Nearly 2-Approximate Distance Oracles in Sub-quadratic Time

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All-Pairs Shortest Paths

- $G = (V, E)$, **un-**directed **un-**weighted
  n vertices & *m* edges
- **Exact** answers require $n^{2.373}$ time [Seidel, 1992]
- Faster running time needs **approx**imations

$$\text{dist}(u, v) \le \text{est}(u, v) \le a \cdot \text{dist}(u, v) + b$$

All-Pairs Shortest Paths

- Some classic results on approximate APSP

running time	(a, b)	reference
$\tilde{O}(n^{7/3})$	**(1, 2)**	[Aingworth, Checkrui, Motwani, 1996]
$\tilde{O}(n^2)$	**(3, 0)**	[Dor, Halperin, Zwick, 2000]
$\tilde{O}(n^2)$	**(2, 1)**	[Berman & Kasiviswanathan, 2007]

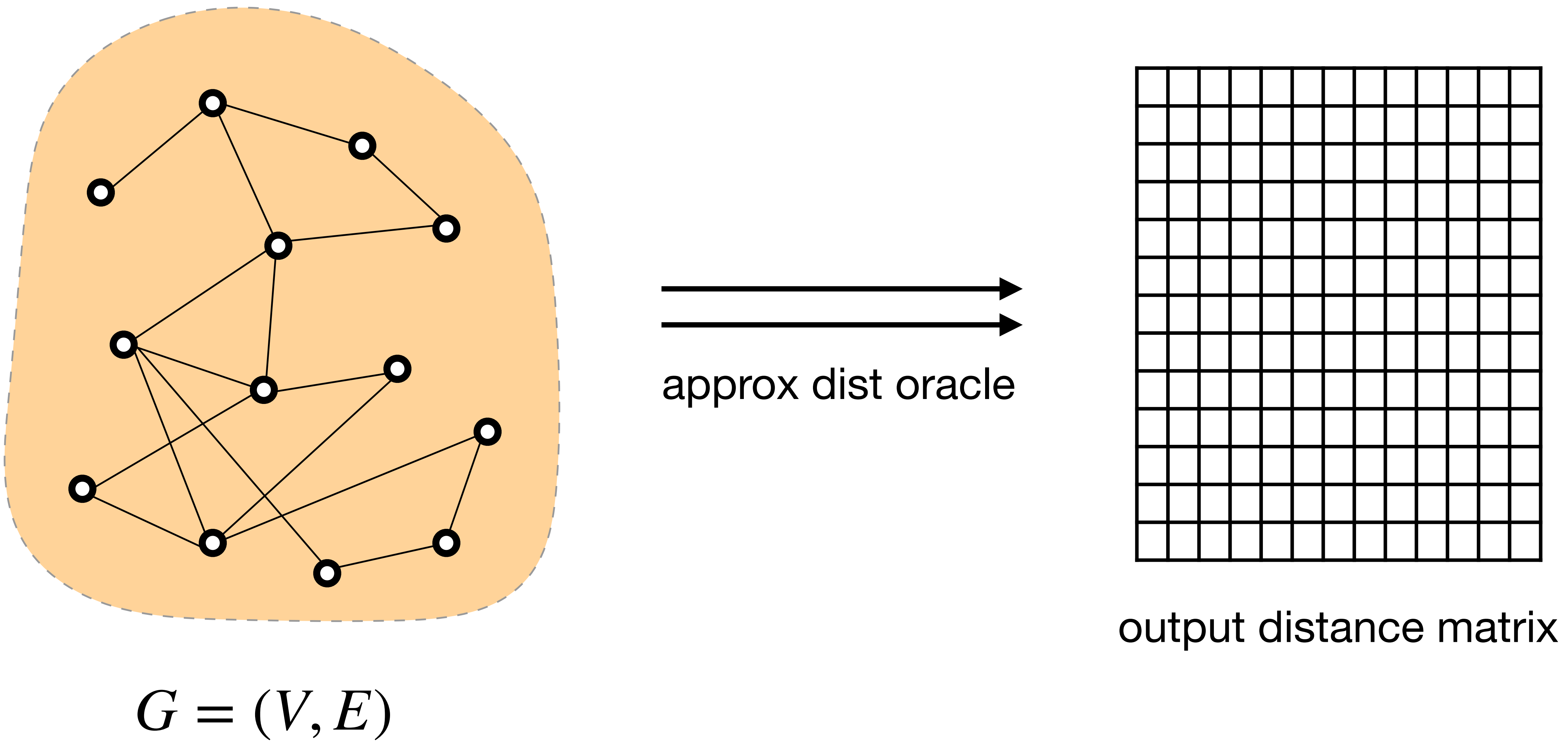
All-Pairs Shortest Paths

- Reduction to approximate distance oracles
- **Approximate Distance Oracles**
  - **Preprocess** input graph $G = (V, E)$
  - **Query** $u, v \in V$, return est(u, v) in $O(1)$ **time**

$$\text{dist}(u, v) \leq \text{est}(u, v) \leq a \cdot \text{dist}(u, v) + b$$

All-Pairs Shortest Paths

- Reduction to approximate distance oracles



All-Pairs Shortest Paths

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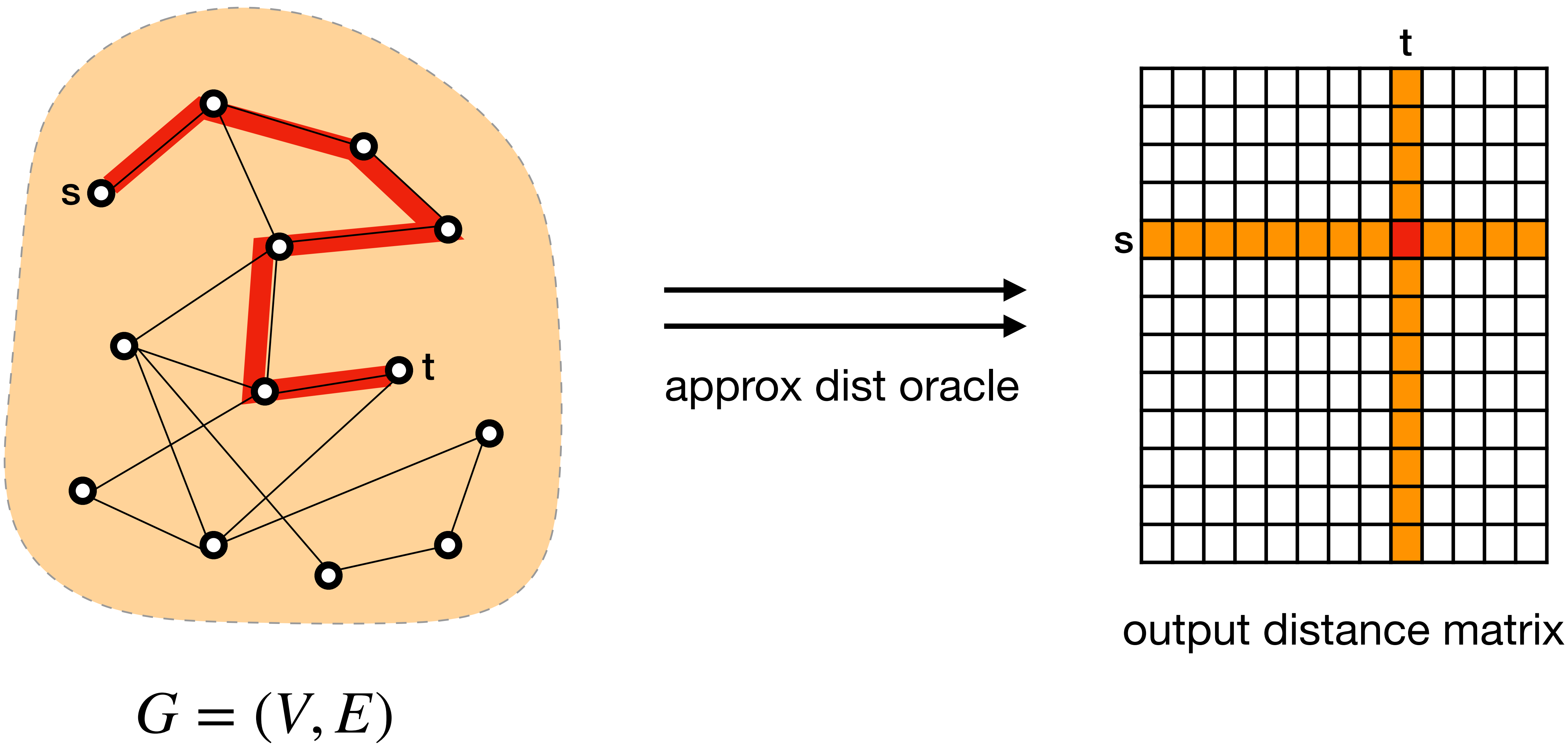

All-Pairs Shortest Paths

- Reduction to approximate distance oracles
- Total runtime = Preprocess + $n^2 \times$ Query
  = Preprocess + $O(n^2)$
- Change of setting:
  - Count preprocessing, but ignore query overhead
  - **Implicit representation** of distance matrix
  - Possibly faster than $O(n^2)$ time

Approximate Distance Oracles

- Approx-DOs with **large** stretch (error)

preprocessing time	(a, b)	space	reference
$O(mn^{1/k})$	(2k-1, 0)	$O(kn^{1+1/k})$	[Thorup & Zwick, 2005]
$O(n^2)$	(2k-1, 0)	$O(kn^{1+1/k})$	[Baswana & Sen, 2006]
$O(\sqrt{k}m + kn^{1+O(1/\sqrt{k})})$	(2k-1, 0)	$O(kn^{1+1/k})$	[Wulff-Nilsen, 2012]

