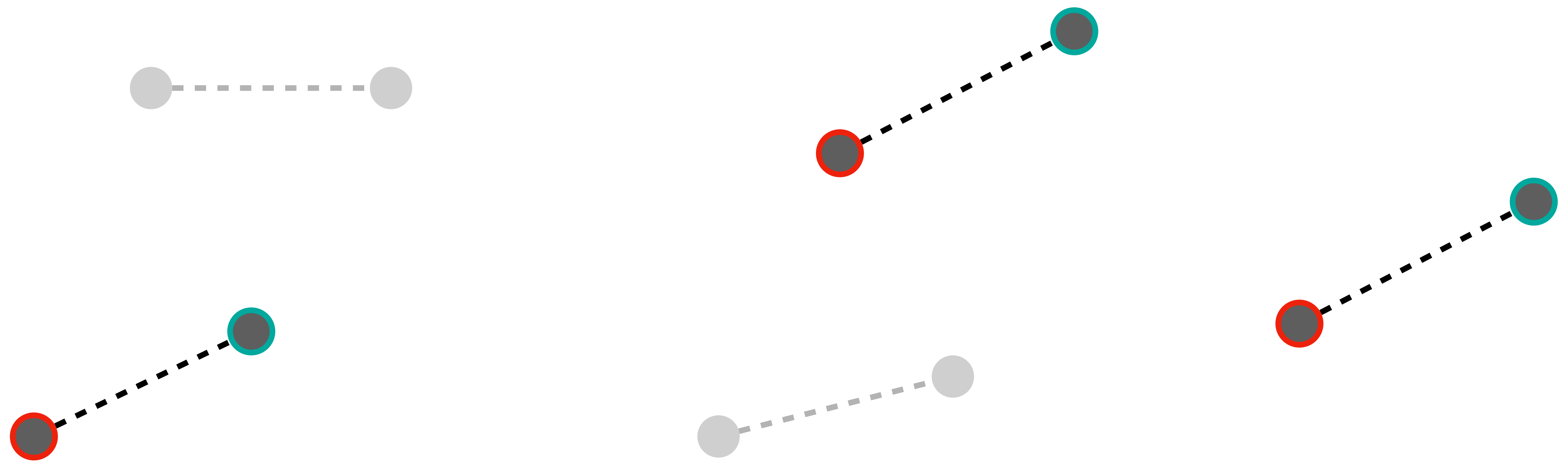

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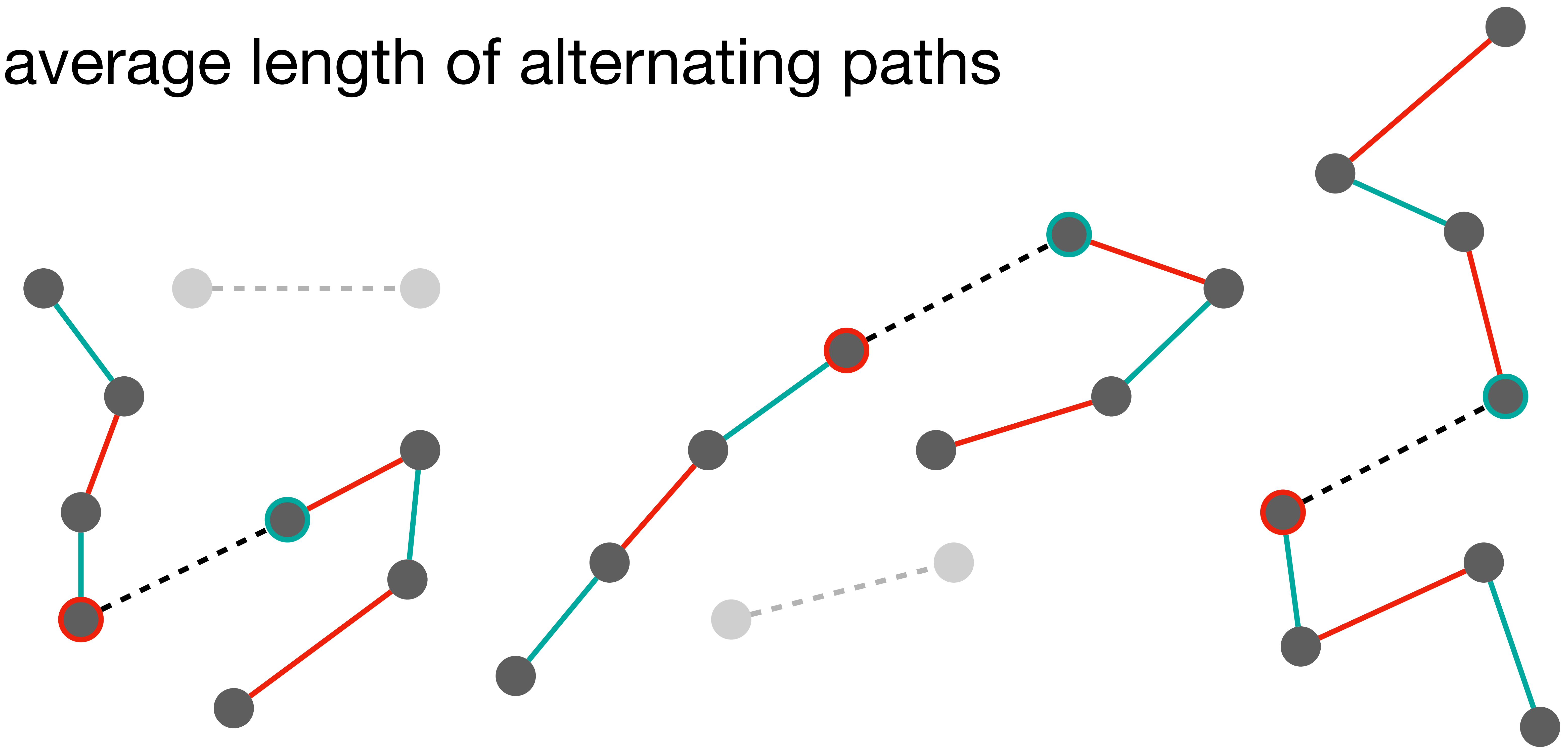

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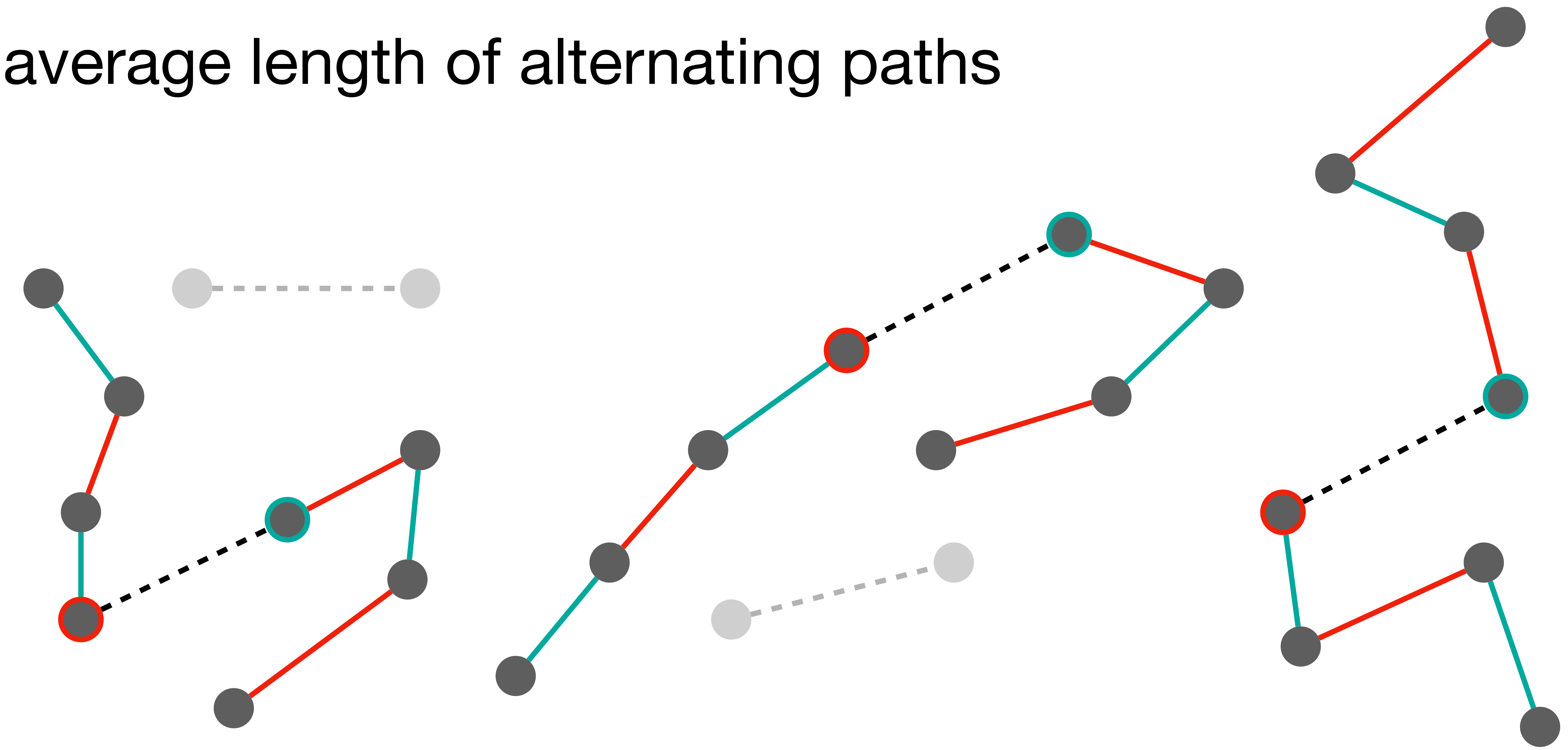
Coloring $O(n)$ edges takes $O(\Delta^2 n)$ time

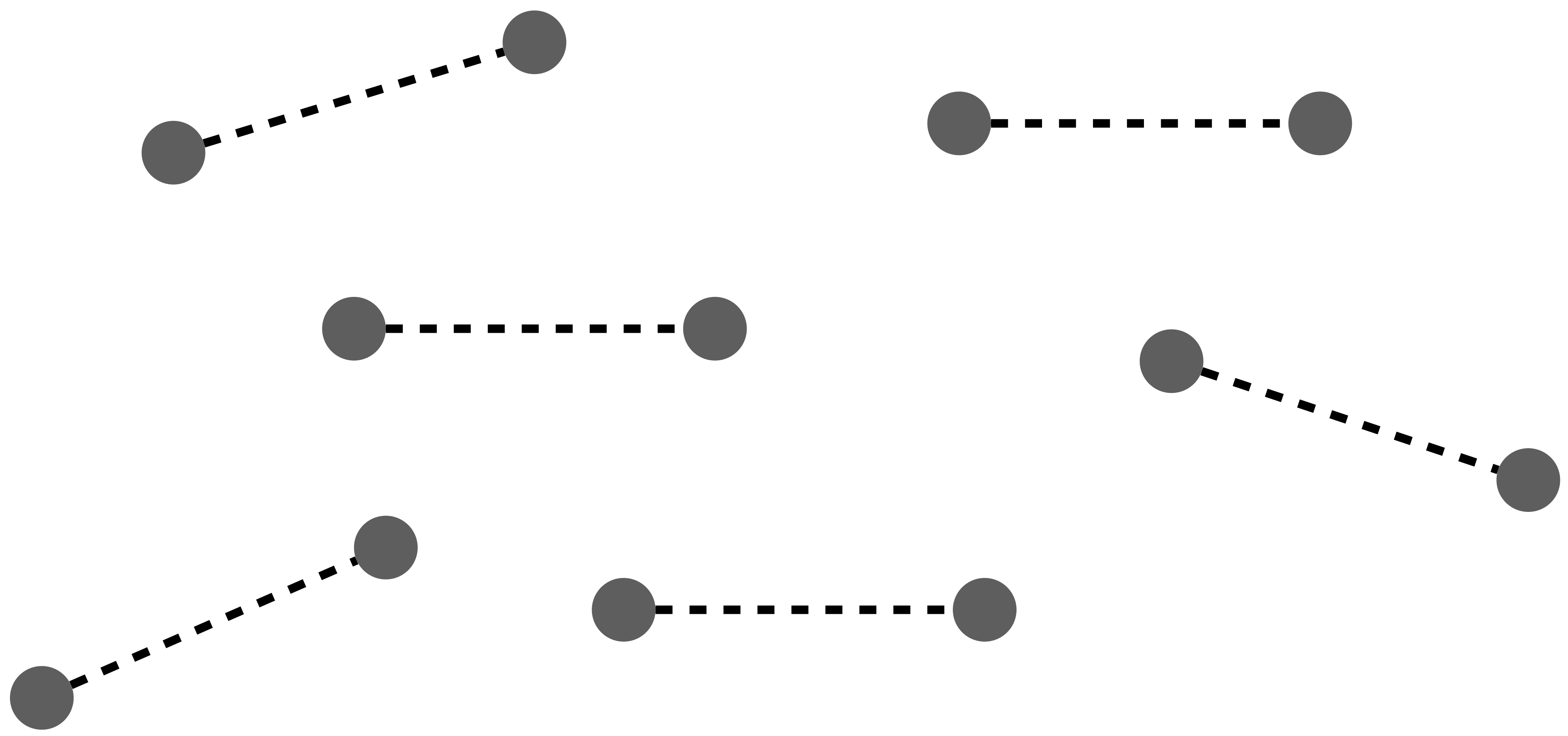


New Approach: Coloring Stars

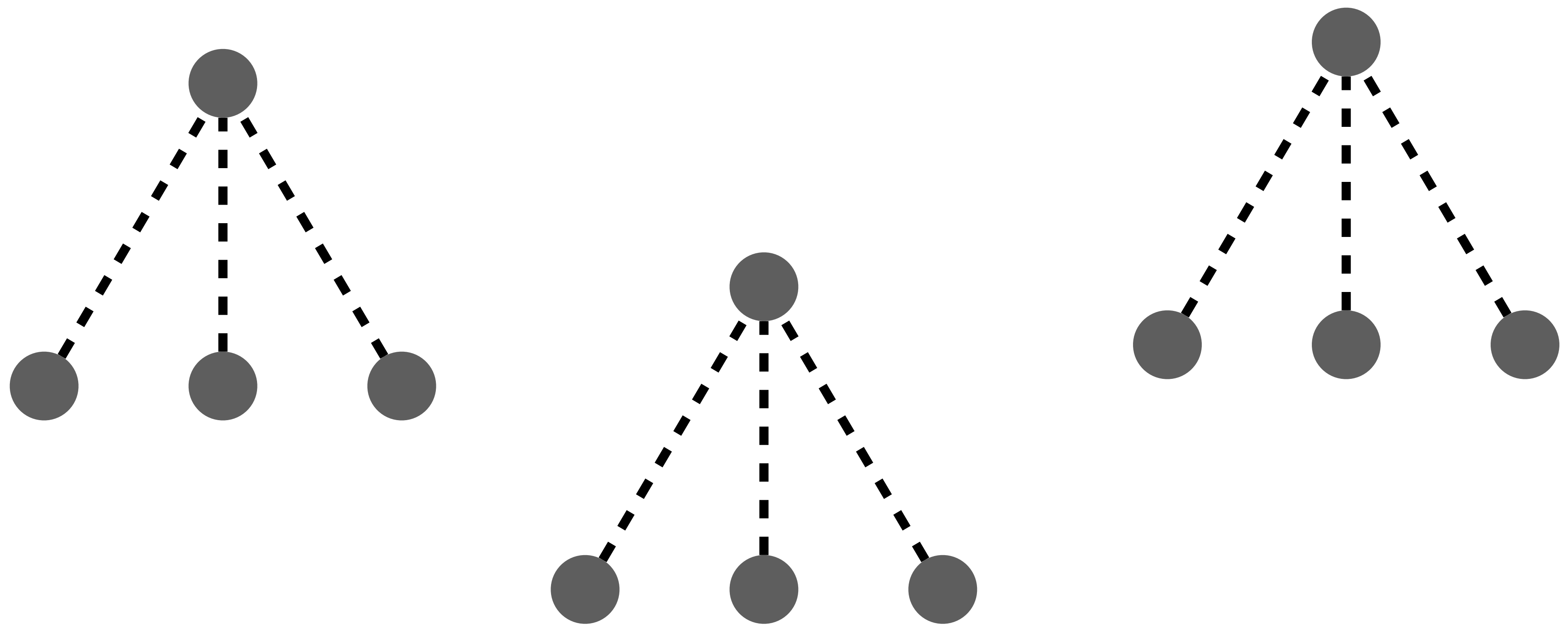
Main observation:

