Improved Distance Sensitivity Oracles via Tree Partitioning

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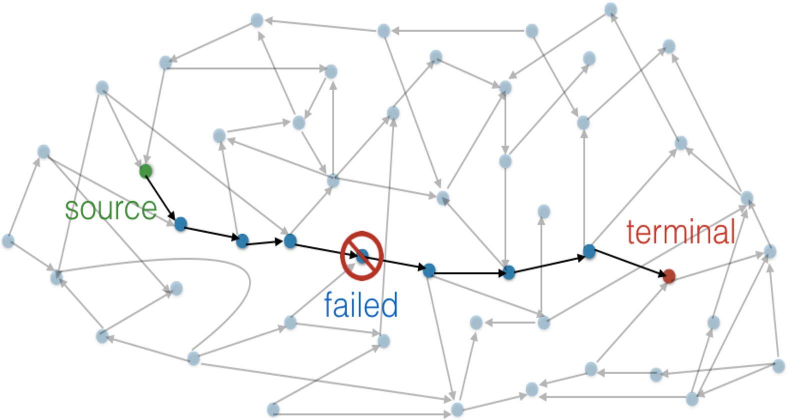
Tsinghua University

Distance sensitivity oralces (DSO)

Preprocess: given a directed graph $G = (V, E)$, edge weights $\omega : E \to \mathbb{R}^+$.
Query: $(s, t, f) \in V^3$
Answer: $\text{dist}_{G \setminus \{f\}}(s, t)$



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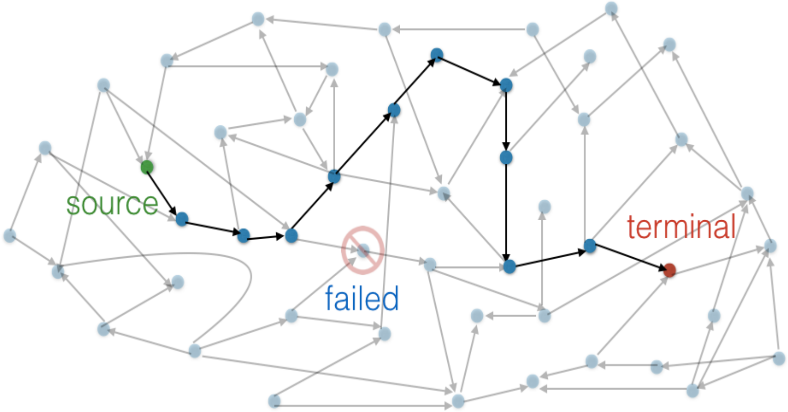

A short history

Define $n = |V|$, $m = |E|$; assume $\mathrm{Im}(\omega) = \{1, 2, \cdots, M\}$.

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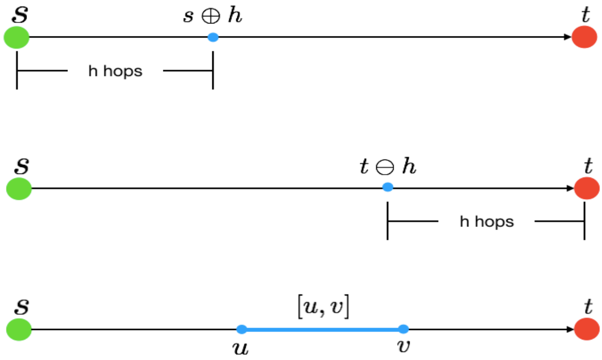
The set-intersection conjecture [PRT12] implies any reachability oracle with constant query time has space $\tilde{\Omega}(n^2)$. So it is not clear if our space upper bound is tight.

Notations

Definition

On shortest path $s \rightsquigarrow t$, for any $h > 0$, define $s \oplus h$ to be the vertex which is h hops after s, and define $t \ominus h$ to be the vertex which is h hops before t. Also, for any u, v, define interval $[u, v]$ in the natural way.