Approximate Distance Oracles

- Approx-DOs with **small** stretch (error)

preprocessing time	(a, b)	space	reference
$\tilde{O}(n^2)$	(2, 3)	$\tilde{O}(n^{5/3})$	[Baswana et al, 2005]
$\text{poly}(n)$	(2, 1)	$\tilde{O}(n^{5/3})$	[Pătraşcu & Roditty, 2010]
∞	(2, 1)	$\tilde{\Omega}(n^{1.5})$	[Pătraşcu & Roditty, 2010]
∞	(2.33, 0)	$\tilde{\Omega}(n^{5/3})$	[Pătraşcu, Roditty, Thorup, 2012]
$\tilde{O}(n^2)$	(2, 1)	$\tilde{O}(n^{5/3})$	[Sommor, 2016]
$\tilde{O}(m + n^{23/12})$	(3, 8)	$\tilde{O}(n^{3/2})$	[Baswana et al, 2008]
$\tilde{O}(m + n^{2-\Omega(\epsilon)})$	$(2 + \epsilon,\ 5)$	$\tilde{O}(n^{11/6})$	[Akav & Roditty, 2020]

Our results

- Approx-DOs with **small** stretch (error)

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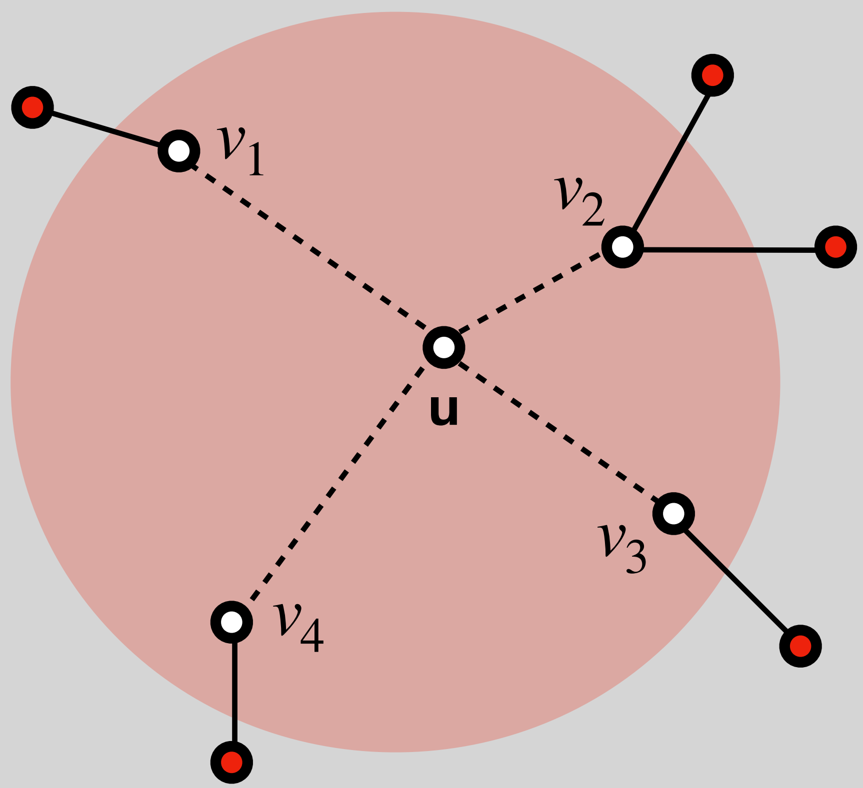
$\tilde{O}(m + n^{1.987})$	(2, 3)	$\tilde{O}(n^{5/3})$	new
$\tilde{O}(m + n^{7/4+O(\epsilon)})$	$(2, 1/\epsilon)$	$\tilde{O}(n^{5/3})$	new
$\tilde{O}(m + n^{5/3+O(\epsilon)})$	$(2 + \epsilon, c(\epsilon))$	$\tilde{O}(n^{5/3})$	new

Overview of [Akav & Roditty, 2020]

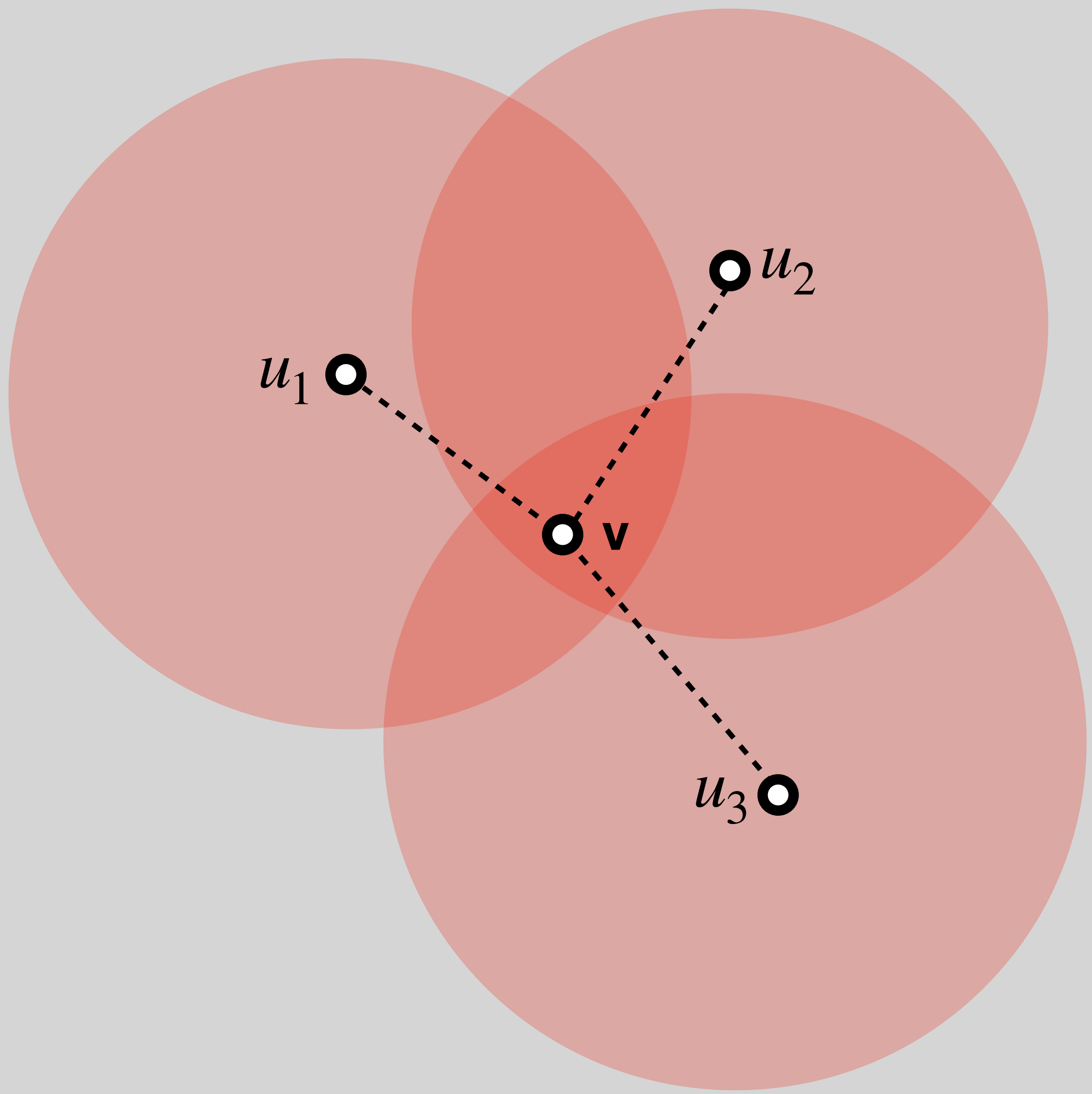
Definition: Balls & Clusters

- Let $S \subseteq V$ be a vertex subset

$\mathbf{B}_S(u) = \{v \mid \text{dist}(u, v) < \text{dist}(u, S)\}$



$\mathbf{C}_S(v) = \{u \mid v \in \mathbf{B}_S(u)\}$

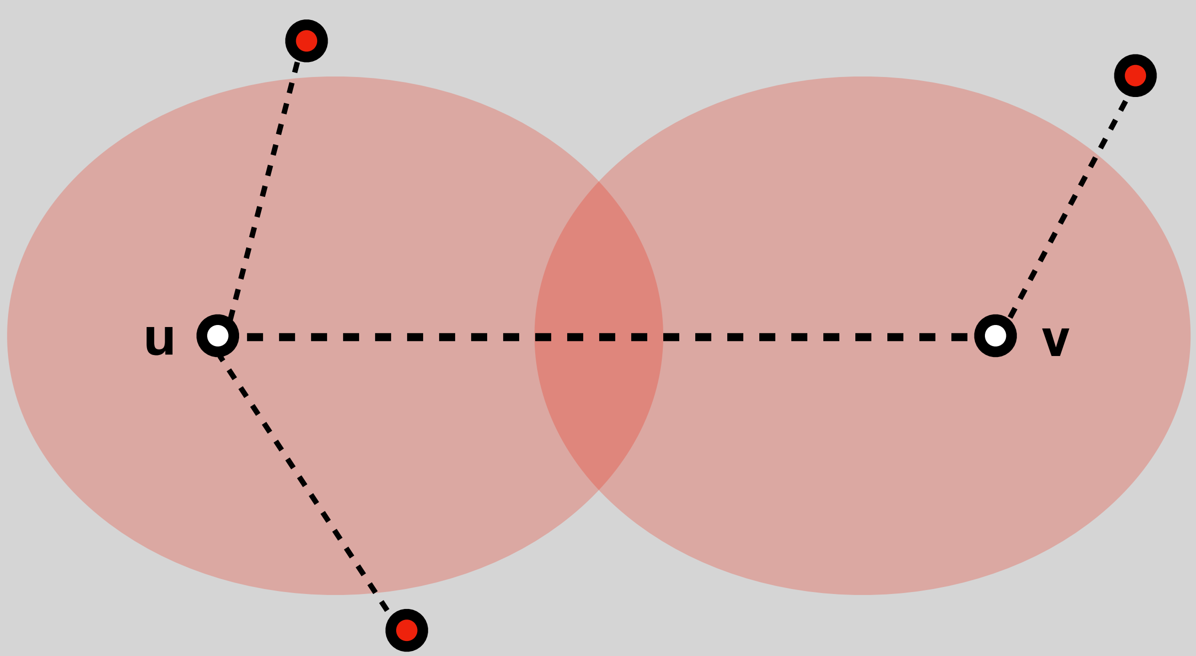


A dichotomy

- Take a random $S \subseteq V$ of size $n^{1-\beta}$ for some $\beta > 1/2$
- Query the distance between $u, v \in V$
- There are two complementary cases below