It could be **easier** if uncolored edges form **star subgraphs**

Reason: An average type contains **more** uncolored edges



uncolored edges
scattered around



uncolored edges
forming **stars**

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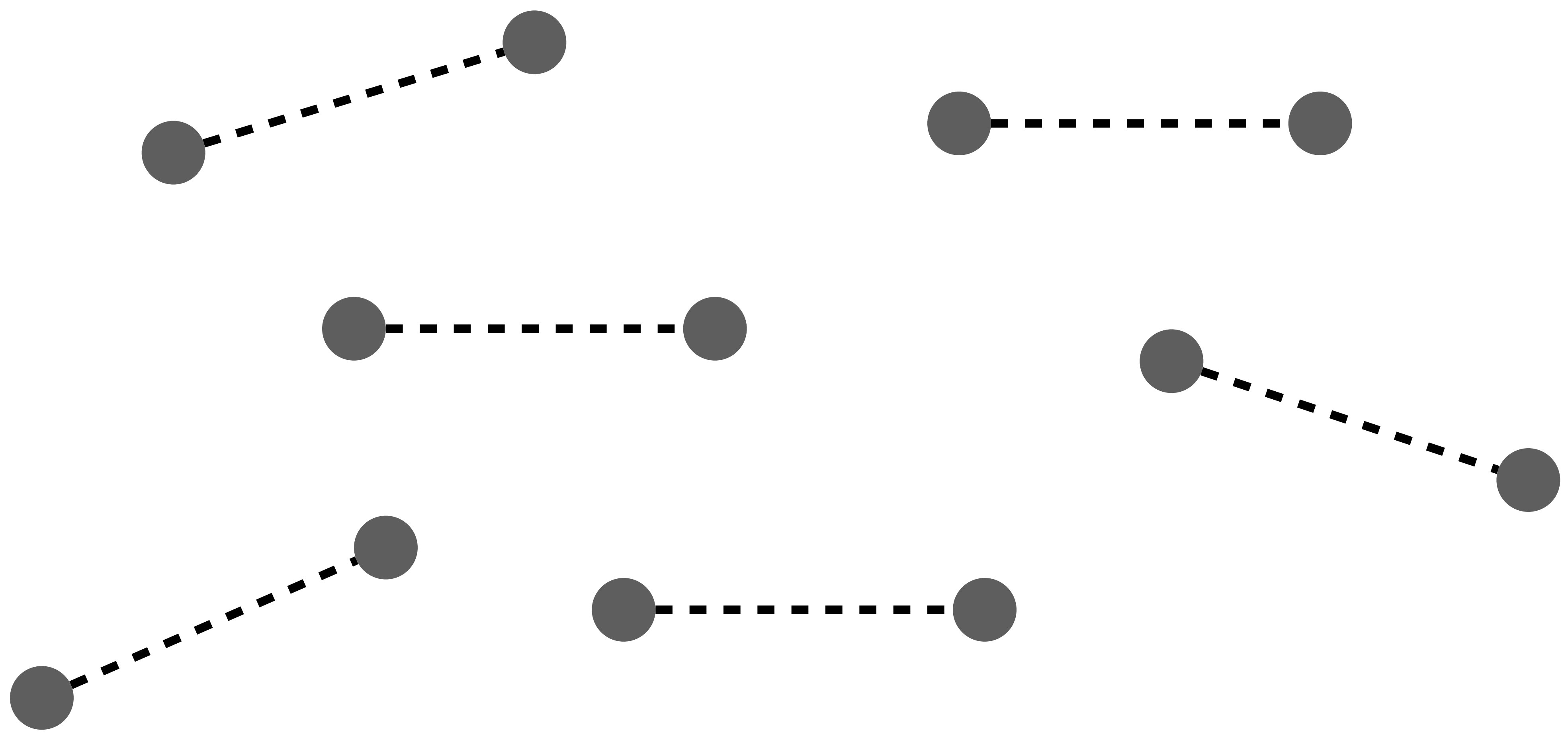
There are ways to reach this **precondition** efficiently



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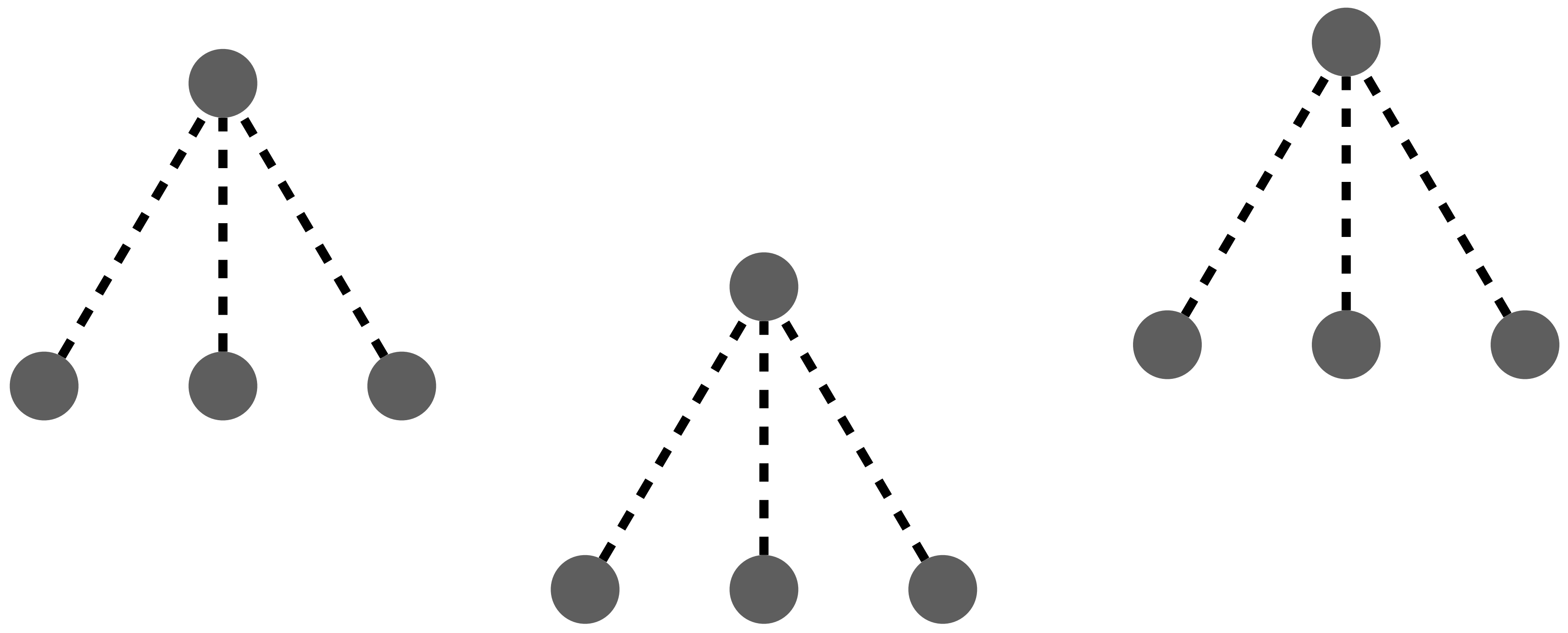

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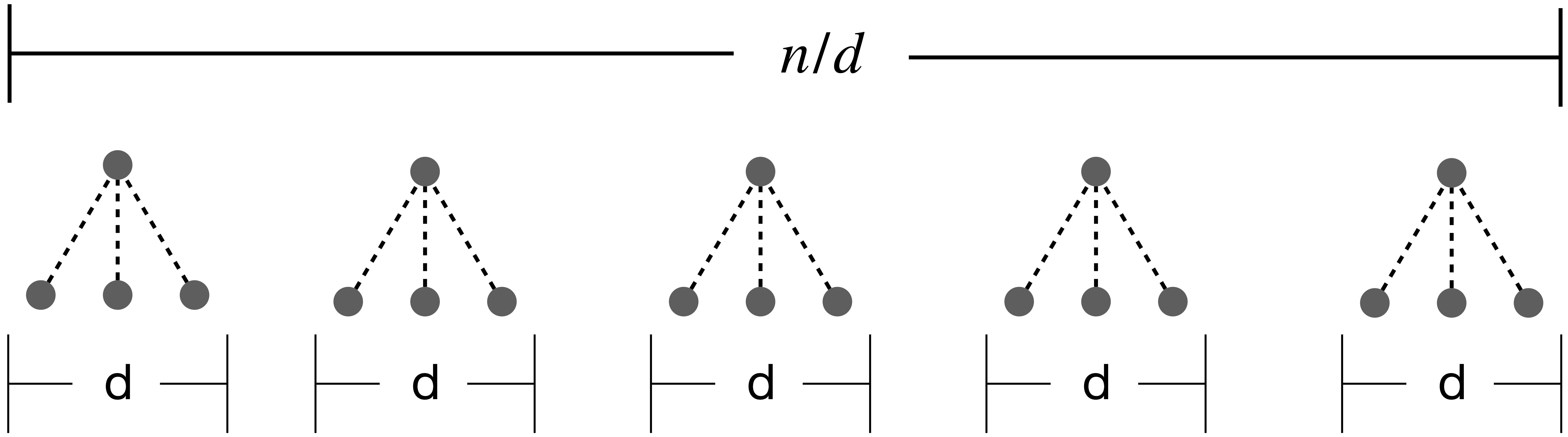

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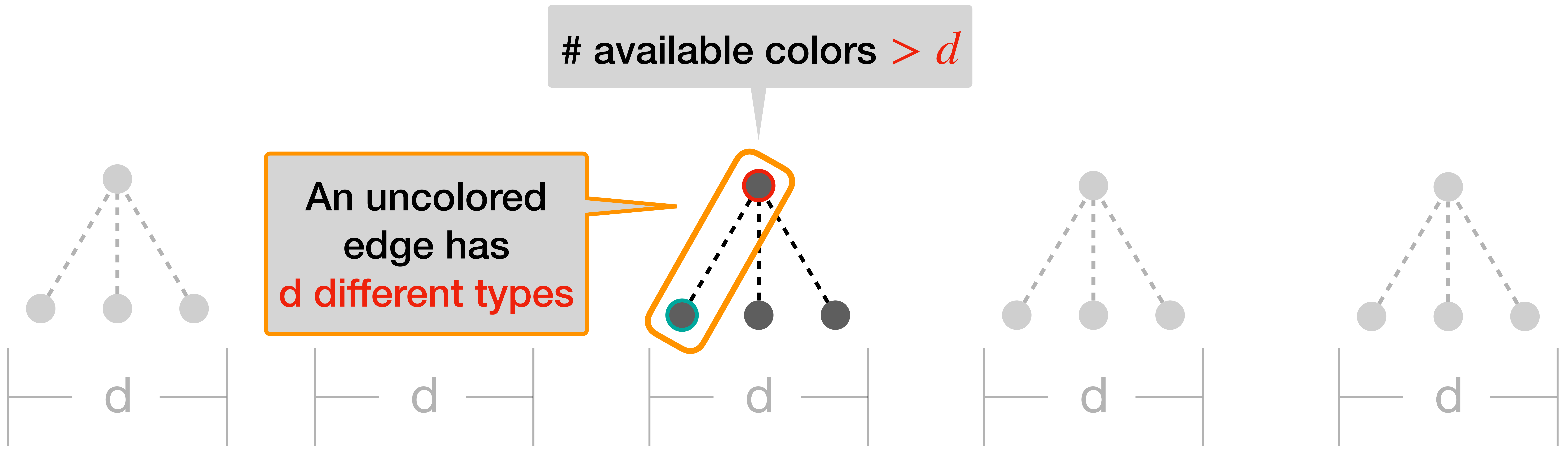