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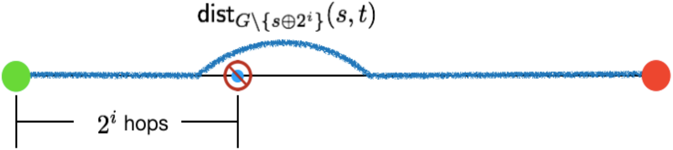

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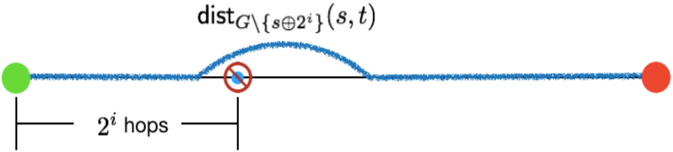

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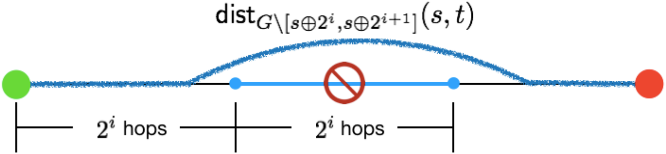

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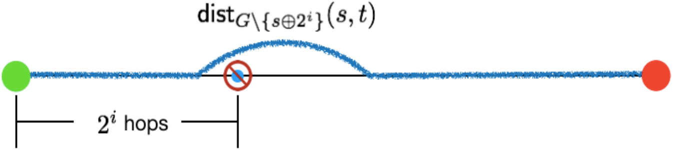

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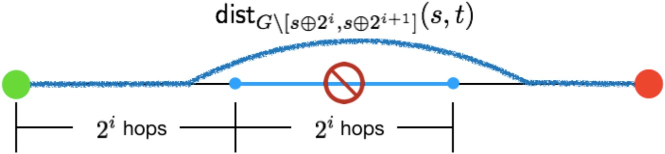

Space complexity: $O(n^2 \log n)$

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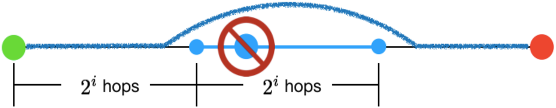

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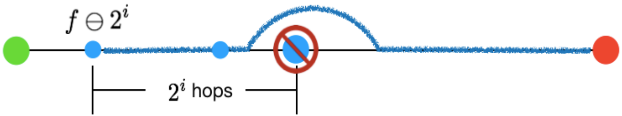

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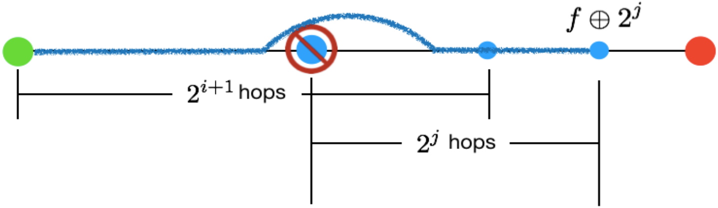

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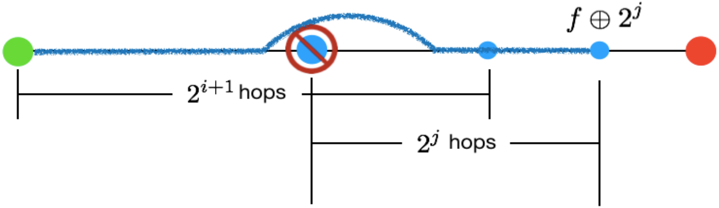

Query time: $O(1)$

Observations

Previous designs of DSO rely on sparse tables. Sparse table stores $O(\log n)$ entries for each pair of source & terminal, thus $\Omega(n^2 \log n)$ in total.