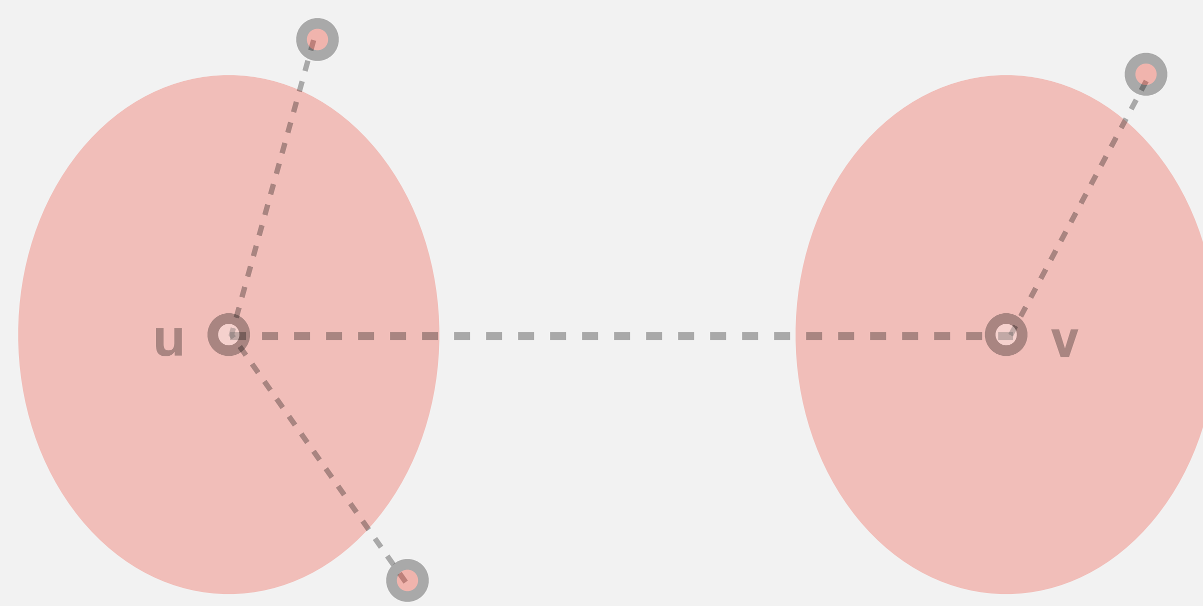
Case 1: **intersecting S-balls**

$\mathrm{B}_S(u) \cap \mathrm{B}_S(v) \neq \varnothing$



Case 2: **disjoint S-balls**

$\mathrm{B}_S(u) \cap \mathrm{B}_S(v) = \varnothing$



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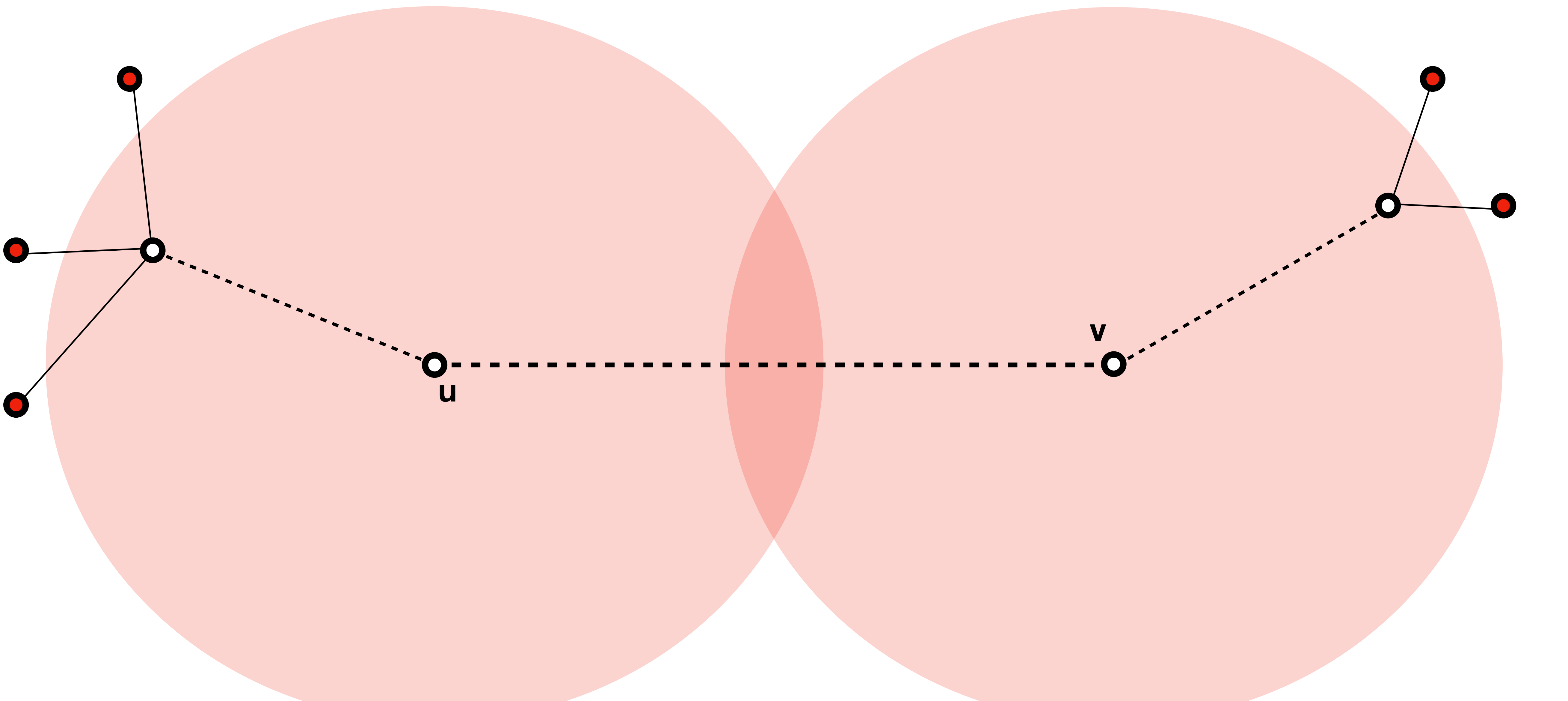
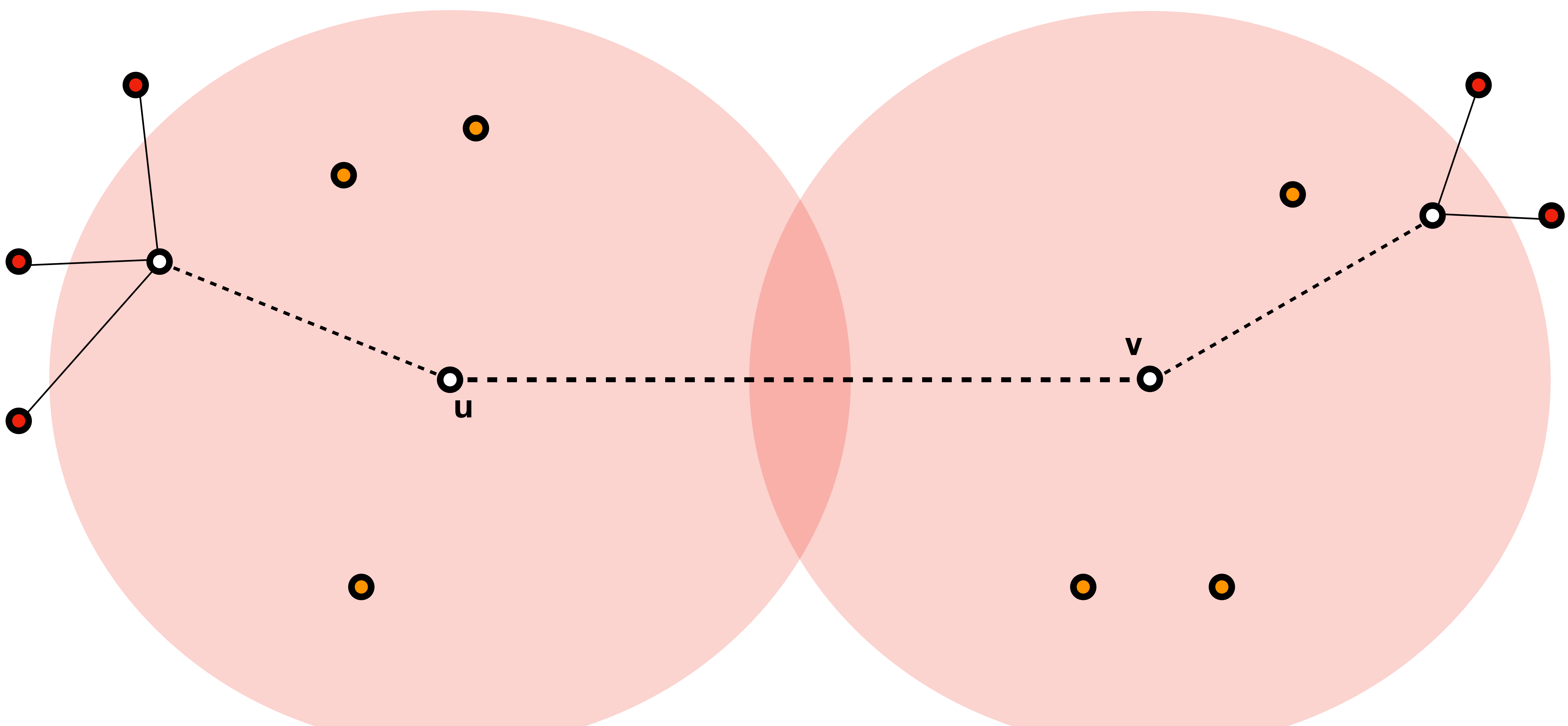