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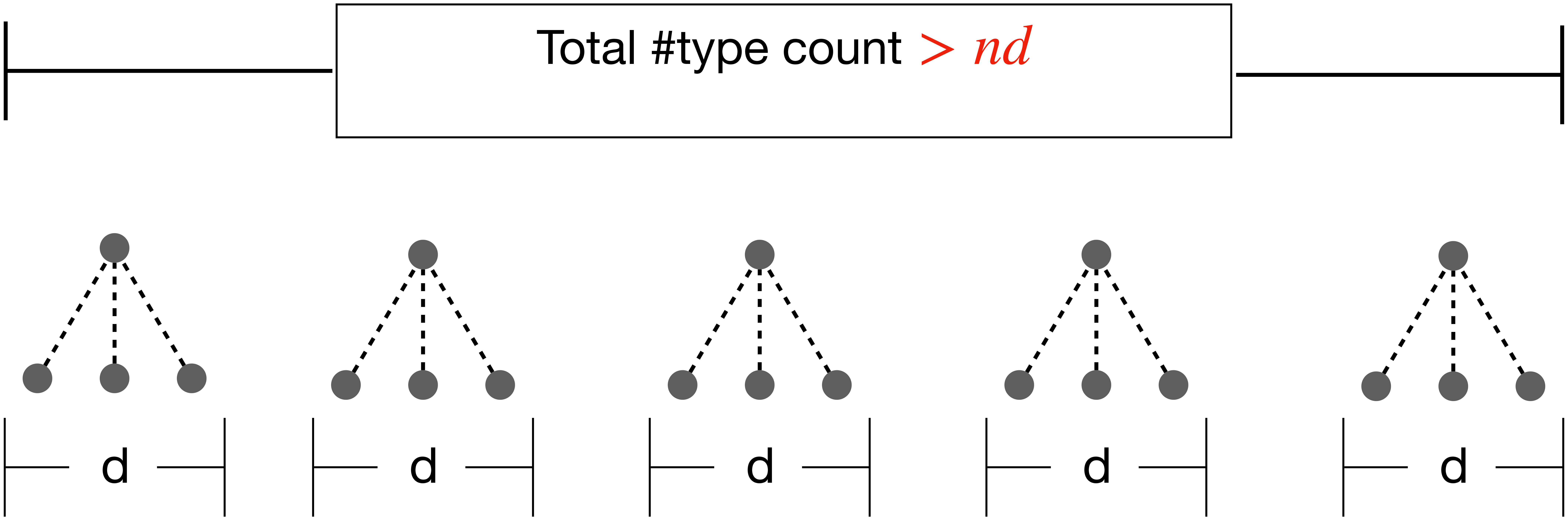

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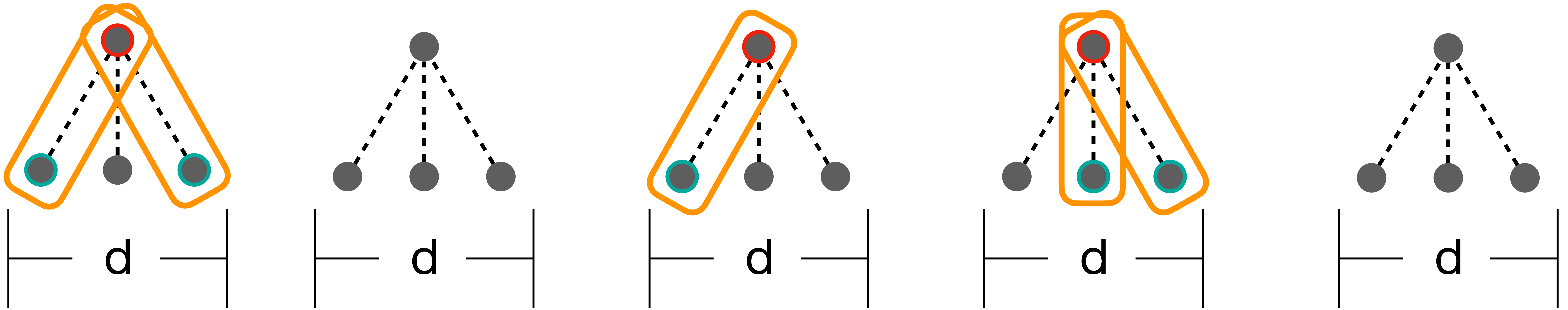

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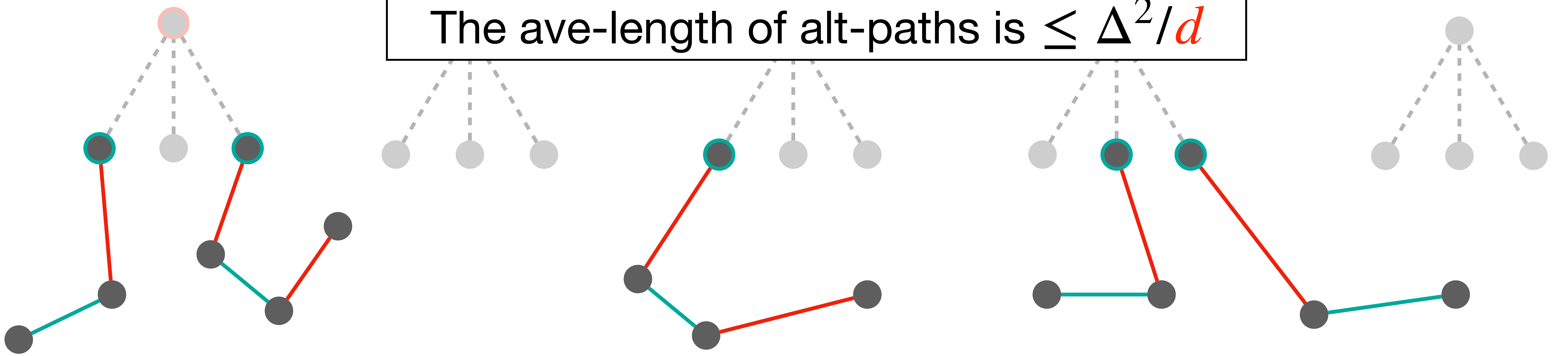
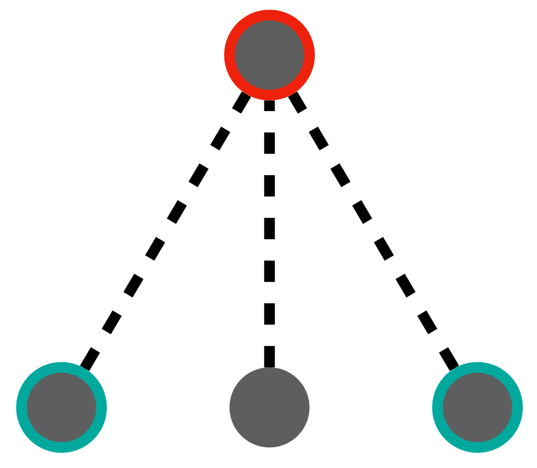
The ave-length of alt-paths is $\leq \Delta^2/d$

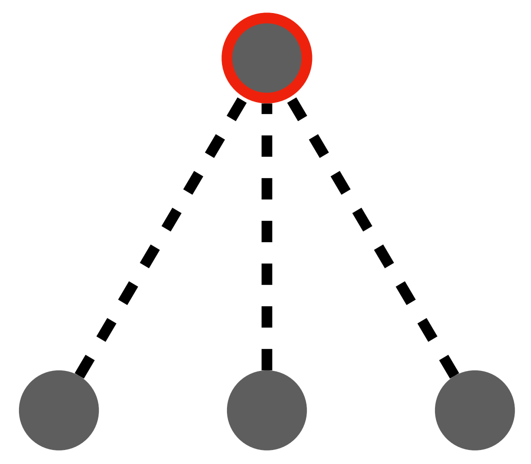


Coloring Stars in General Graphs

Issue in general graphs:

- Edge types are defined by the **Vizing fan** structure
- Edges in the star might generate the **same alternating path**





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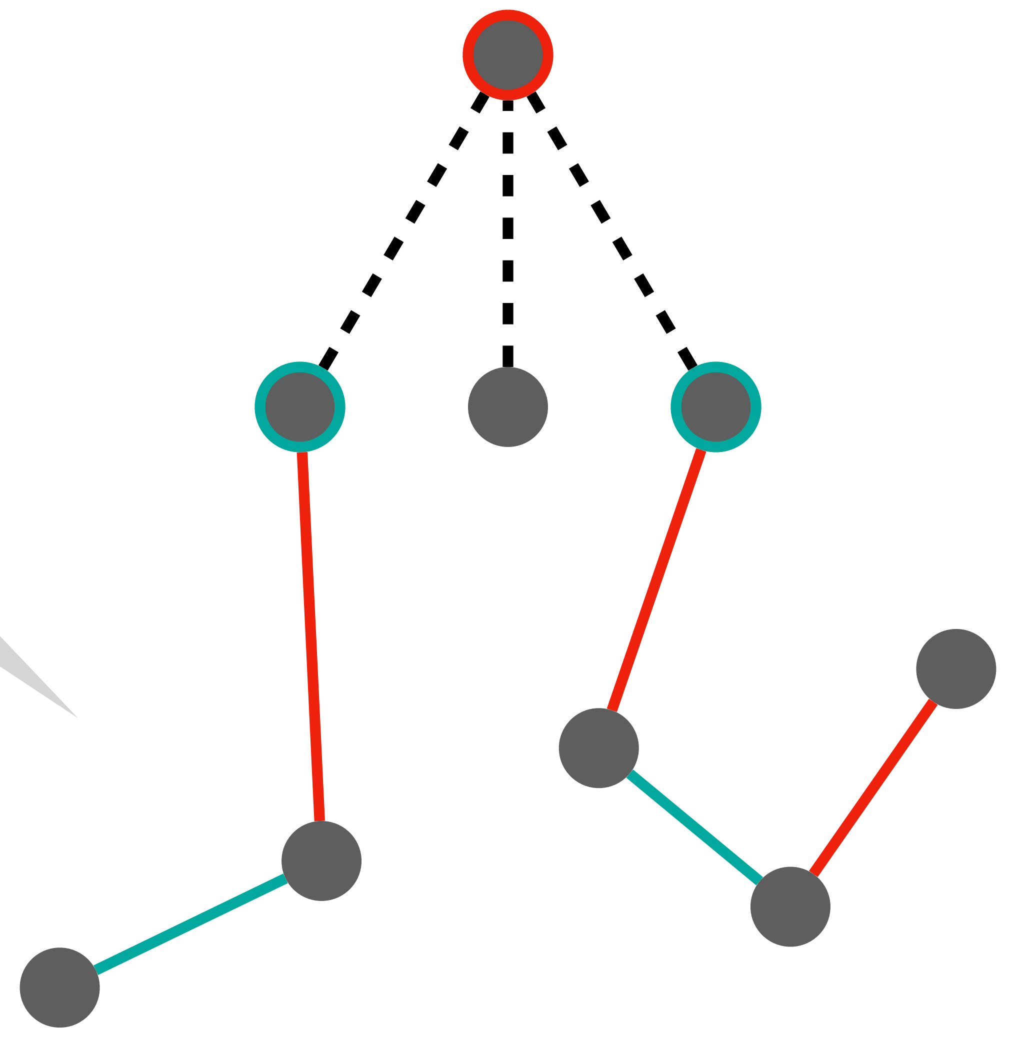

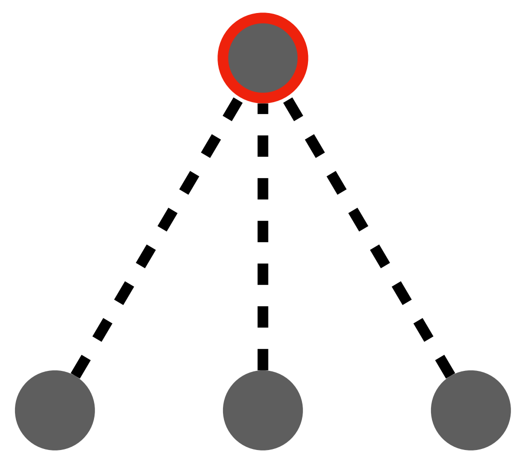

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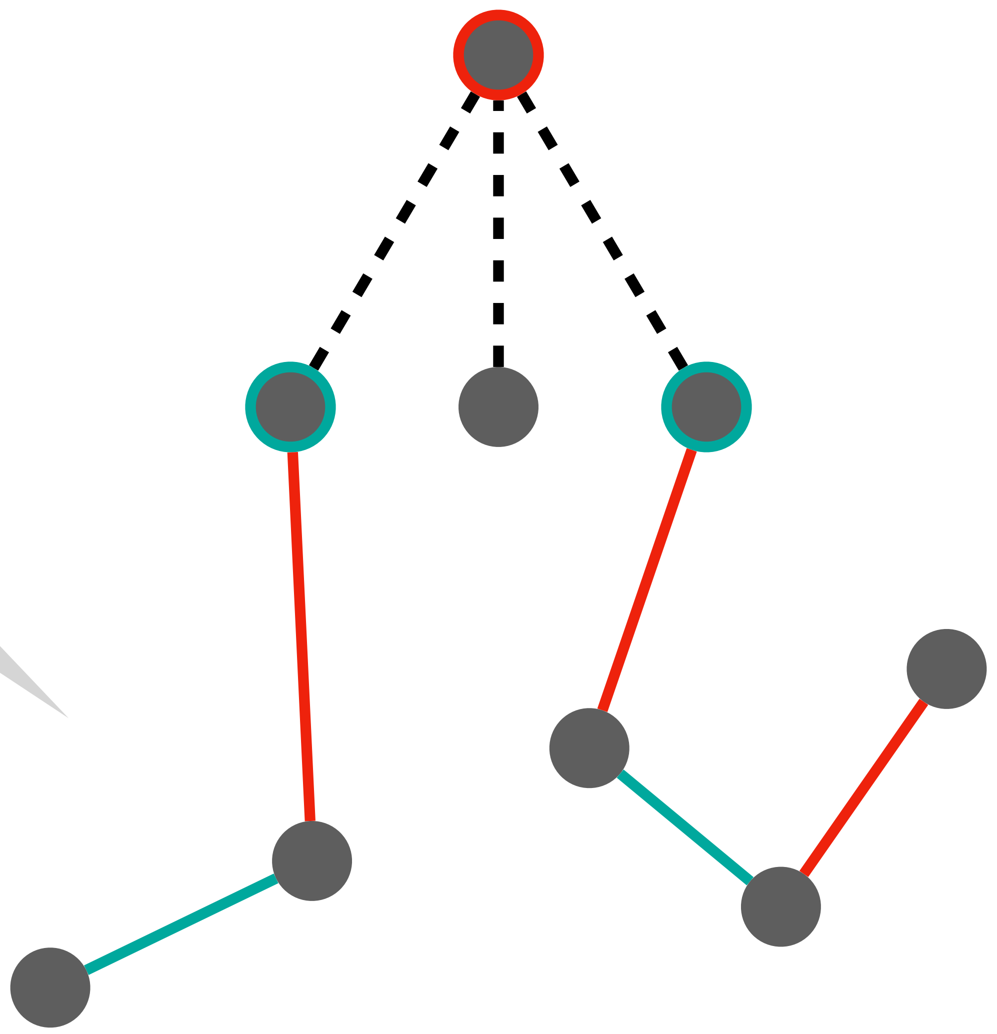