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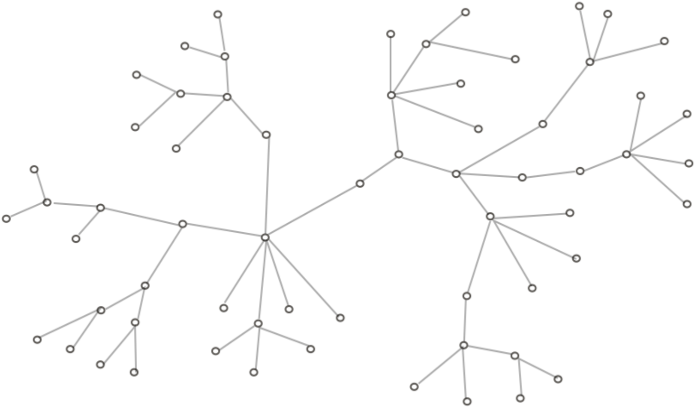
How to select special terminals?

Tree partition lemma

Lemma

Let T be a spanning tree on n vertices. For any integer $2 \leq k \leq n$, we can select a subset of $\leq 3k - 5$ vertices whose removal partitions T into subtrees of size $\leq n/k$.



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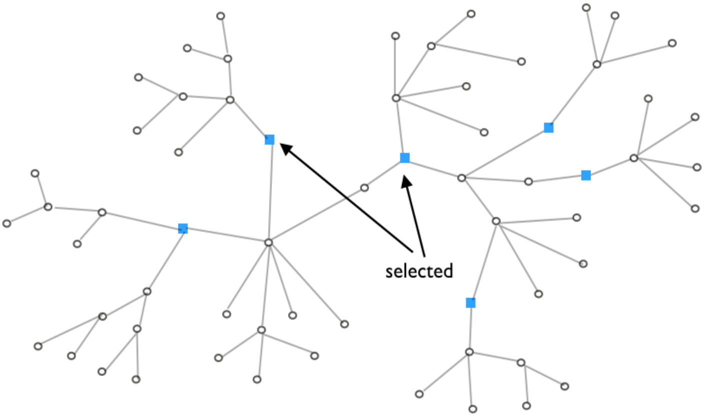

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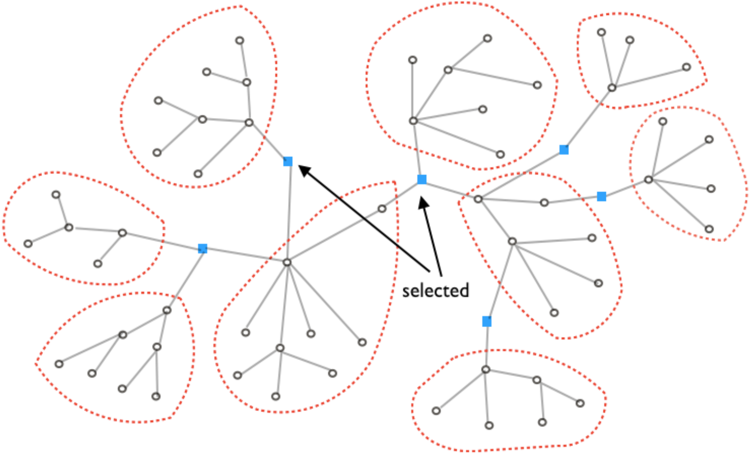
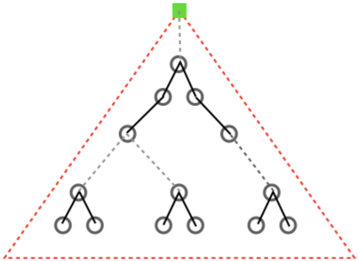
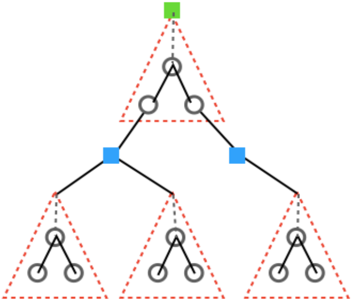


Partition single-source shortest paths trees

Data structure: For every $s \in V$, let T_s be the single-source shortest paths tree rooted at s, and apply Tree-partition Lemma on T_s with parameter n/L.