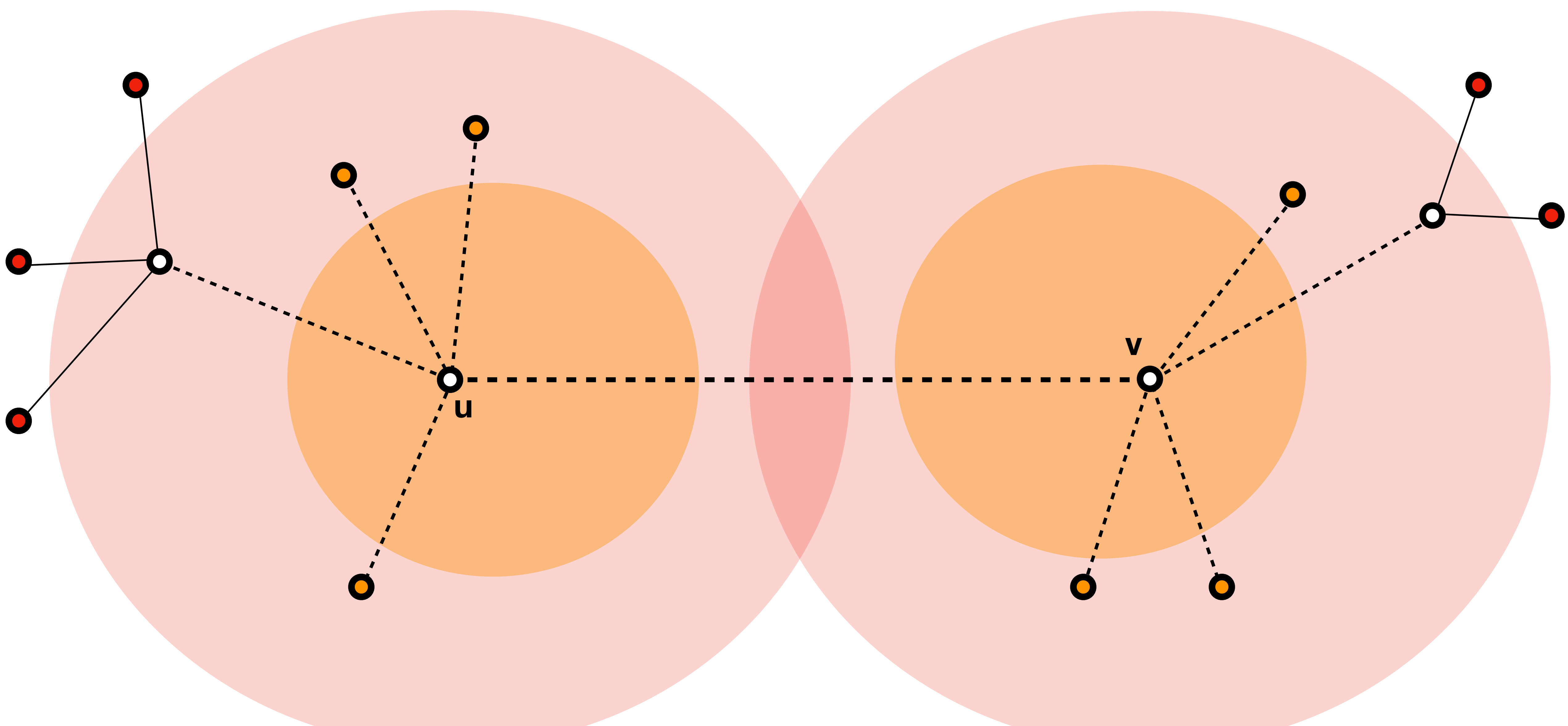
Case 1: intersecting S-balls

- Take a random $T \subseteq V$ of size $n^{1-\alpha}$, $\alpha < 1/2 < \beta$



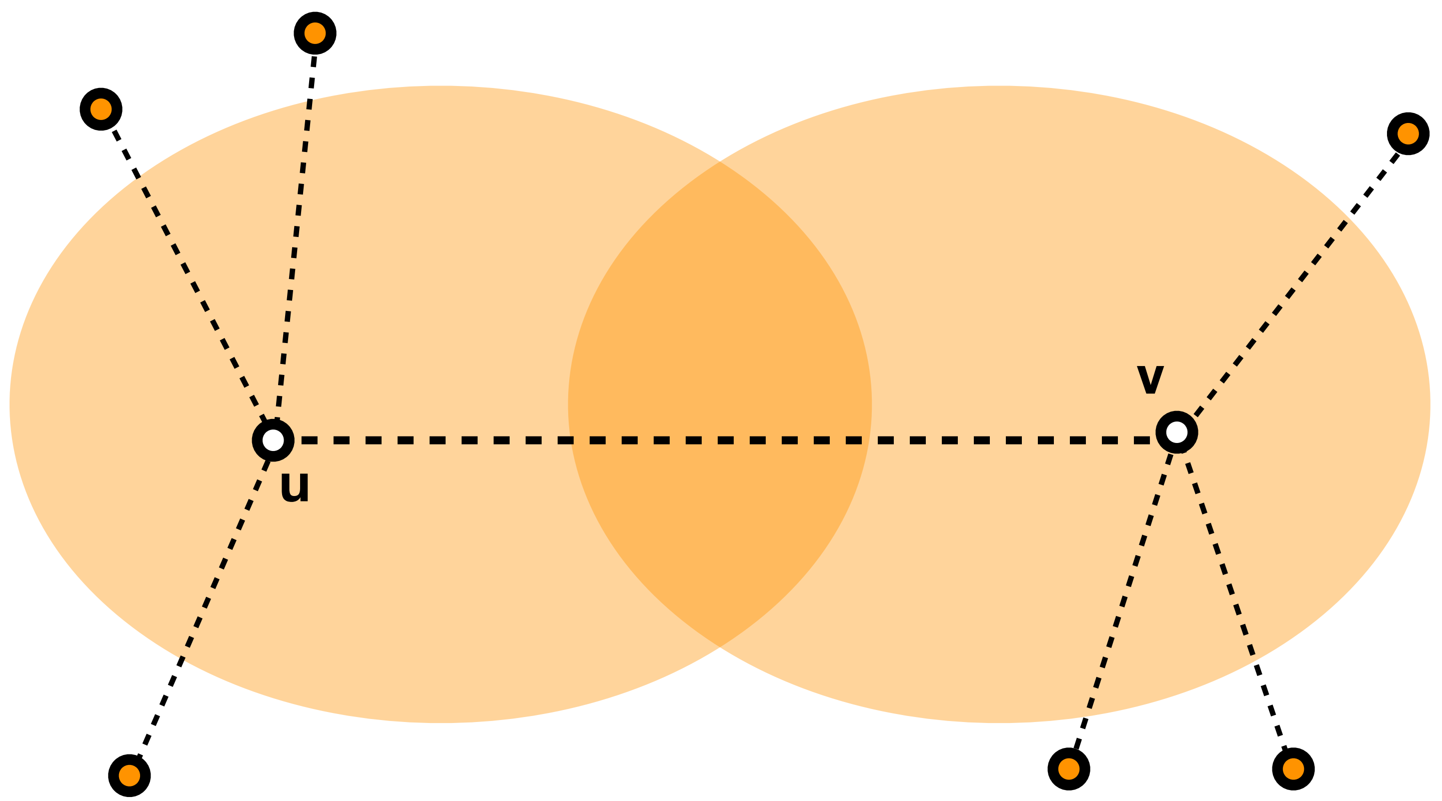
Case 1: intersecting S-balls

- Take a random $T \subseteq V$ of size $n^{1-\alpha}$, $\alpha < 1/2 < \beta$
- Consider two smaller balls $\mathsf{B}_T(u), \mathsf{B}_T(v)$