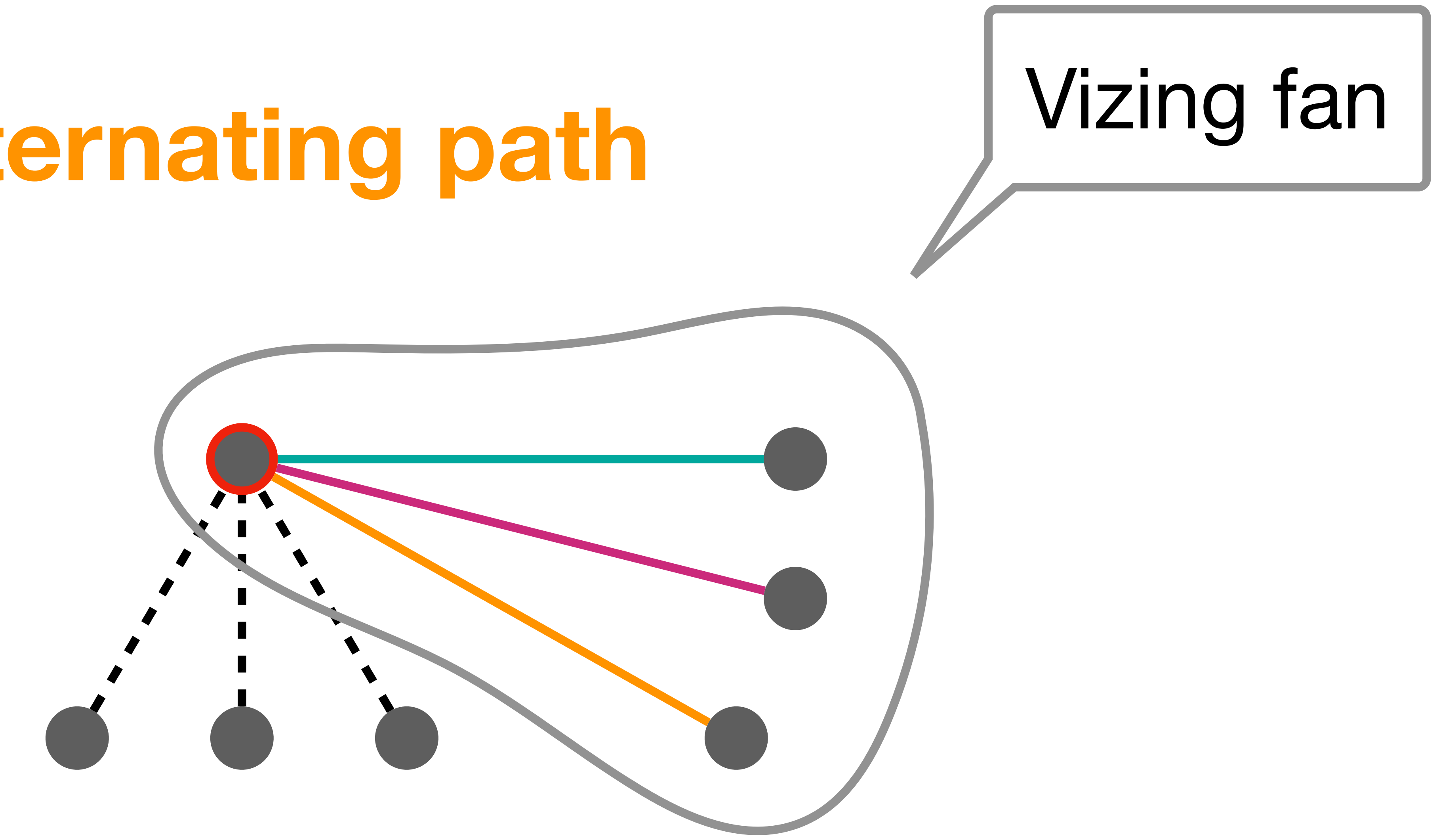

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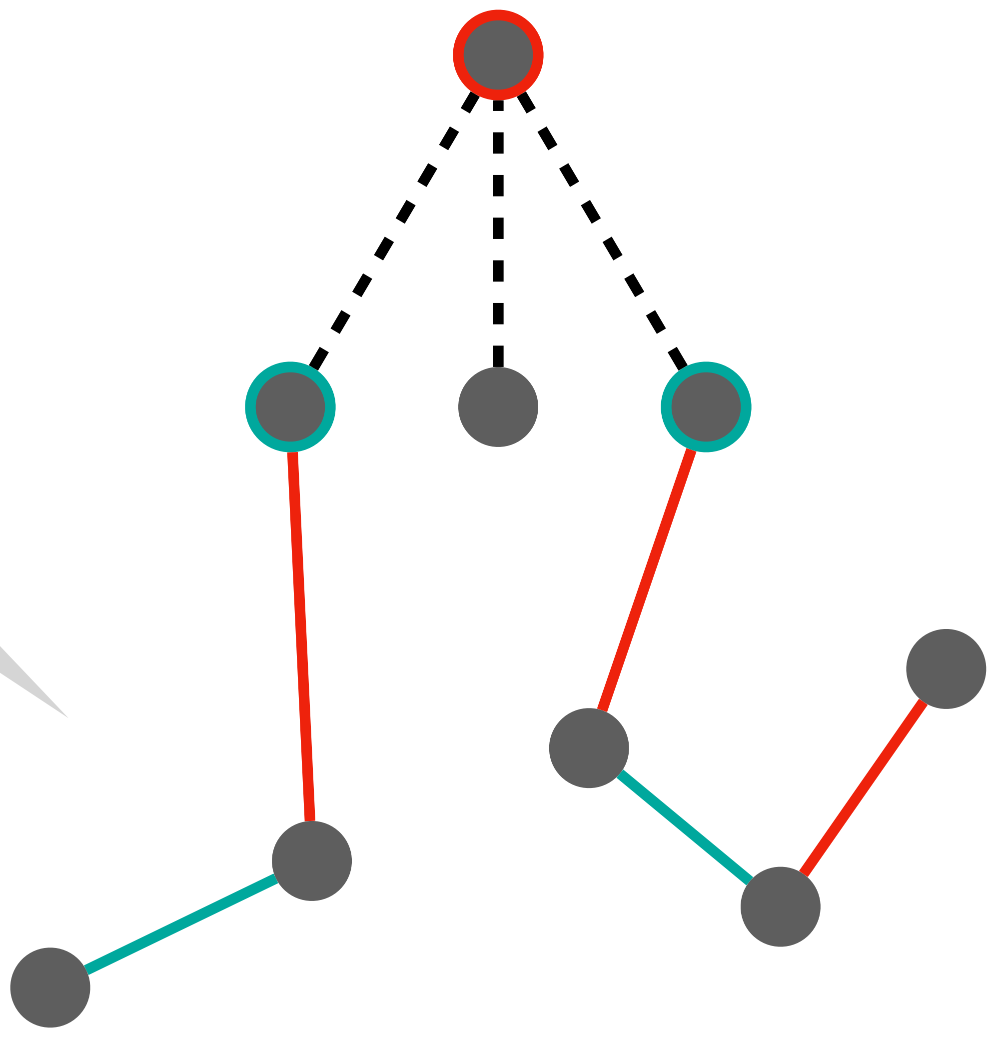

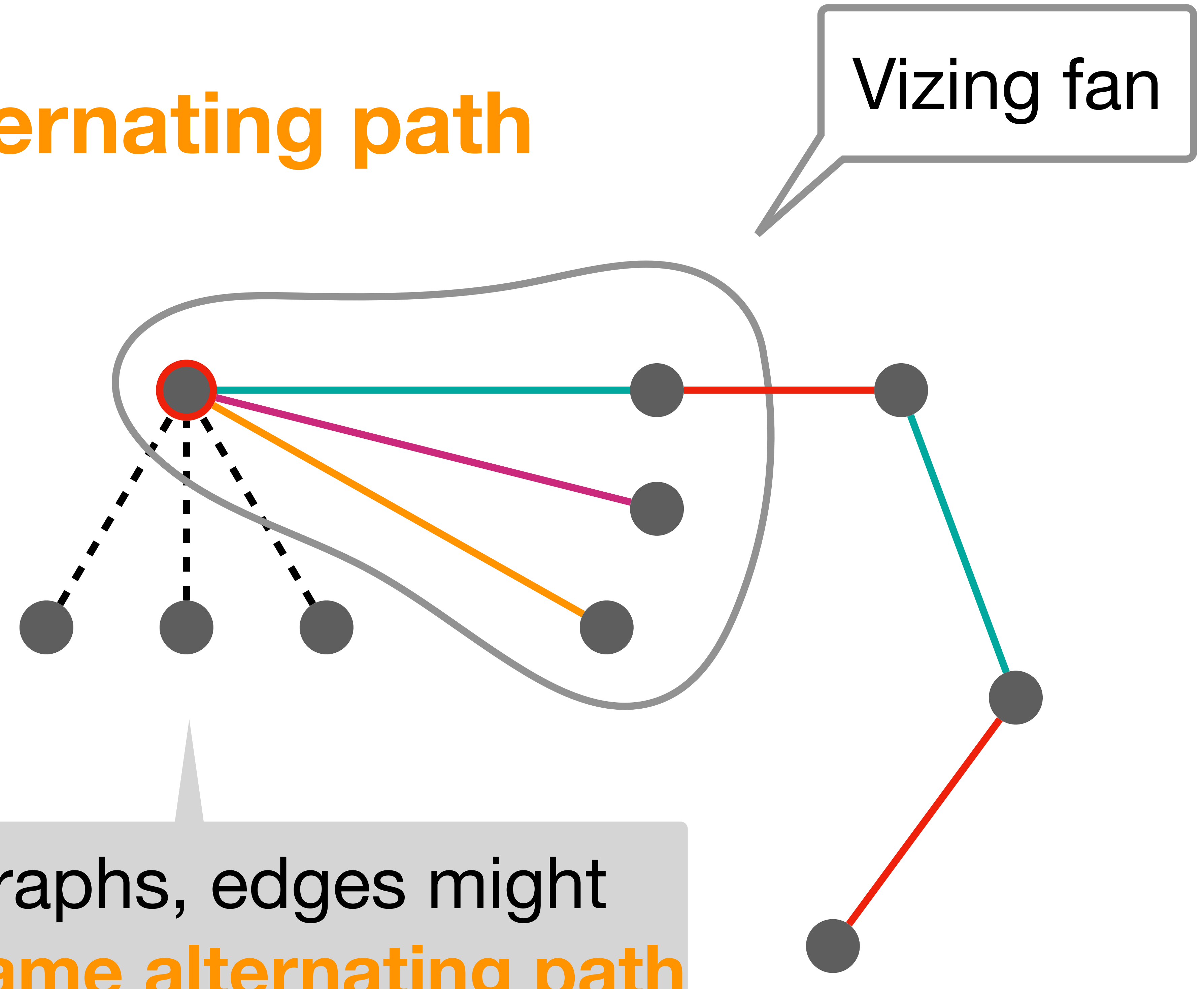

In **general** graphs, edges might generate the **same alternating path**

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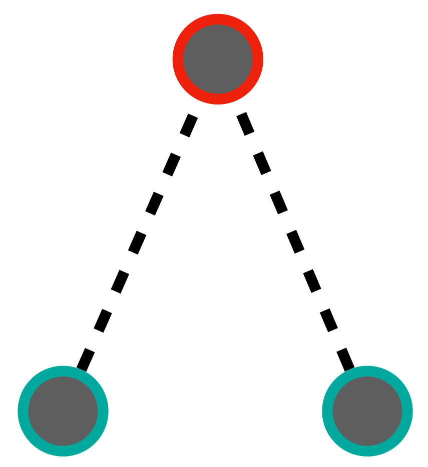
Observation:

- Use **u-fans** to avoid Vizing fans and go back to bipartite case [Gabow et al., 1985]



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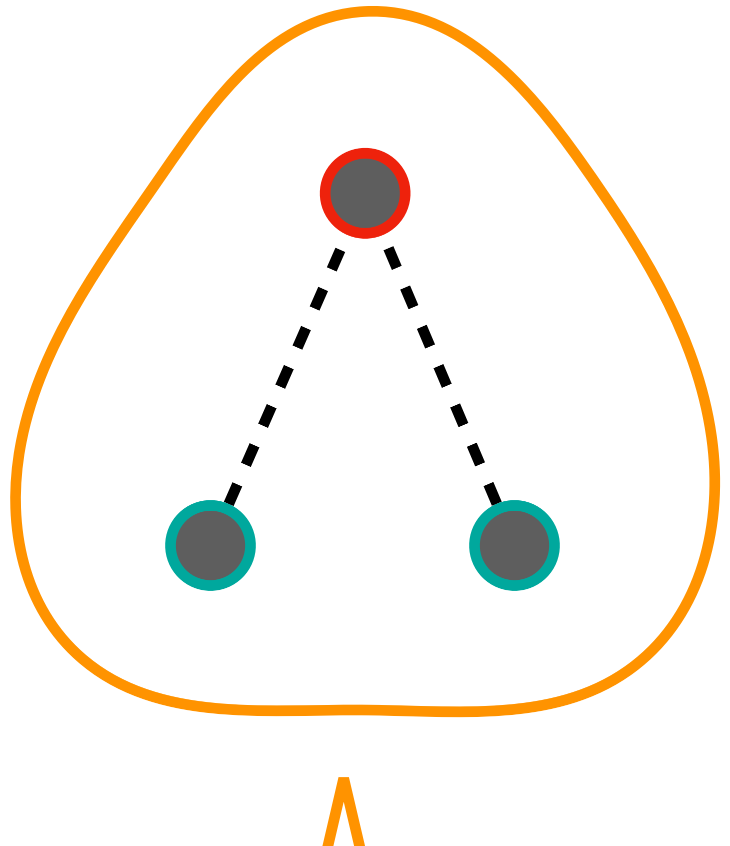

u-fan: a pair of leaves
missing the same color

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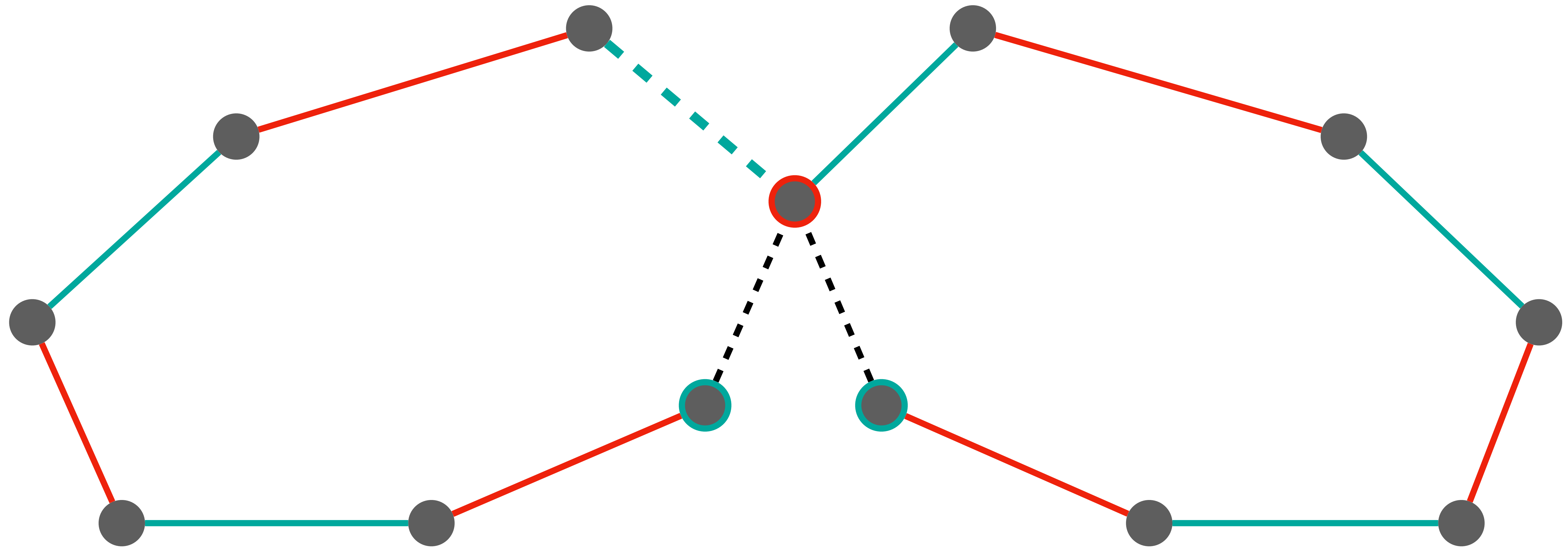