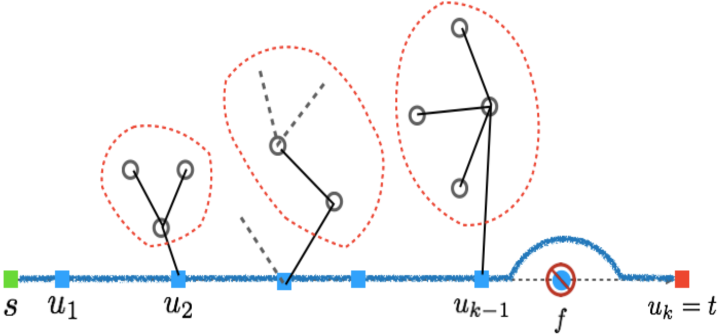
An SSSP tree.



Blue squares are selected vertices.

Sparser table:

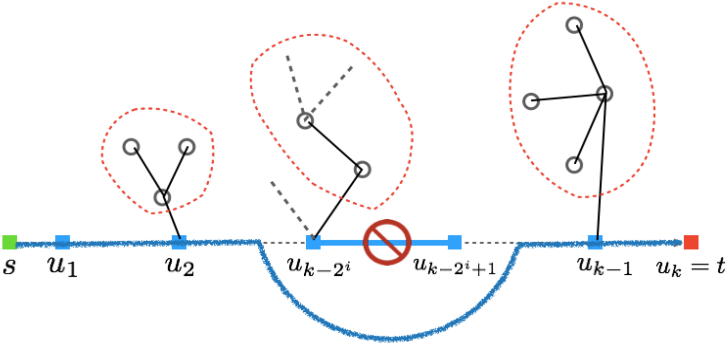
(i) For every selected t, $\text{dist}_{G \setminus \{f\}}(s, t)$.



$u_1, u_2, \cdots, u_k$ are selected in SSSP tree T_s.

Sparser table:

(ii) For every selected t, $\text{dist}_{G\setminus[u_{k-2^i},u_{k-2^i+1}]}(s,t)$.



$u_1, u_2, \cdots, u_k$ are selected in SSSP tree T_s.

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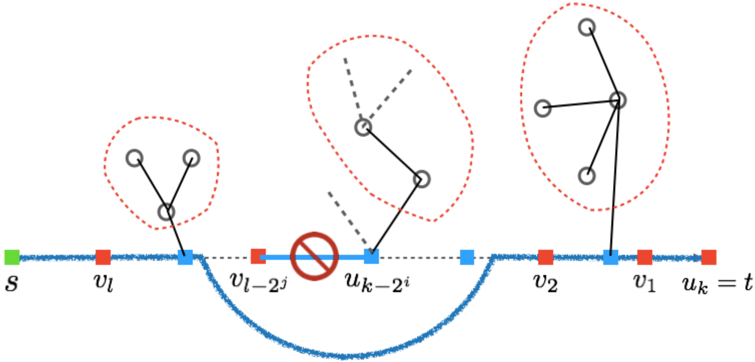

$u_1, u_2, \cdots, u_k$ are selected in SSSP tree T_s.
$v_1, v_2, \cdots, v_l$ are selected in **reverse** SSSP tree $\widehat{\mathrm{T}}_t$.

Sparser table

Space complexity: $O(n^2 \log^2 n / L)$

If $L = \log^2 n$, then the data structure so far occupies space $O(n^2)$.

t, f are not in the same subtree

Query algorithm: On input (s, t, f).

First consider when a **subtree-path** containing f is skipped over.