Case 1: intersecting S-balls

- **Easy sub-case:** $\mathsf{B}_T(u) \cap \mathsf{B}_T(v) \neq \emptyset$



<u>Case 1:</u> intersecting S-balls

- **Easy sub-case:** $\mathsf{B}_T(u) \cap \mathsf{B}_T(v) \neq \emptyset$

- **Solution:** Explicitly store all sets $\mathsf{B}_T(u) \cap \mathsf{B}_T(v)$,
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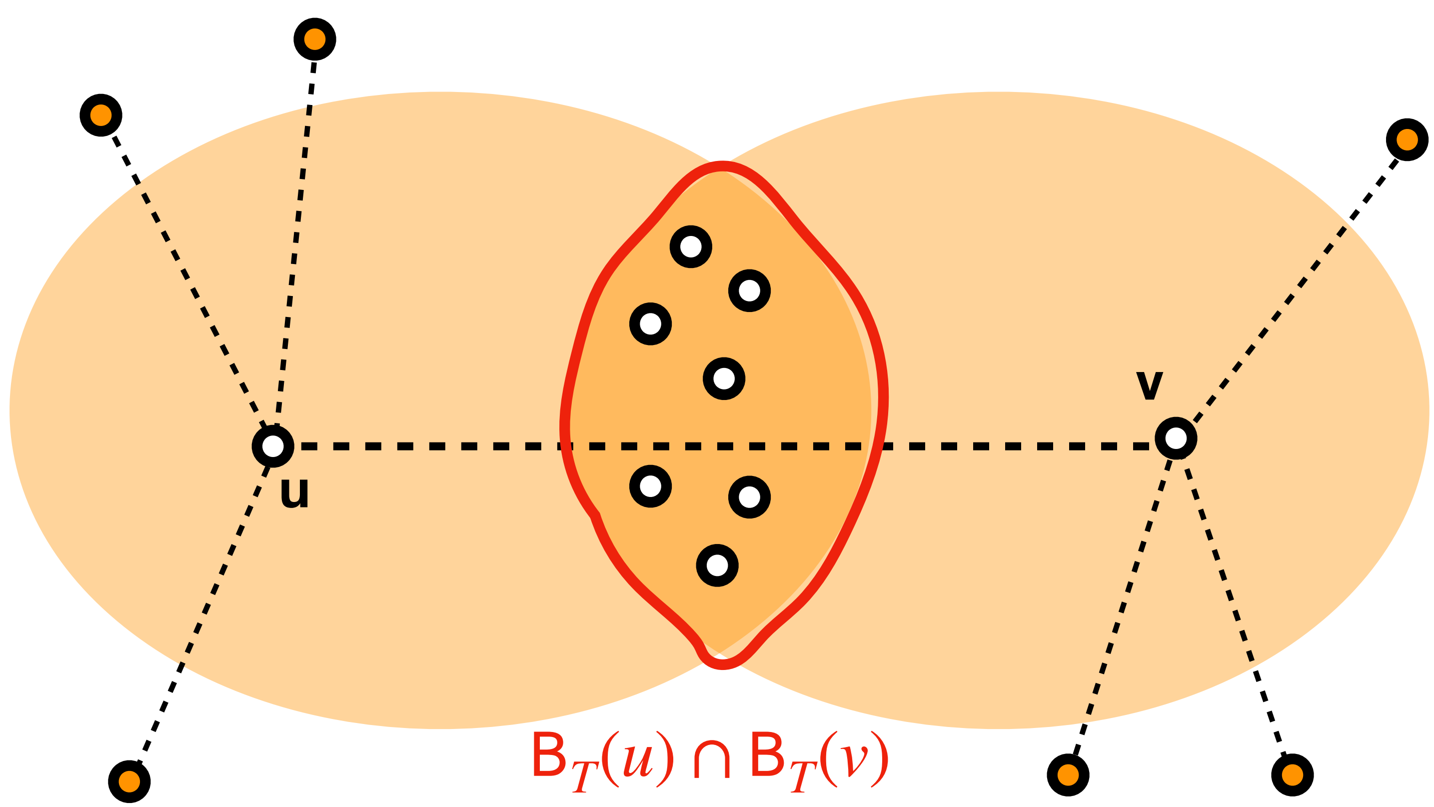

$\mathsf{B}_T(u) \cap \mathsf{B}_T(v)$

Case 1: intersecting S-balls

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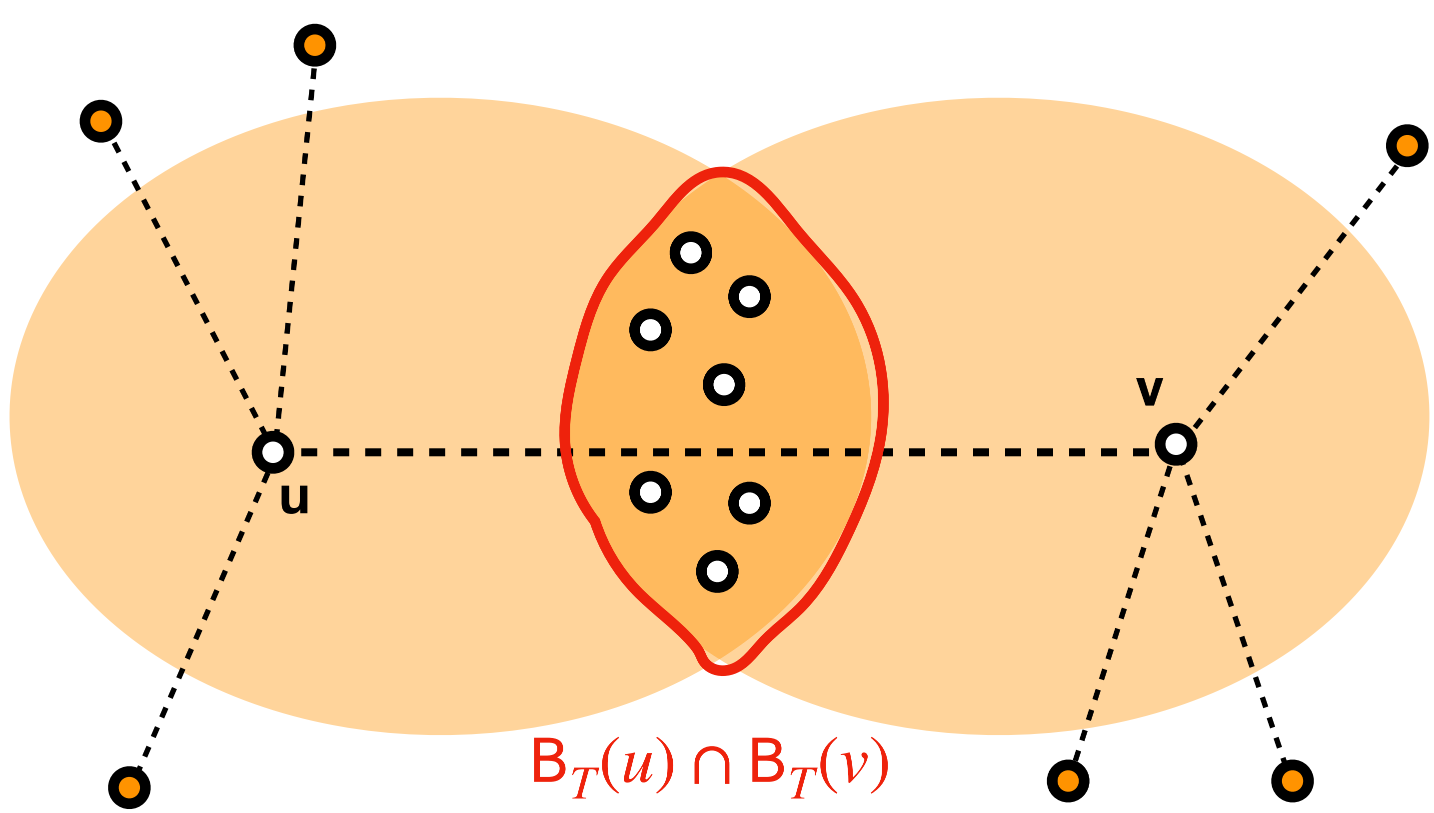


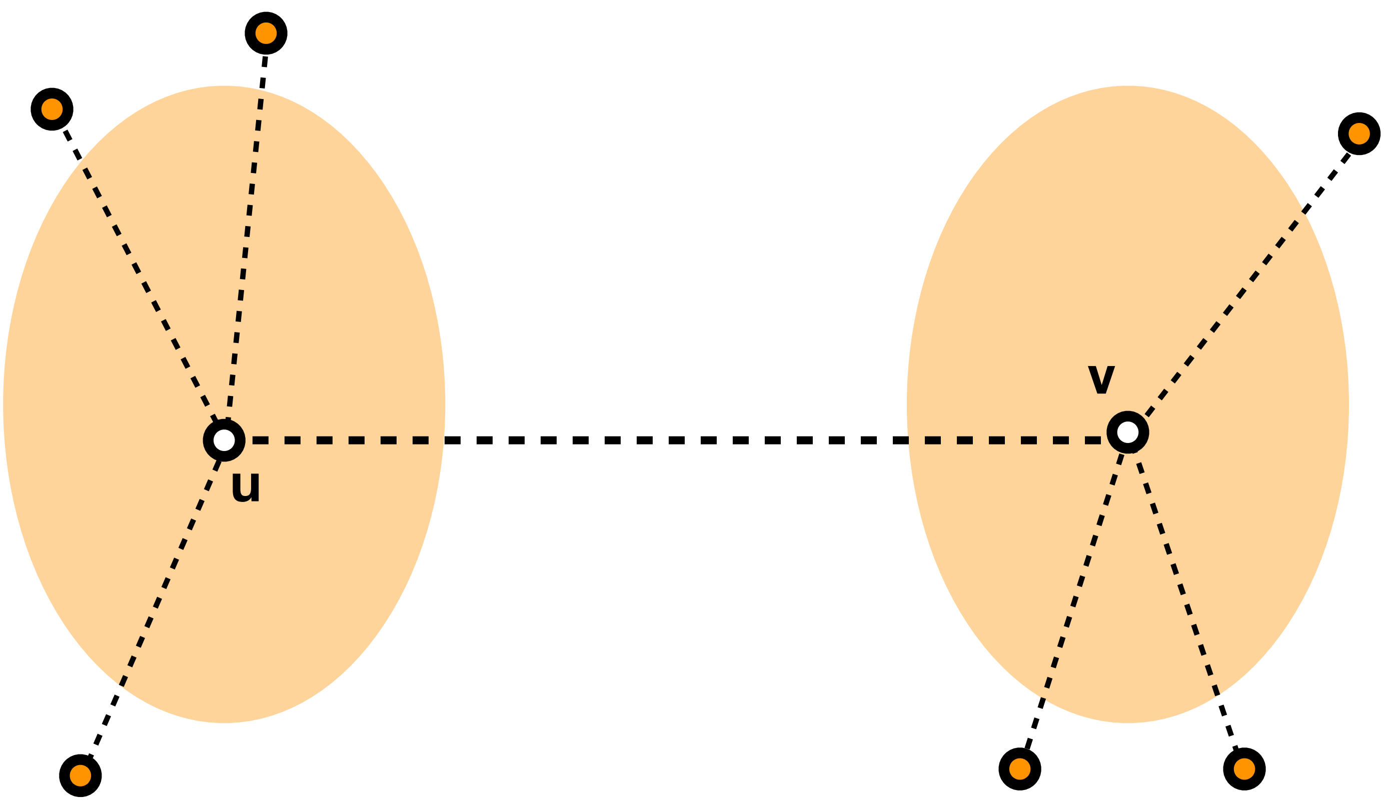
Each vertex w appears in $|C_T(w)| \leq n^\alpha$ T-balls