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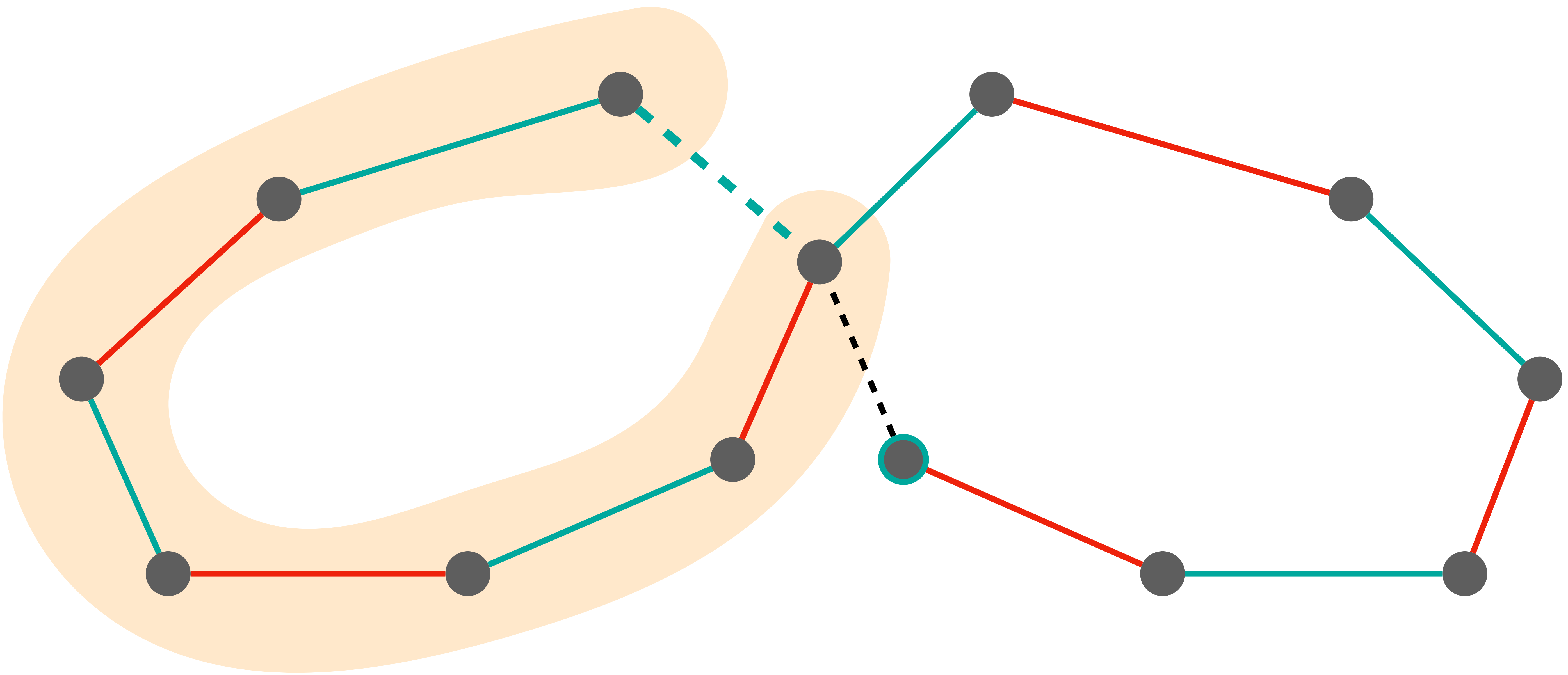

Flip this path and extend the color

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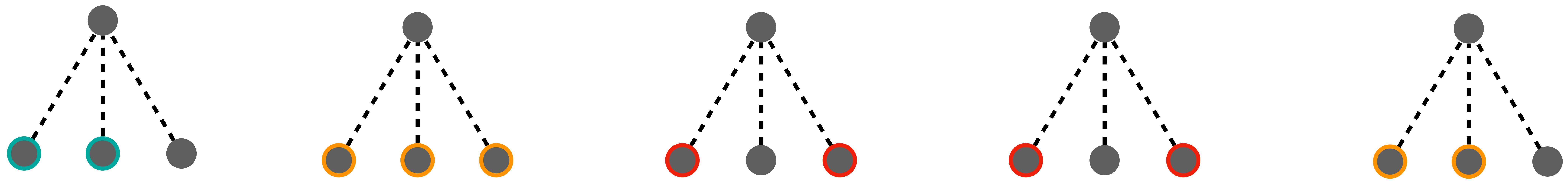

Coloring Stars in General Graphs

Issue in general graphs:

- Edge types are defined by the **Vizing fan** structure
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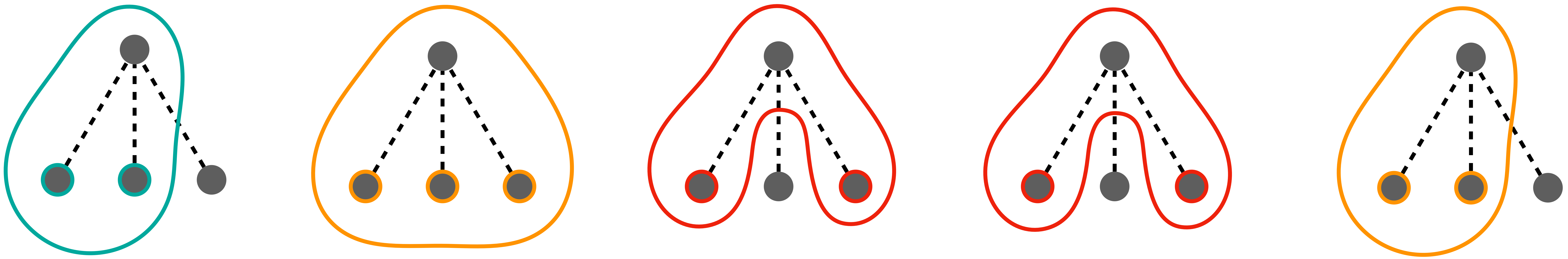
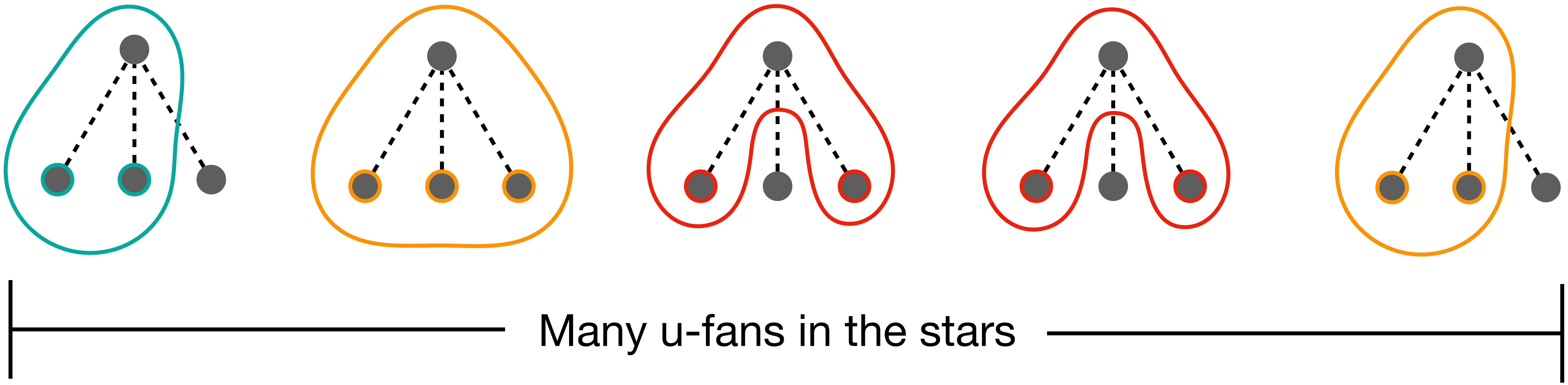


Many u-fans in the stars

Coloring Stars in General Graphs

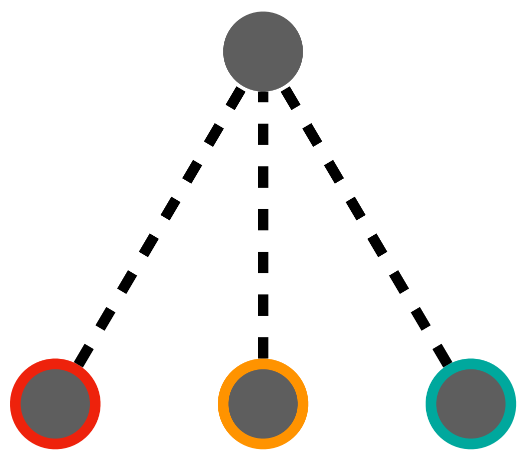
<u>Issue in general graphs:</u>

- ~~Edge types are defined by the **Vizing fan** structure~~ Use u-fans instead
- ~~Edges in the star might generate the **same alternating path**~~ Bipartite analysis

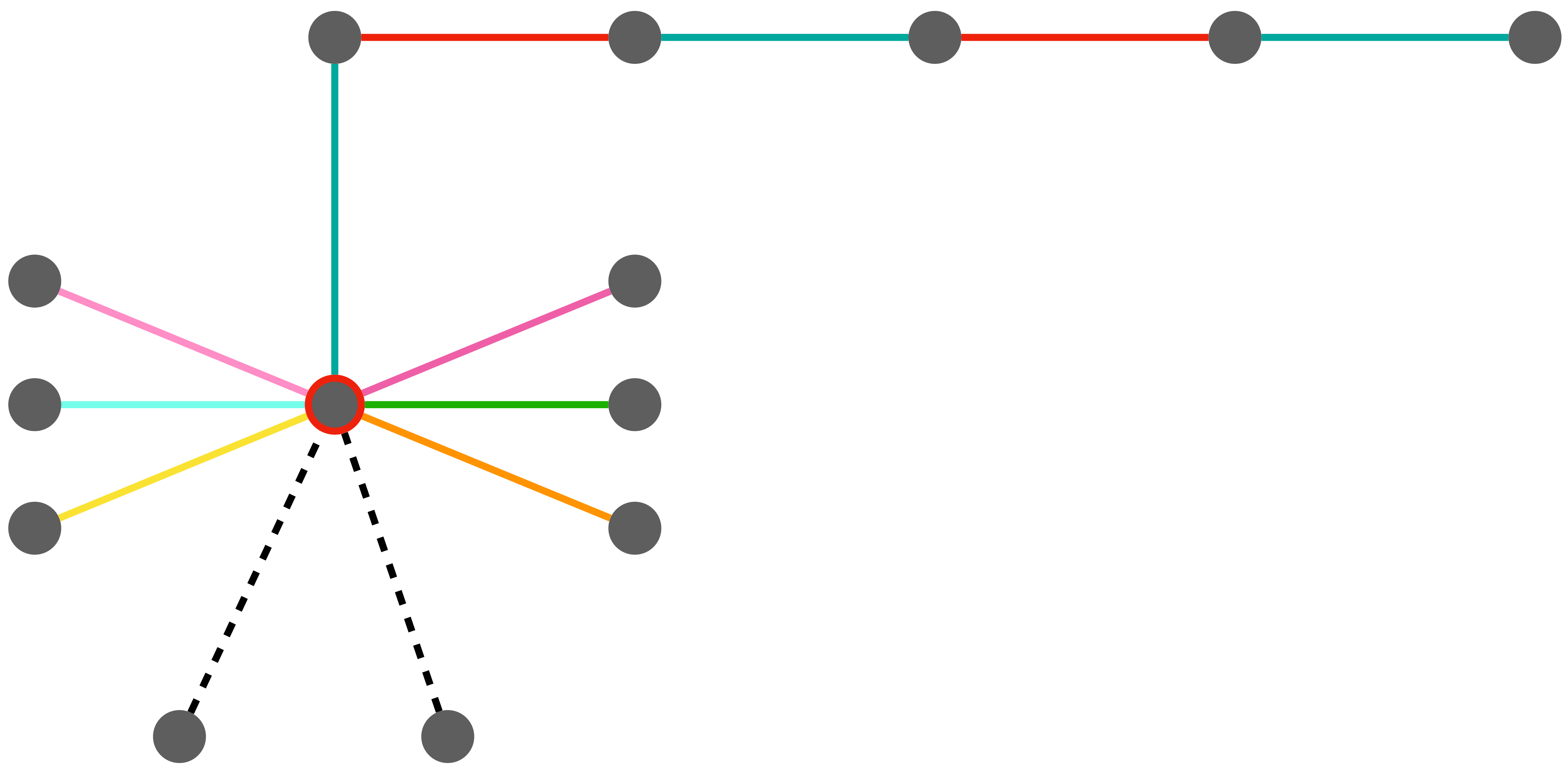


Coloring Stars in General Graphs

What if leaf **missing colors are different** in the same star?