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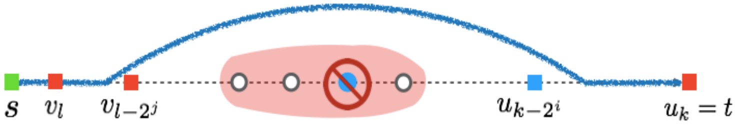

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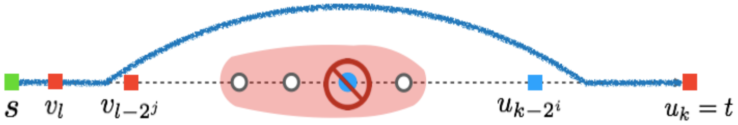

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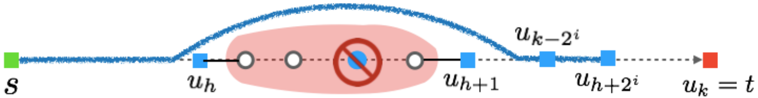

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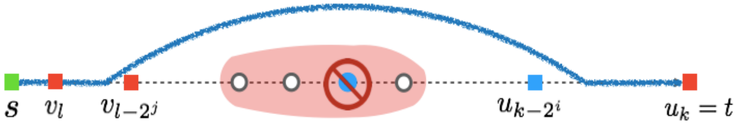

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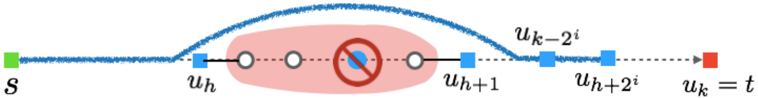

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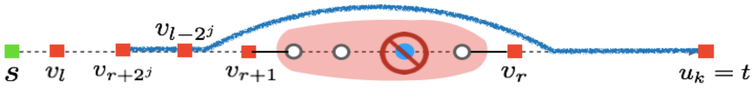

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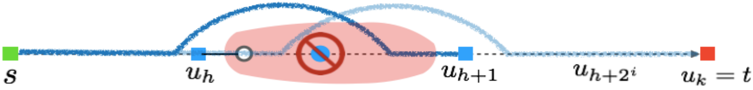

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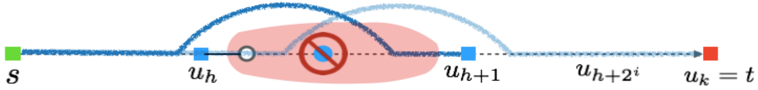

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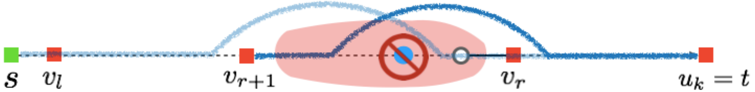

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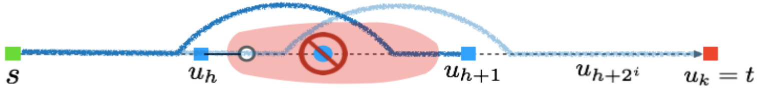

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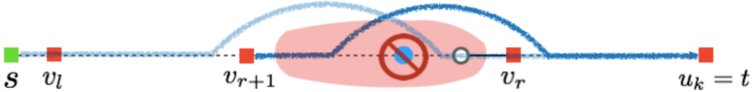

Now what if t, f are in the same subtree?

An $O(n^2 \log\log n)$-space DSO

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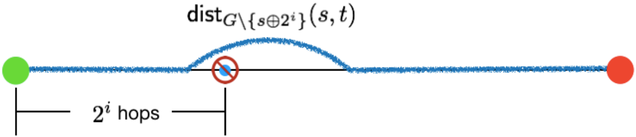

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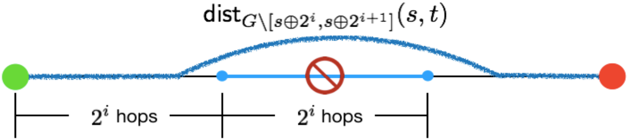

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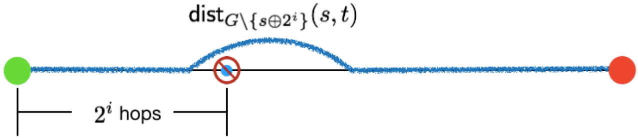

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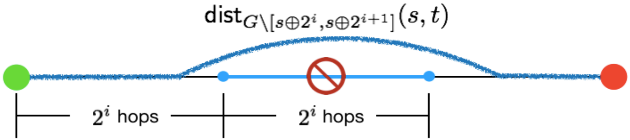

Space complexity: $O(n^2 \log L)$

An $O(n^2 \log\log n)$-space DSO

Query algorithm: Suppose t, f are in the same subtree.