Total size of intersections:
$$\sum_{u,v} |B_T(u) \cap B_T(v)| \leq n^{1+2\alpha}$$

Recalling $\alpha < 1/2$, space is still **sub-quadratic**

Case 1: intersecting S-balls

- **Hard sub-case:** $\mathsf{B}_T(u) \cap \mathsf{B}_T(v) = \emptyset$



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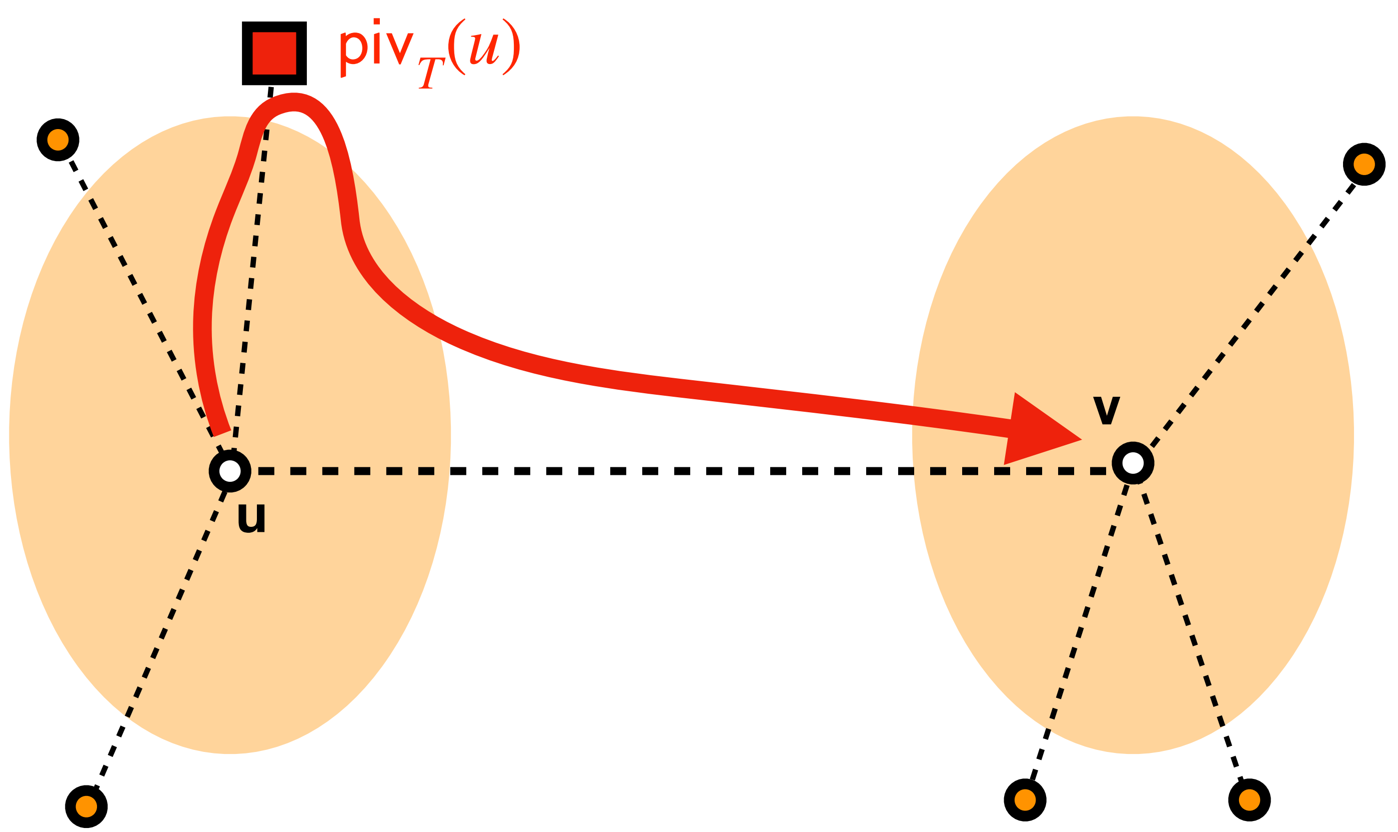

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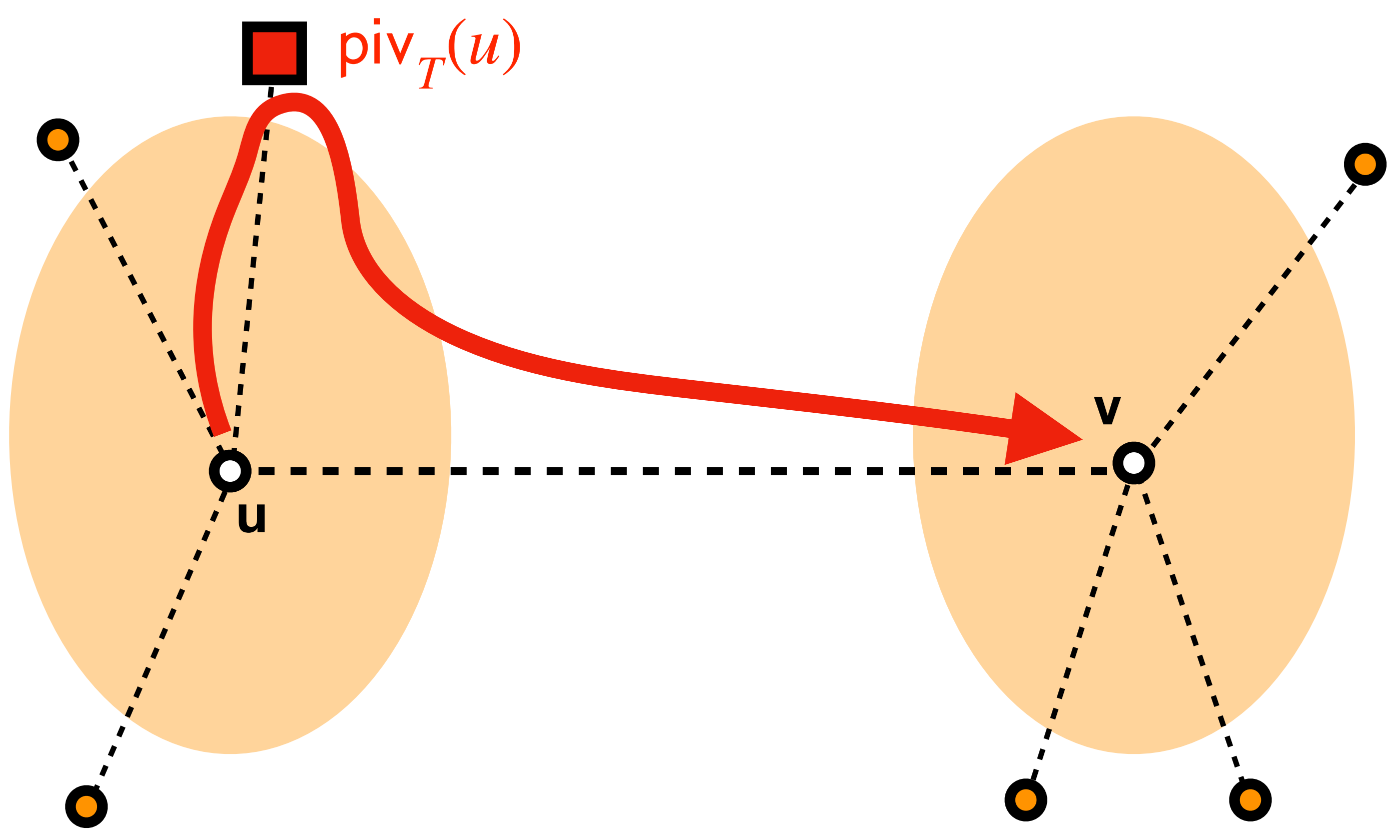

Storing all values of $\mathrm{dist}(\mathrm{piv}_T(u), v)$ takes $n^{2-\alpha}$ space ✓