Suppose two uncolored edges share a Vizing chain



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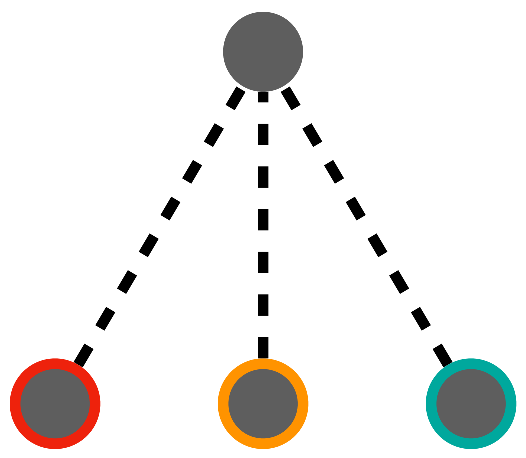

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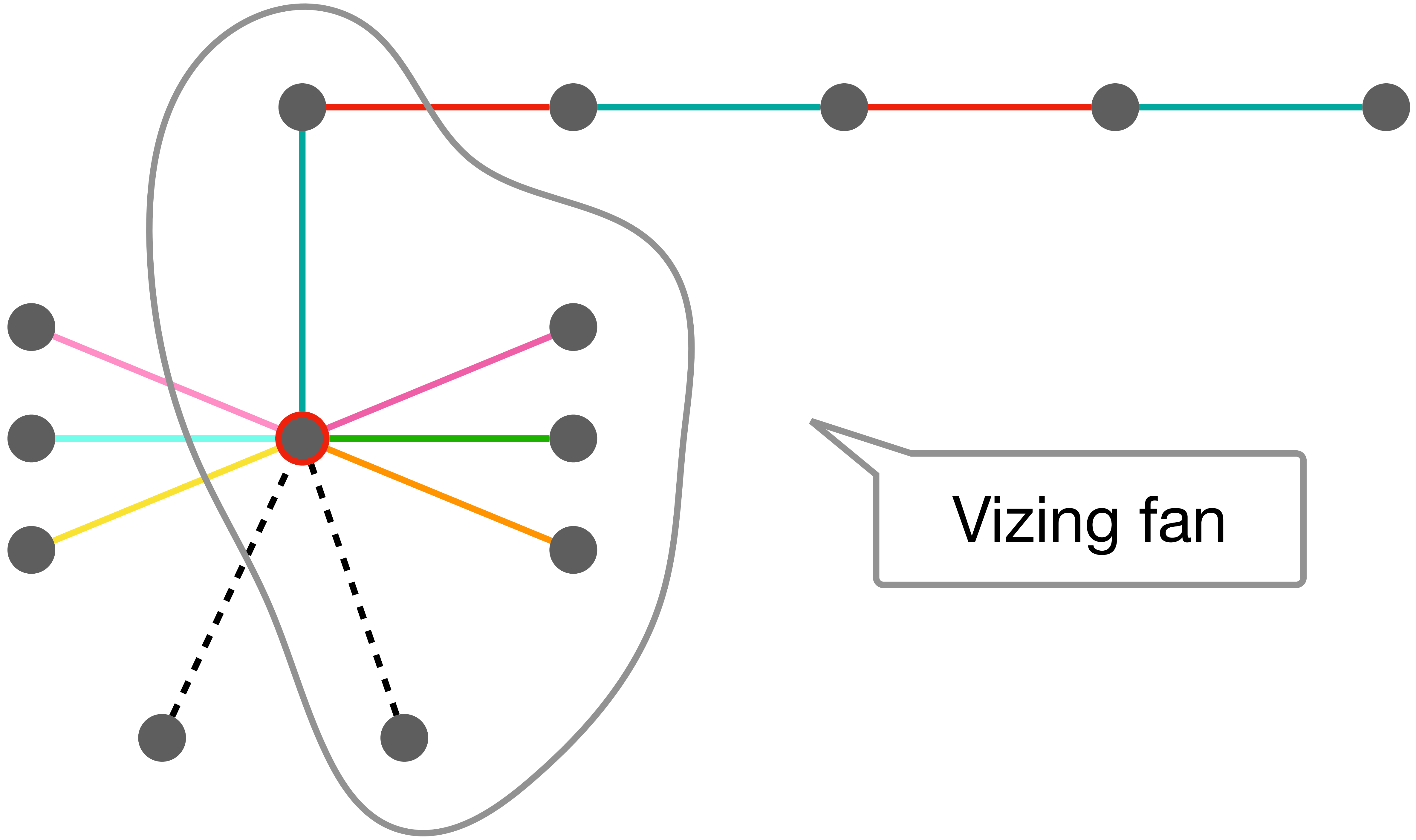

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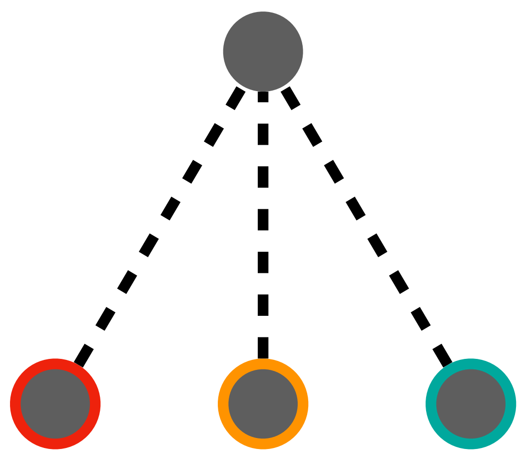

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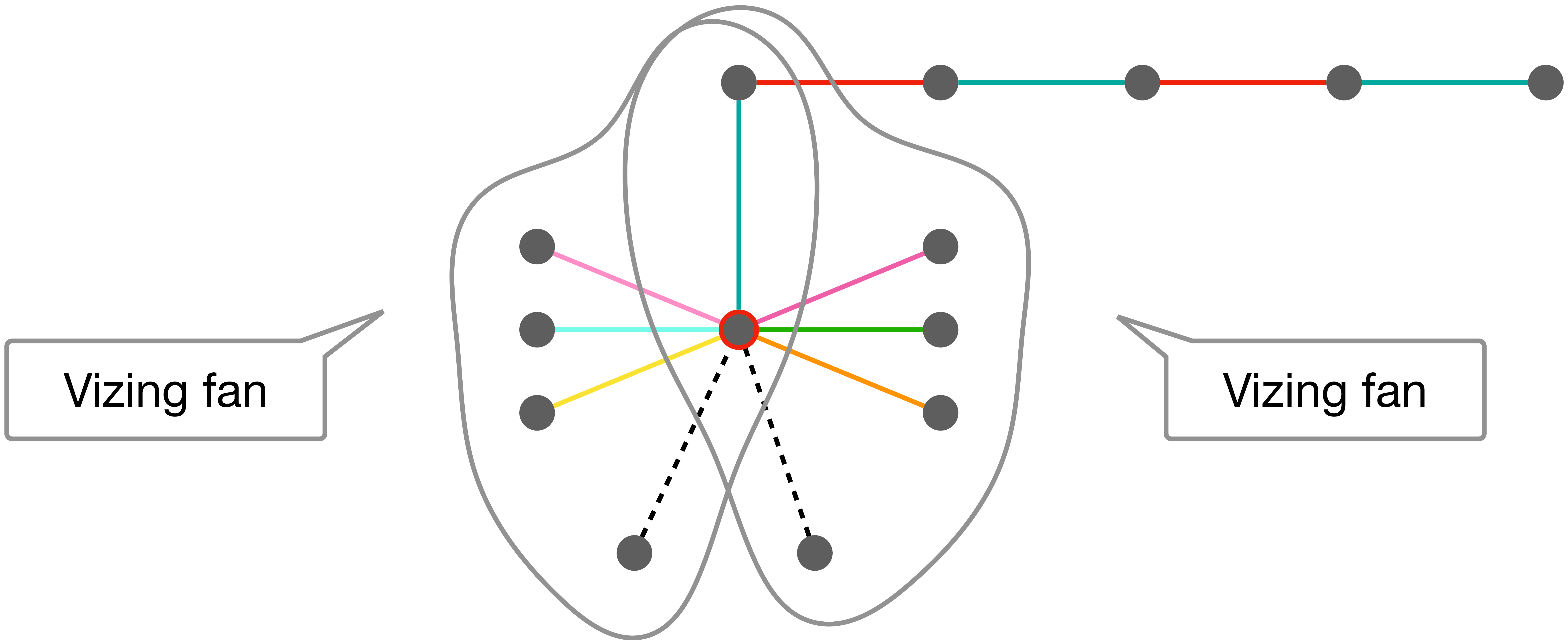

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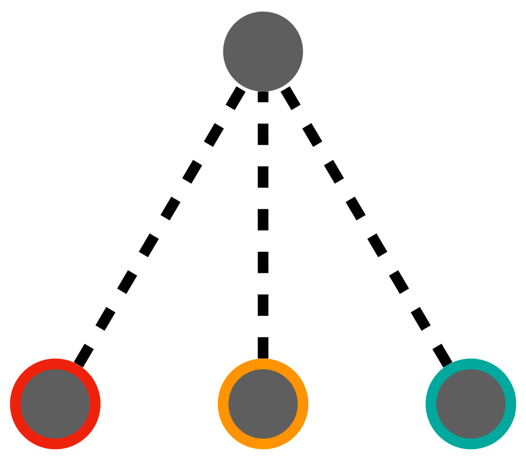

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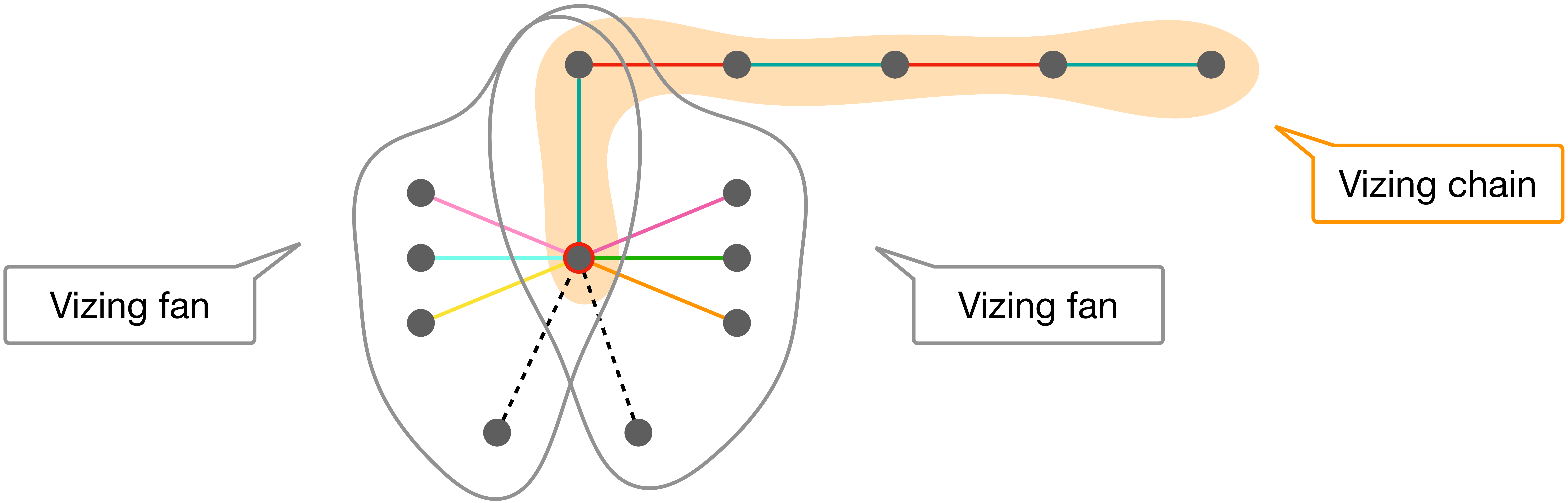

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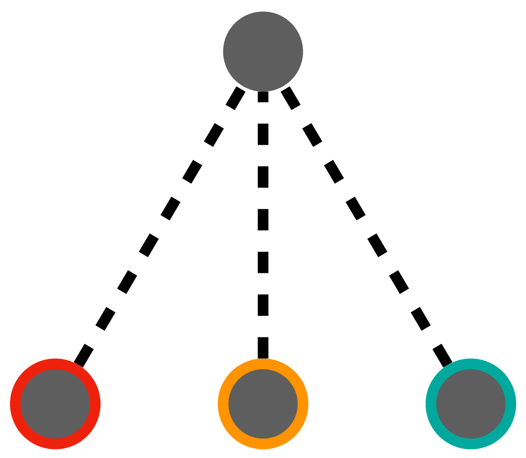

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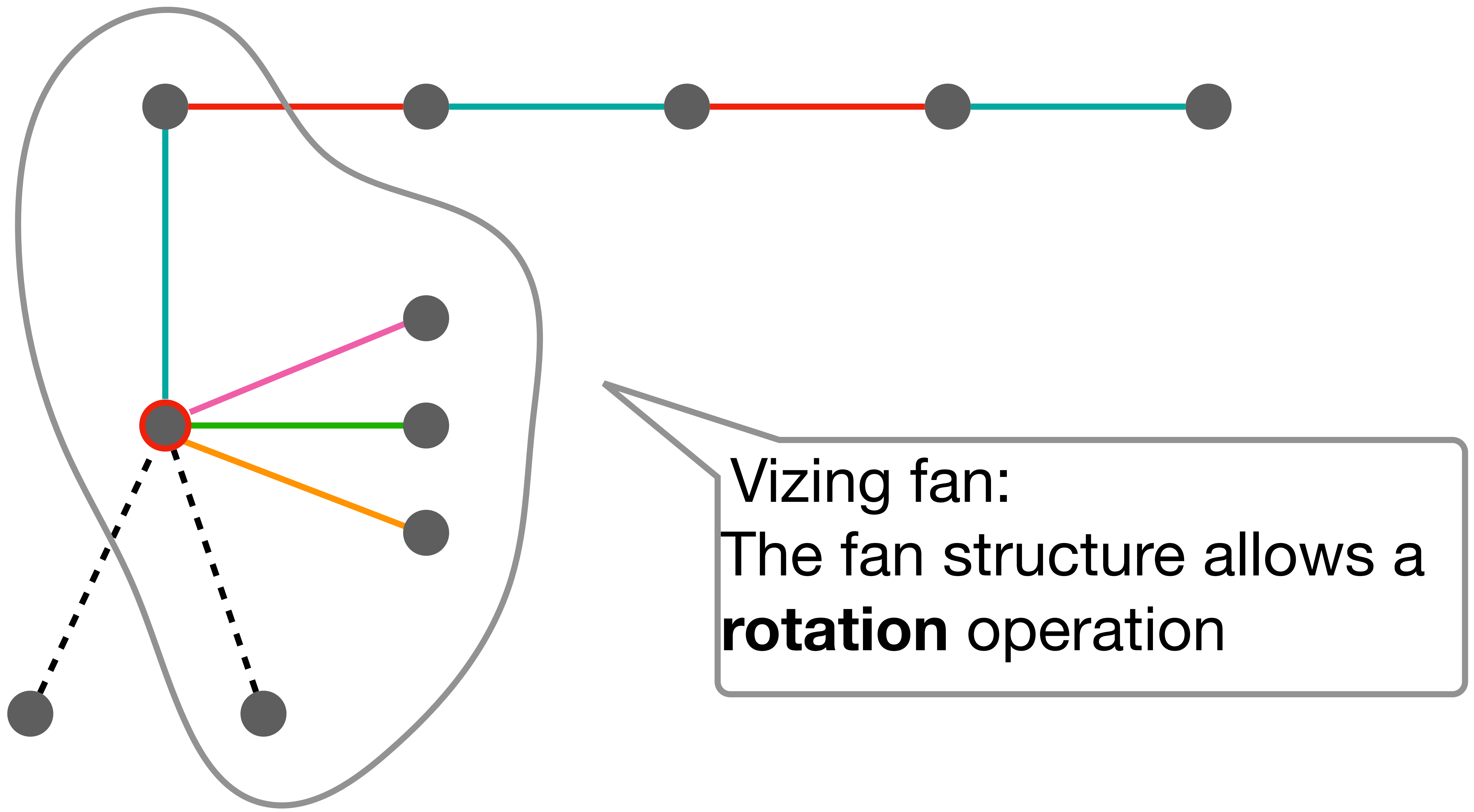