Analysis from [DT02] still works!

Query time: $O(1)$

How to obtain an $O(n^2)$-space DSO?

Assume an $\Omega(\log n)$-RAM model.

Rough idea

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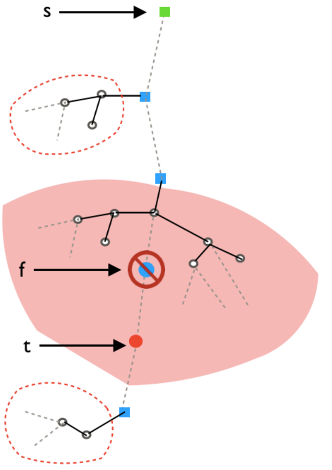
- Further partition subtrees into even smaller ones.
- Apply the bit-tricks ("Four Russians").

Two-level partition of SSSP trees

Hard cases: t, f lie in the same subtree.



Two-level partition of SSSP trees

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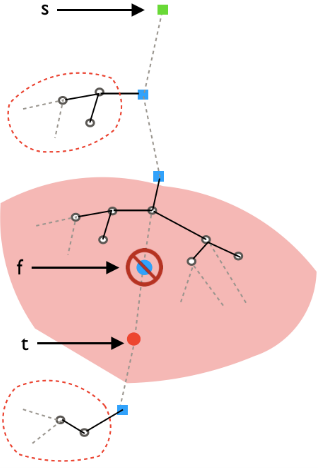
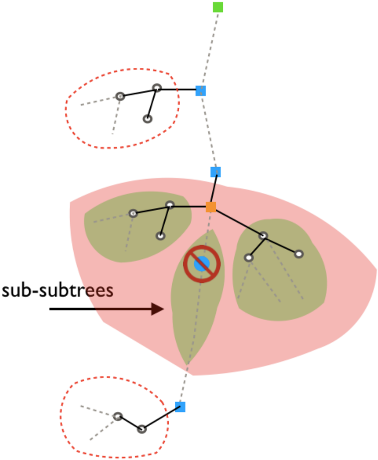
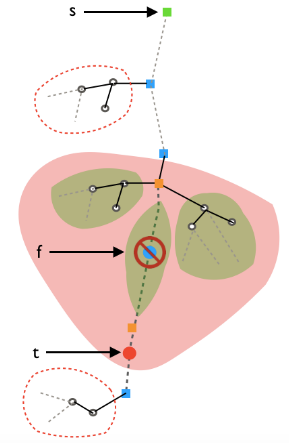


Two-level partition: Tree partition with $L' = \log^2 L$.



Two-level partition of SSSP trees

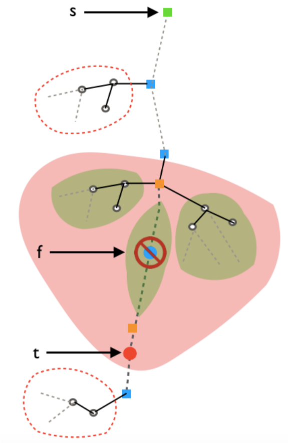
Easy cases: If t, f lie in different sub-subtrees, solve it using a truncated version of the data structure in previous slides.

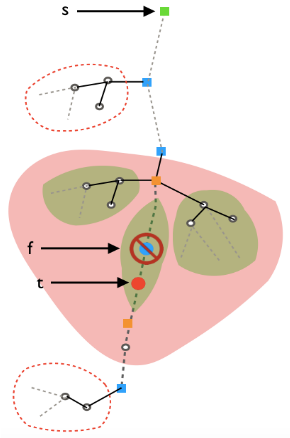


Two-level partition of SSSP trees

Easy cases: If t, f lie in different sub-subtrees, solve it using a truncated version of the data structure in previous slides.