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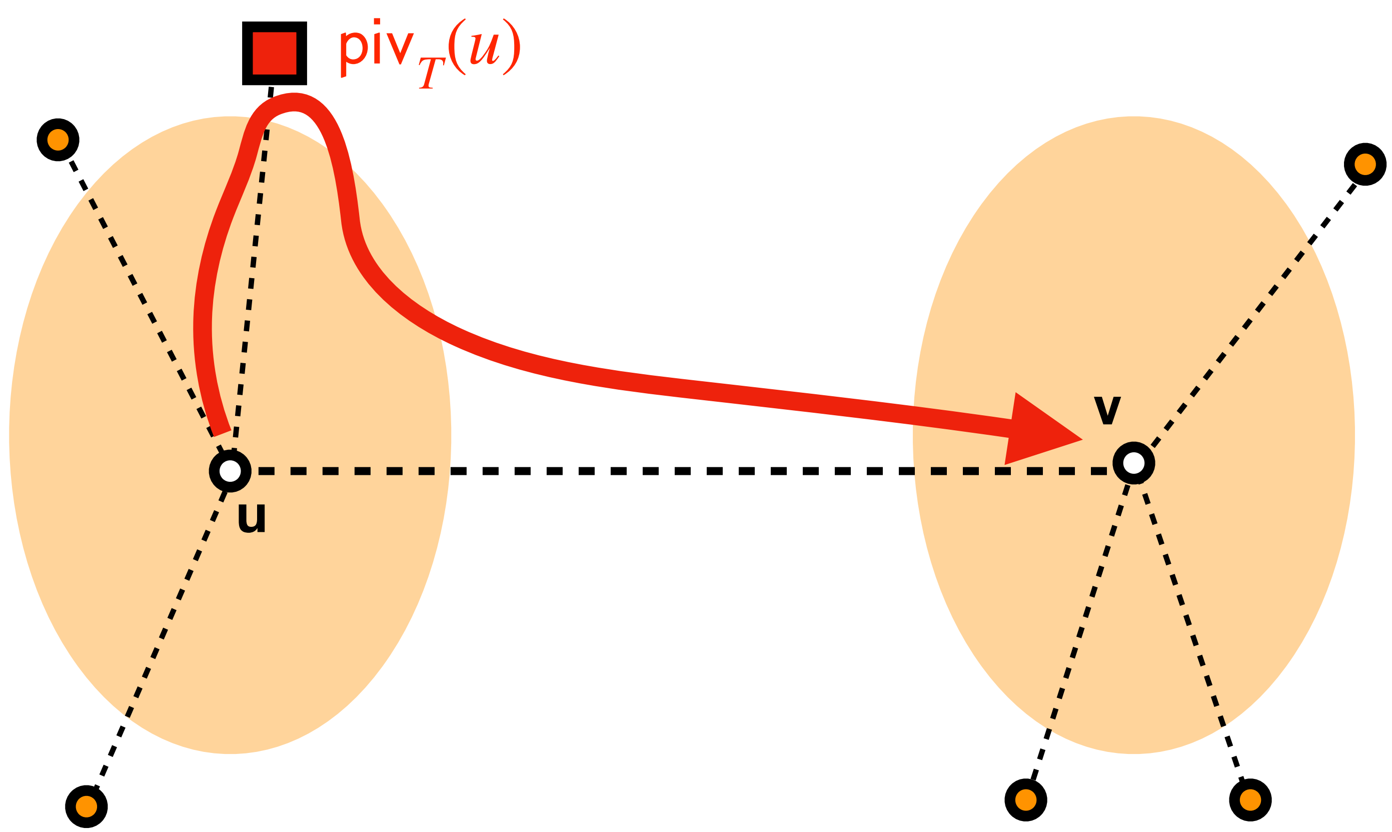


Storing all values of $\mathsf{dist}(\mathsf{piv}_T(u), v)$ takes $n^{2-\alpha}$ space

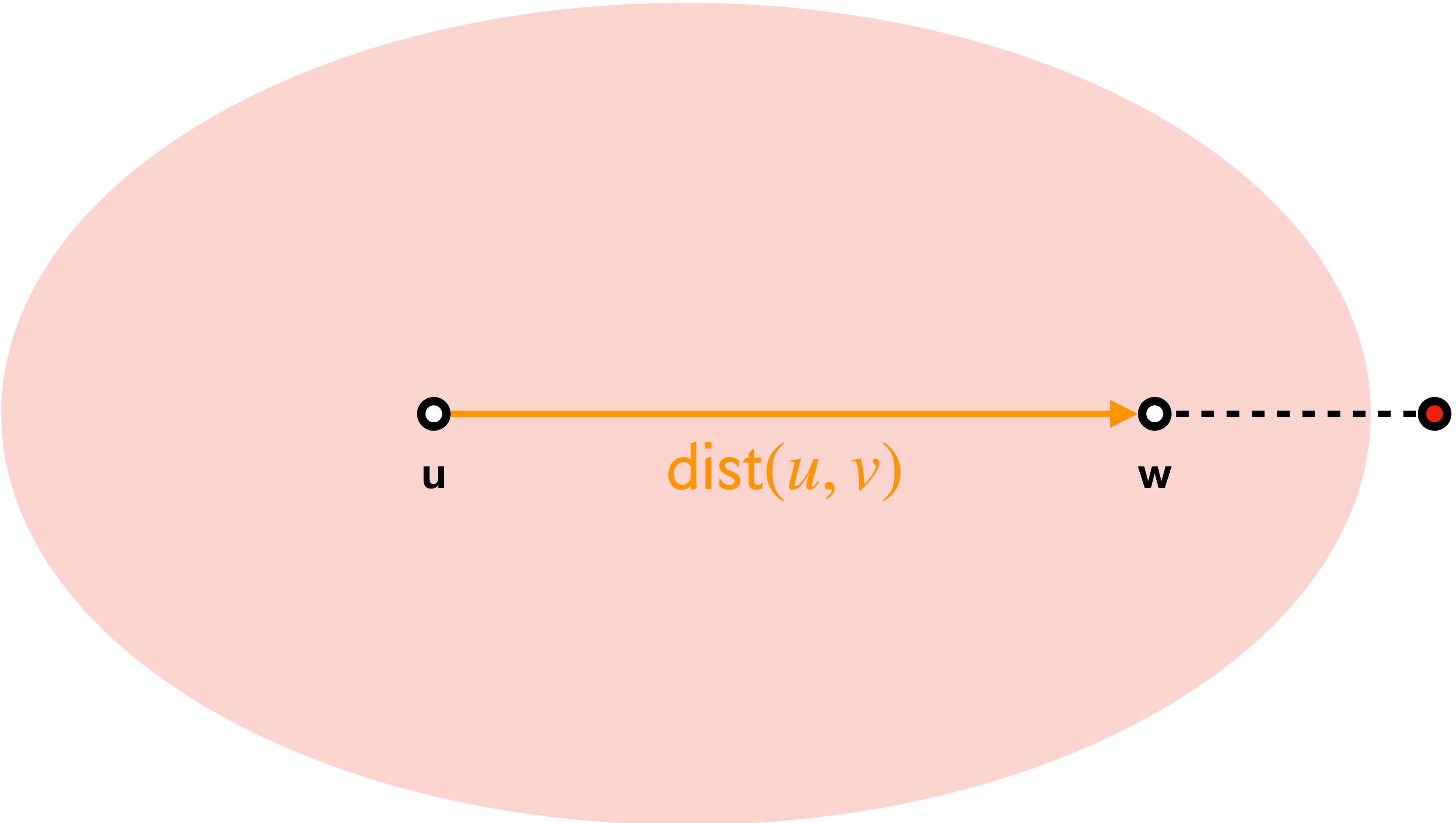
Computing SSSP from all T-vertices takes $mn^{1-\alpha}$ time

Subcase: disjoint T-balls

- **Difficulty:** SSSP from all pivots is expensive
- **Solution:** Sparsification within S-balls [AR'20]

<u>Subcase:</u> disjoint T-balls

- **Lemma [Akav & Roditty, 2020]**
  There is an ***emulator*** $H = (V, F)$ of size $n^{1+\beta/2}$, s.t. for any $w \in \mathbf{B}_S(u)$, $\text{dist}(u, w) \leq \text{dist}(u, S) - 3$, we have: $\text{dist}_H(u, w) \leq \text{dist}(u, w) + 2$



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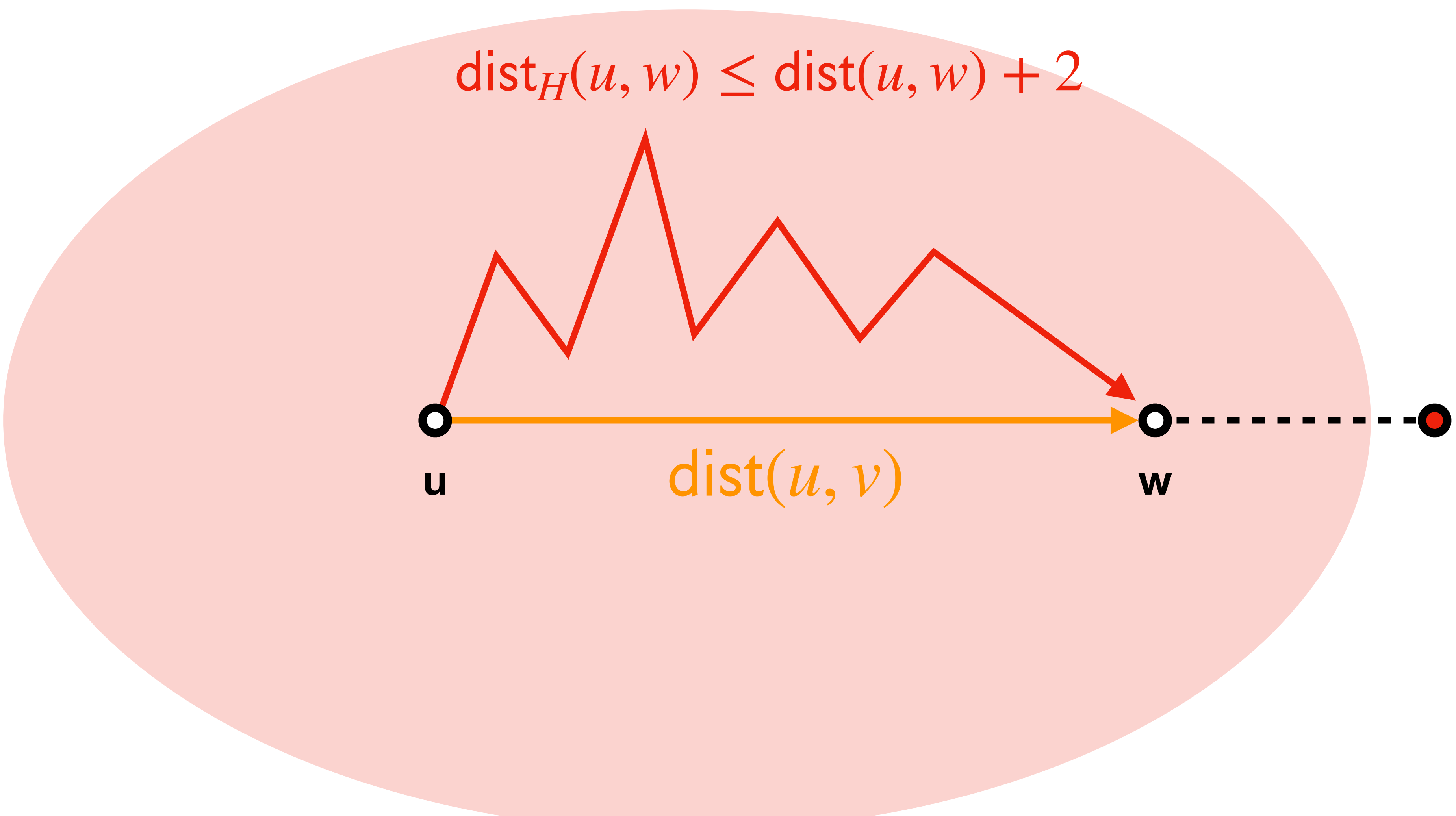