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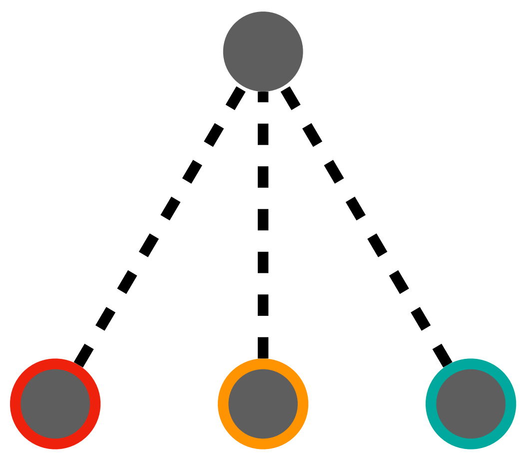

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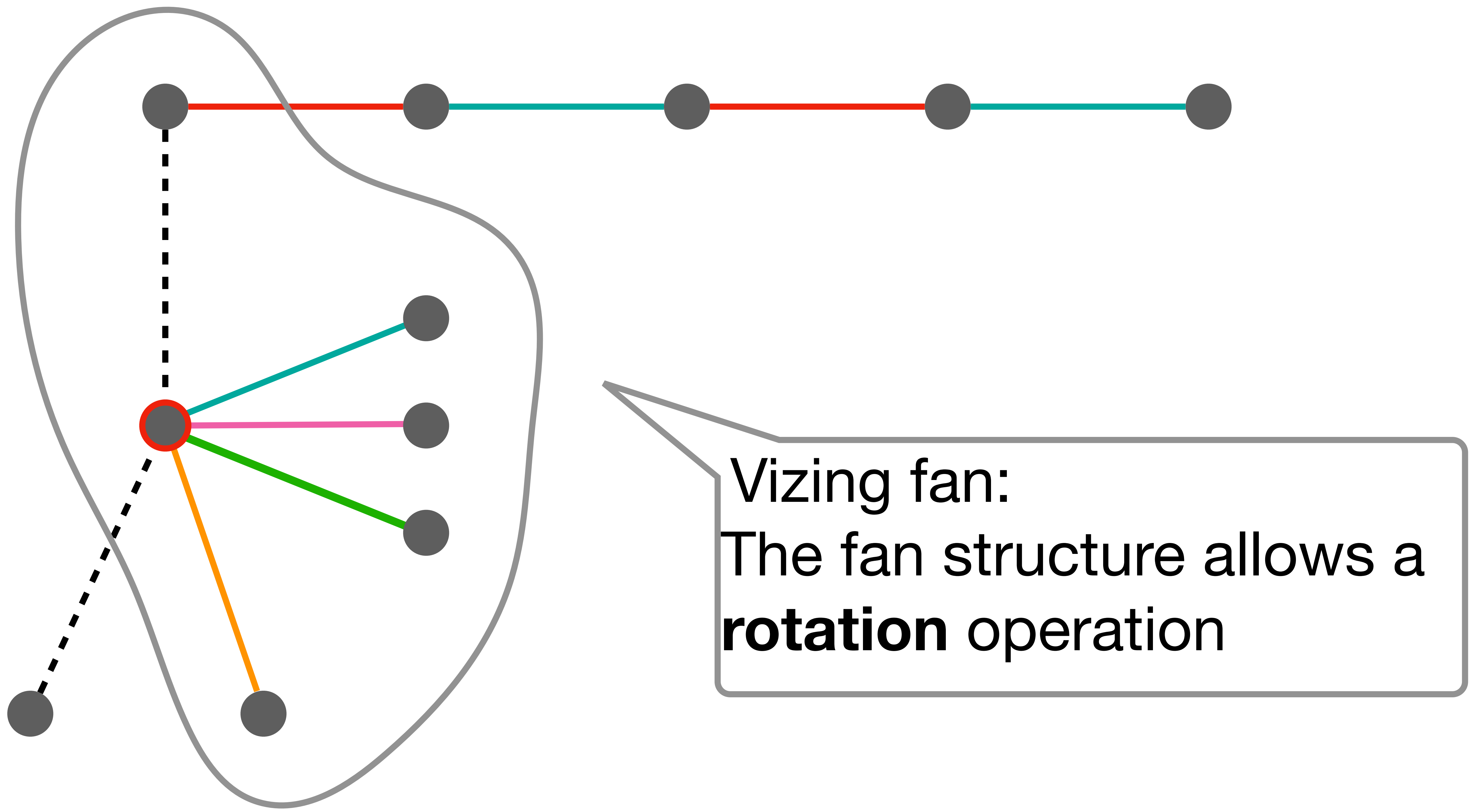

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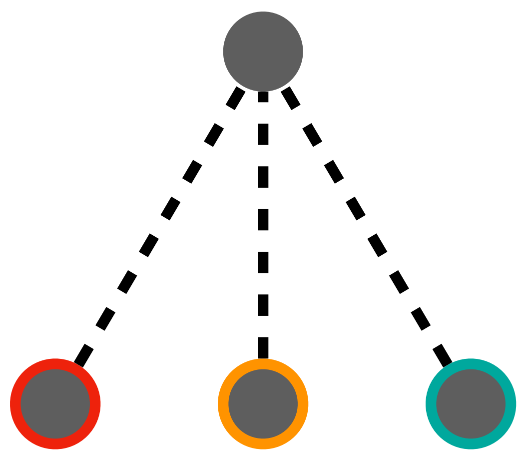

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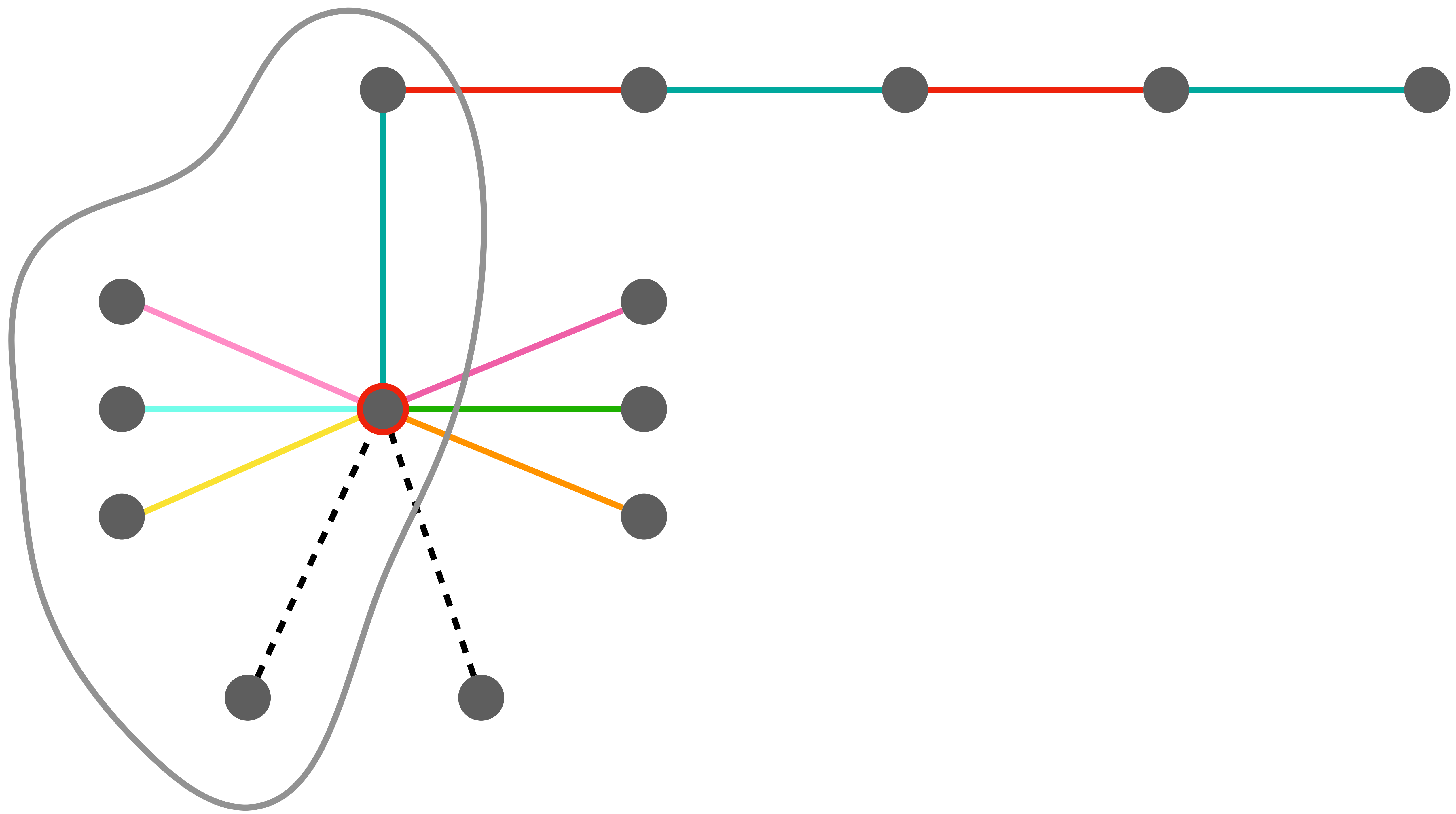

Vizing fan:
The fan structure allows a **rotation** operation

Coloring Stars in General Graphs

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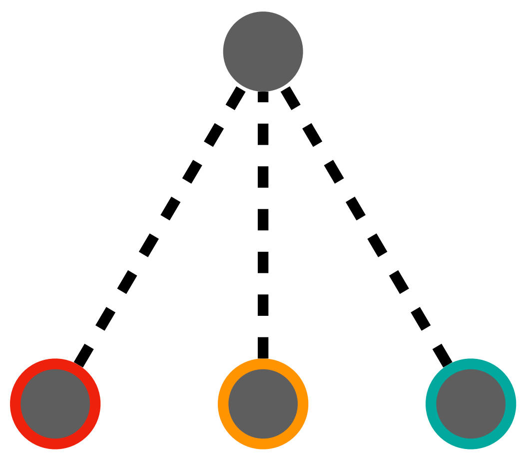

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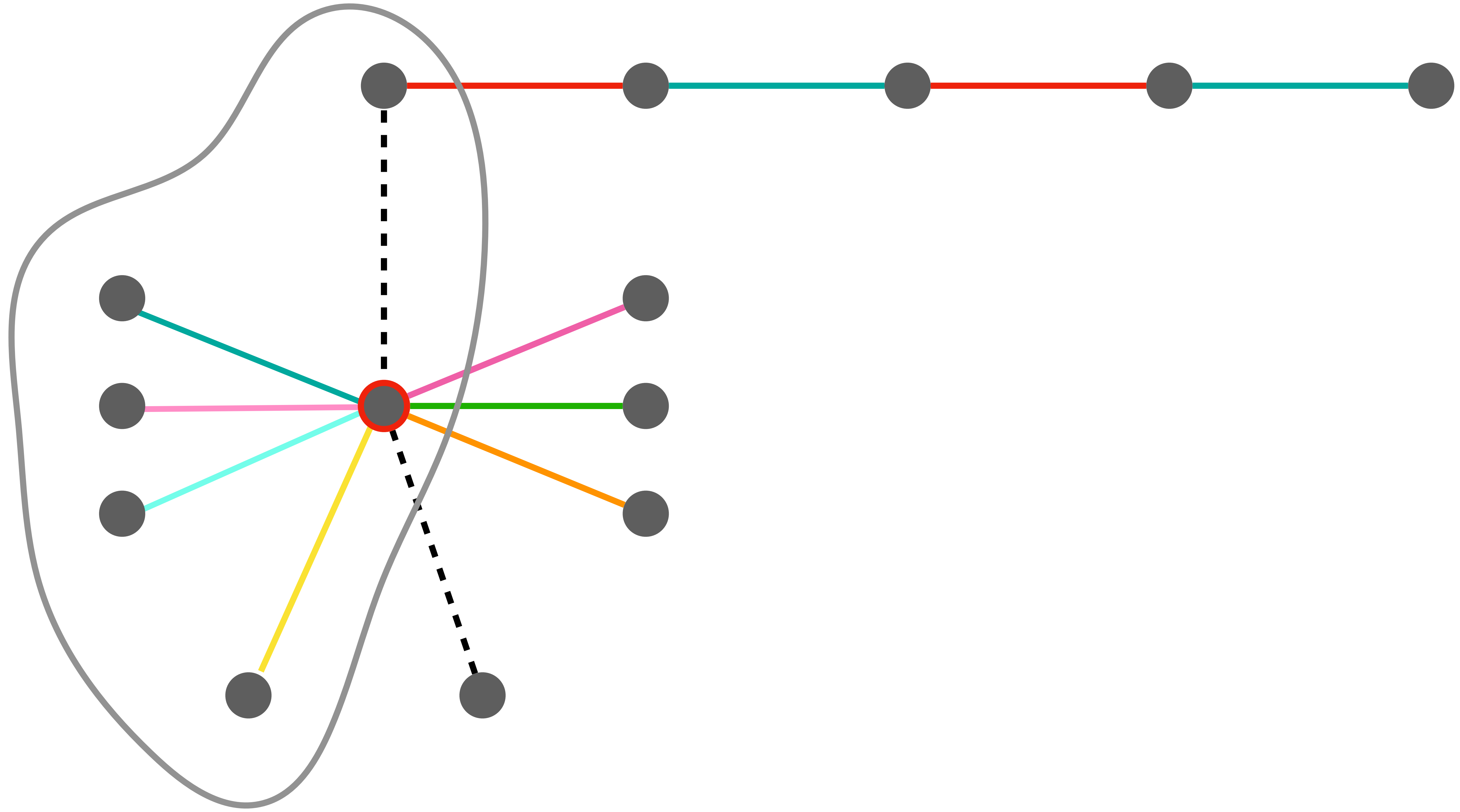