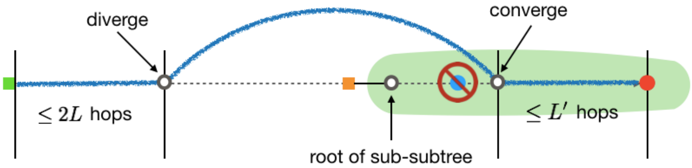
Hard cases: If t, f lie in the same sub-subtree, solve it later using the tabulation technique ("Four Russians").





Conclude with bit-tricks

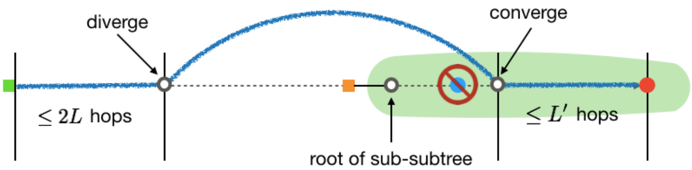
Use an $o(n)$-space table to store **all** replacement paths of the following kind; don't care about other replacement paths.



- The replacement path is encoded as:
  (1) # hops before diverge; (2) # hops after converge.

Conclude with bit-tricks

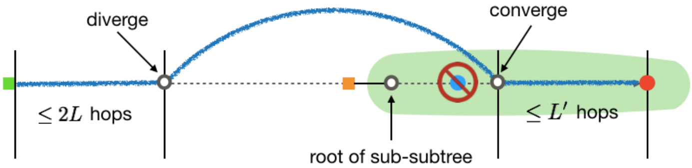
Use an $o(n)$-space table to store **all** replacement paths of the following kind; don't care about other replacement paths.



- The replacement path is encoded as: (1) # hops before diverge; (2) # hops after converge.
- (1)(2) can be stored in $\log(2L \cdot L')$-bits.

Conclude with bit-tricks

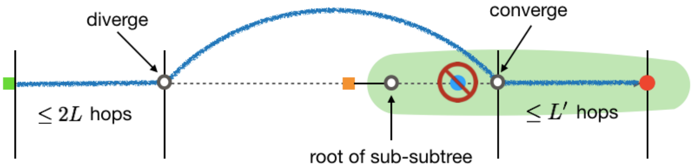
Use an $o(n)$-space table to store **all** replacement paths of the following kind; don't care about other replacement paths.



- Configuration of this sub-subtree can be stored in $O((L')^2 \cdot \log(2L \cdot L')) = O(\log \log^5 n)$ bits.

Conclude with bit-tricks

Use an $o(n)$-space table to store **all** replacement paths of the following kind; don't care about other replacement paths.



- Configuration of this sub-subtree can be stored in $O((L')^2 \cdot \log(2L \cdot L')) = O(\log\log^5 n)$ bits.
- All possible sub-subtree configurations can be stored in an indexable table of size $2^{O(\log\log^5 n)} = o(n)$.

Thanks for listening

References I

Aaron Bernstein and David Karger.
Improved distance sensitivity oracles via random sampling.
In *Proceedings of the nineteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 34–43. Society for Industrial and Applied Mathematics, 2008.

Aaron Bernstein and David Karger.
A nearly optimal oracle for avoiding failed vertices and edges.
In *Proceedings of the forty-first annual ACM symposium on Theory of computing*, pages 101–110. ACM, 2009.

Camil Demetrescu and Mikkel Thorup.
Oracles for distances avoiding a link-failure.
In *Proceedings of the thirteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 838–843. Society for Industrial and Applied Mathematics, 2002.

References II

Fabrizio Grandoni and Virginia Vassilevska Williams.
Improved distance sensitivity oracles via fast single-source replacement paths.
In *Foundations of Computer Science (FOCS), 2012 IEEE 53rd Annual Symposium on*, pages 748–757. IEEE, 2012.

Mihai Patrascu, Liam Roditty, and Mikkel Thorup.
A new infinity of distance oracles for sparse graphs.
In *Foundations of Computer Science (focs), 2012 Ieee 53rd Annual Symposium on*, pages 738–747. IEEE, 2012.