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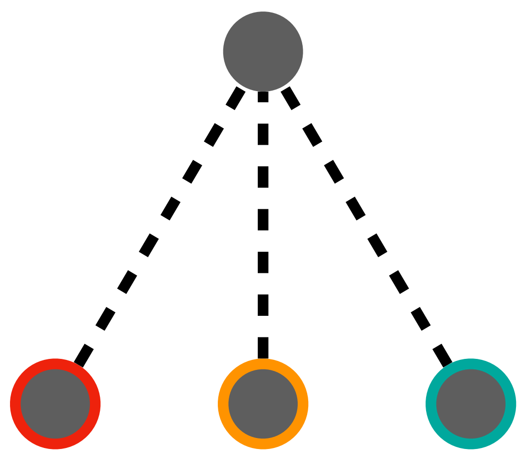

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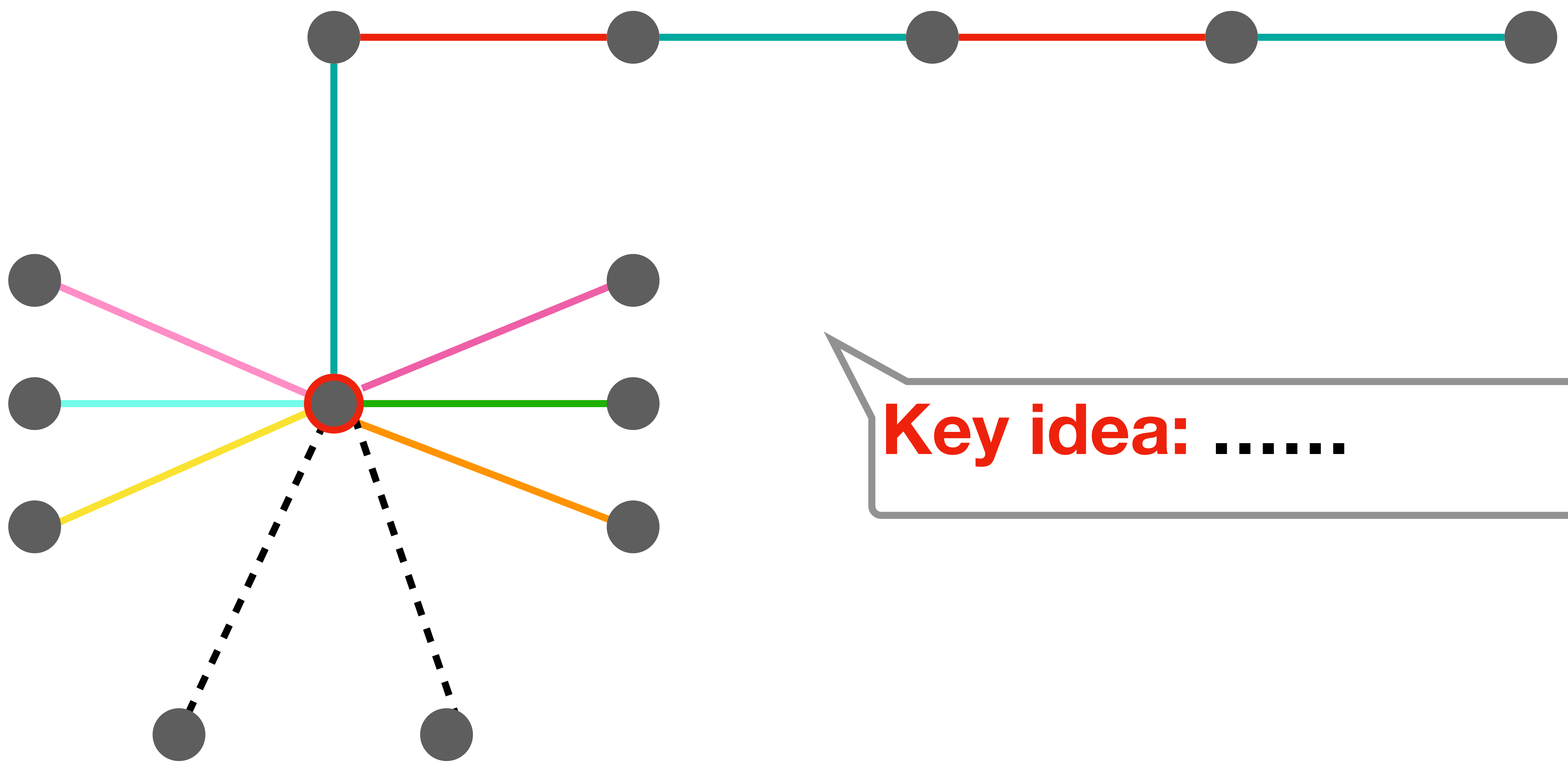

Key idea: ……

Coloring Stars in General Graphs

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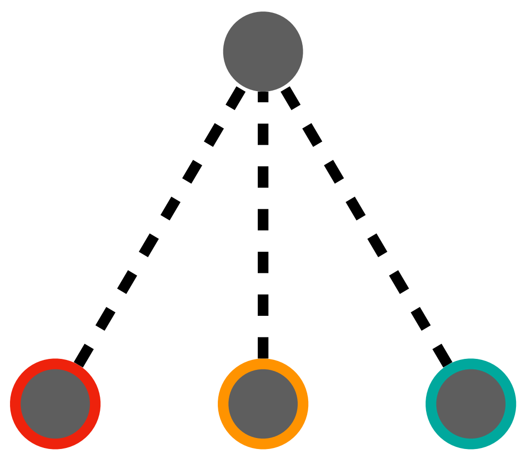

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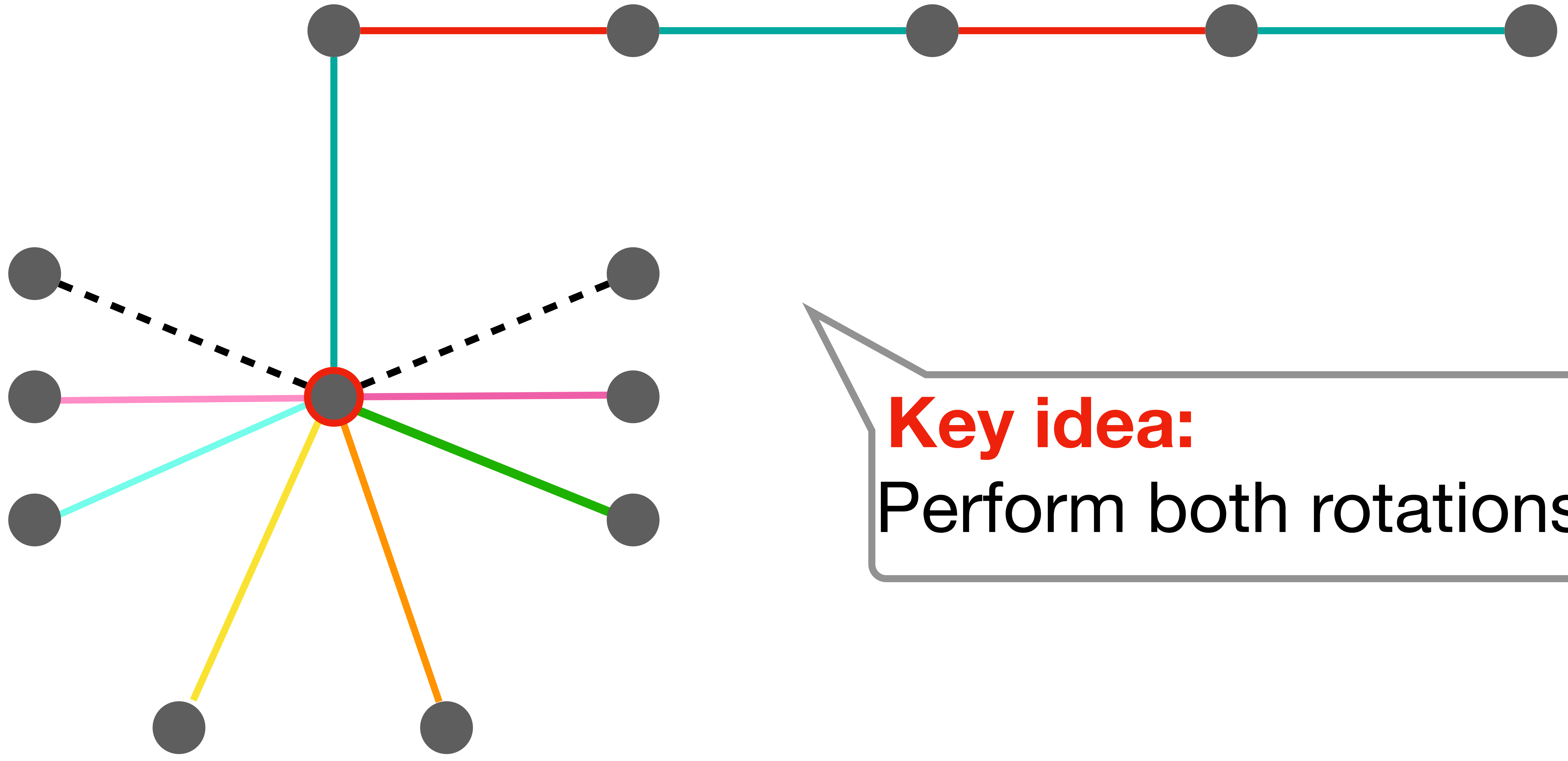

Key idea:
Perform both rotations except for the last edge

Coloring Stars in General Graphs

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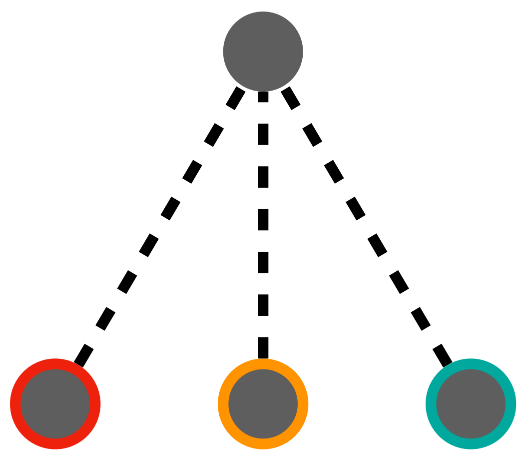

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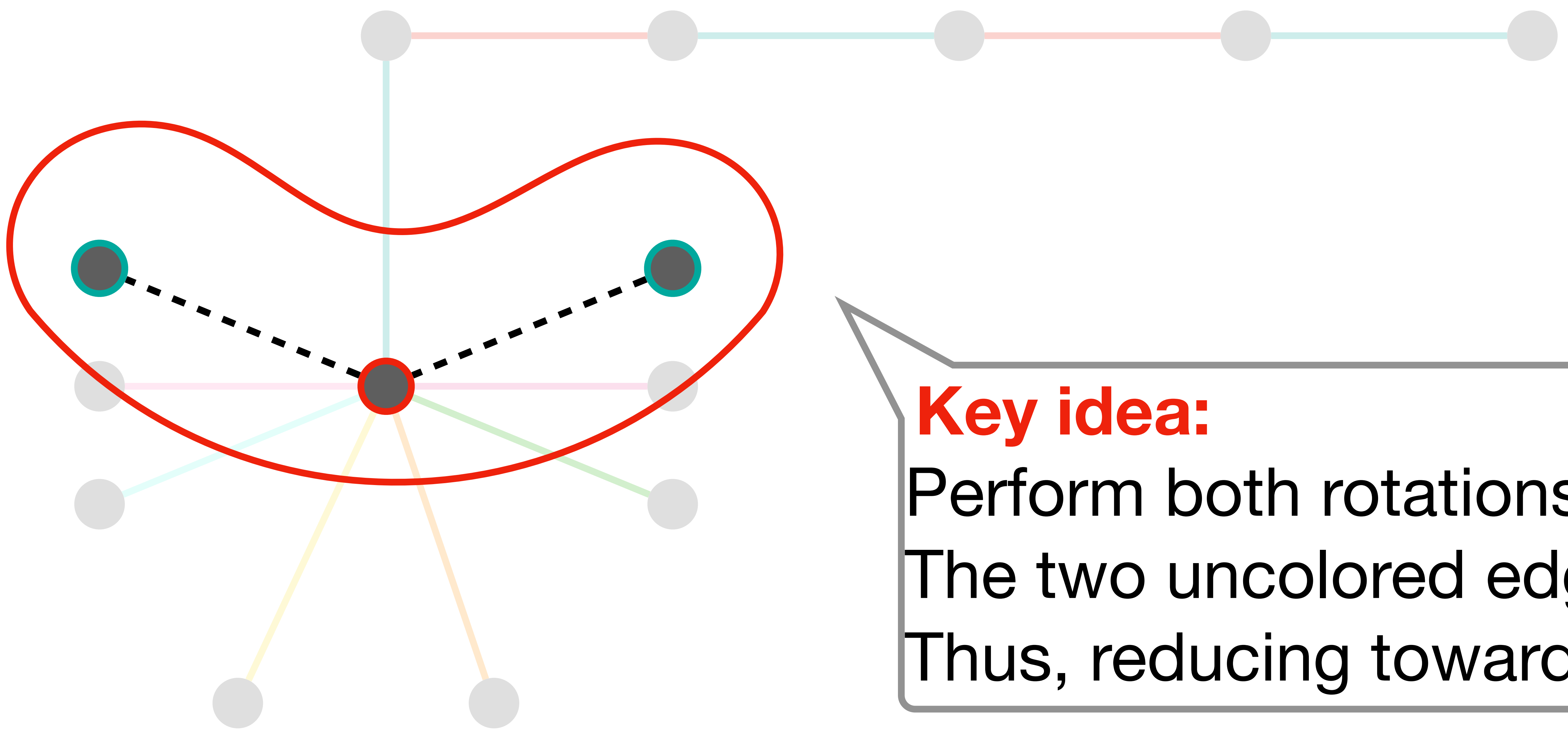

Key idea:
Perform both rotations except for the last edge
The two uncolored edges make a new **u-fan**
Thus, reducing towards the **bipartite** case

Conclusion

- Main result: $(\Delta + 1)$-edge coloring faster than the classical $mn^{1/2}$ bound
- Open question: Near-linear runtime?
- Other questions: Dynamic? Parallel?

Conclusion

- Main result: $(\Delta + 1)$-edge coloring faster than the classical $mn^{1/2}$ bound
- ~~Open question: Near-linear runtime?~~ Solved by more recent [ABBCSZ’24]
- Other questions: Dynamic? Parallel?