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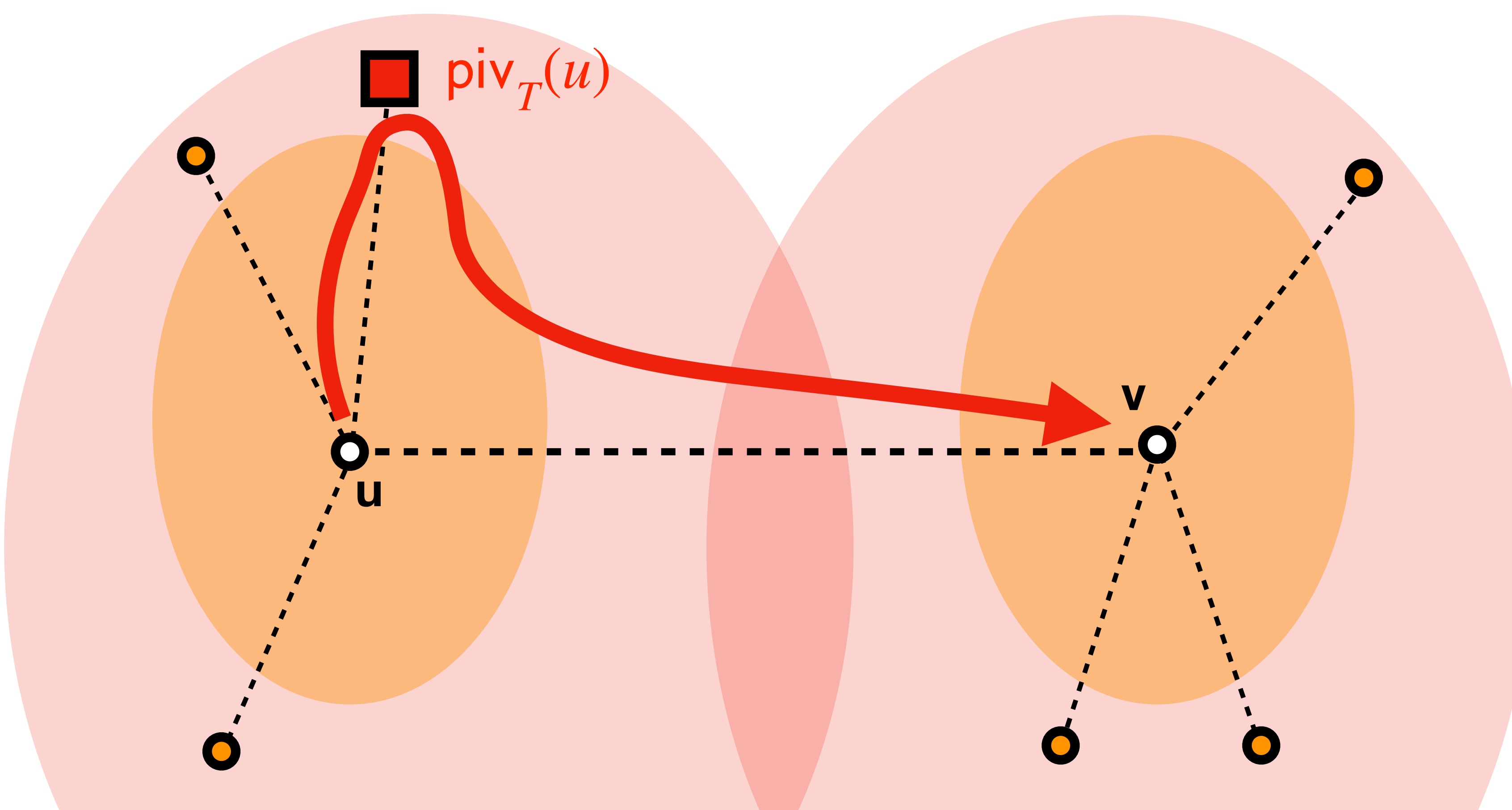
- SSSP in ***emulator H*** from all pivots takes time $n^{2-\alpha+\beta/2<2}$



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- SSSP in ***emulator H*** from all pivots takes time $n^{2-\alpha+\beta/2<2}$

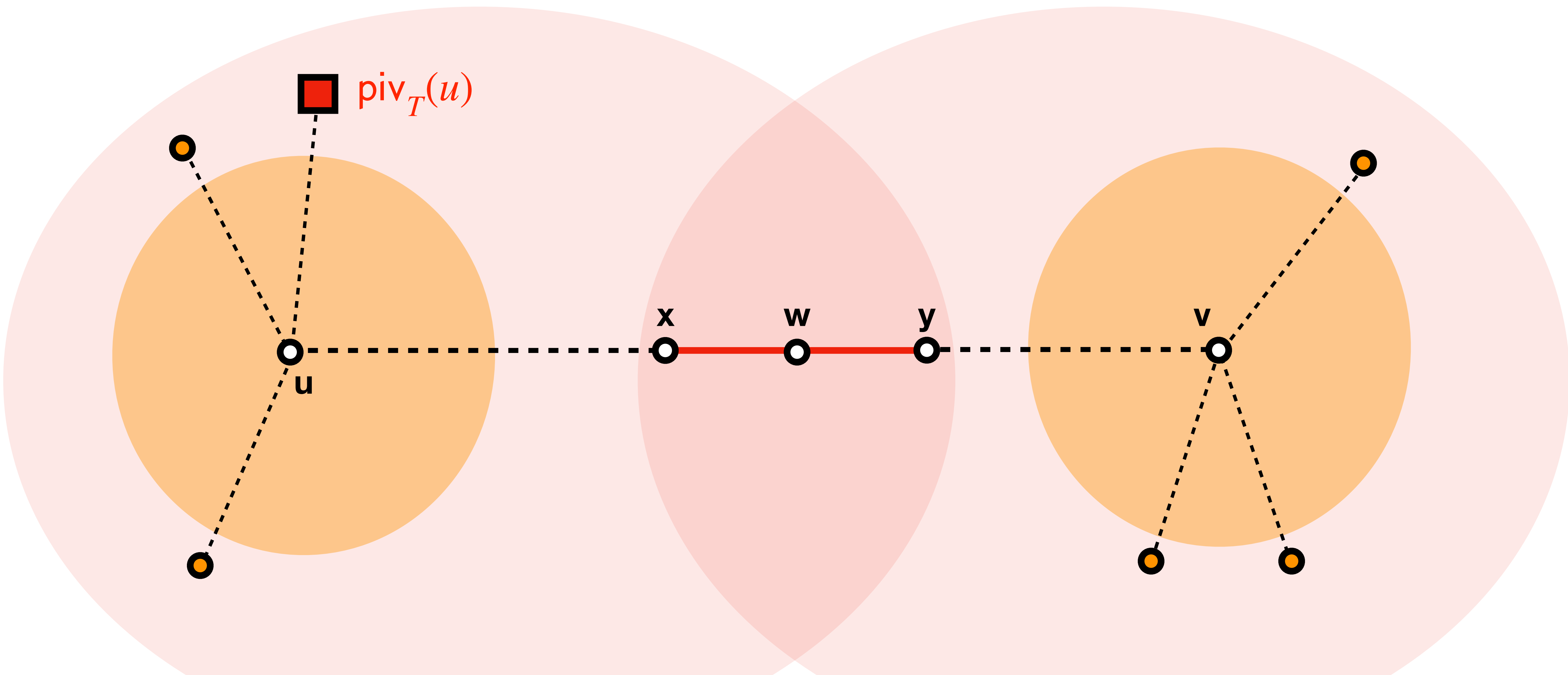
- Drawback:
  Emulator $H = (V, F)$ incurs **many additive errors**, leading to $(2+\epsilon,\ 5)$ or $(2,\ 10+)$ stretch in the end

- How to achieve $(2,\ 3)$ as in [Baswana et al, 2005]?

Our improvement

Shaving additive errors

- One of the hard cases where the additive error is large when applying **Lemma [AR'20]**
- $B_S(u) \cap B_S(v)$ contains exactly 2 edges on a shortest path



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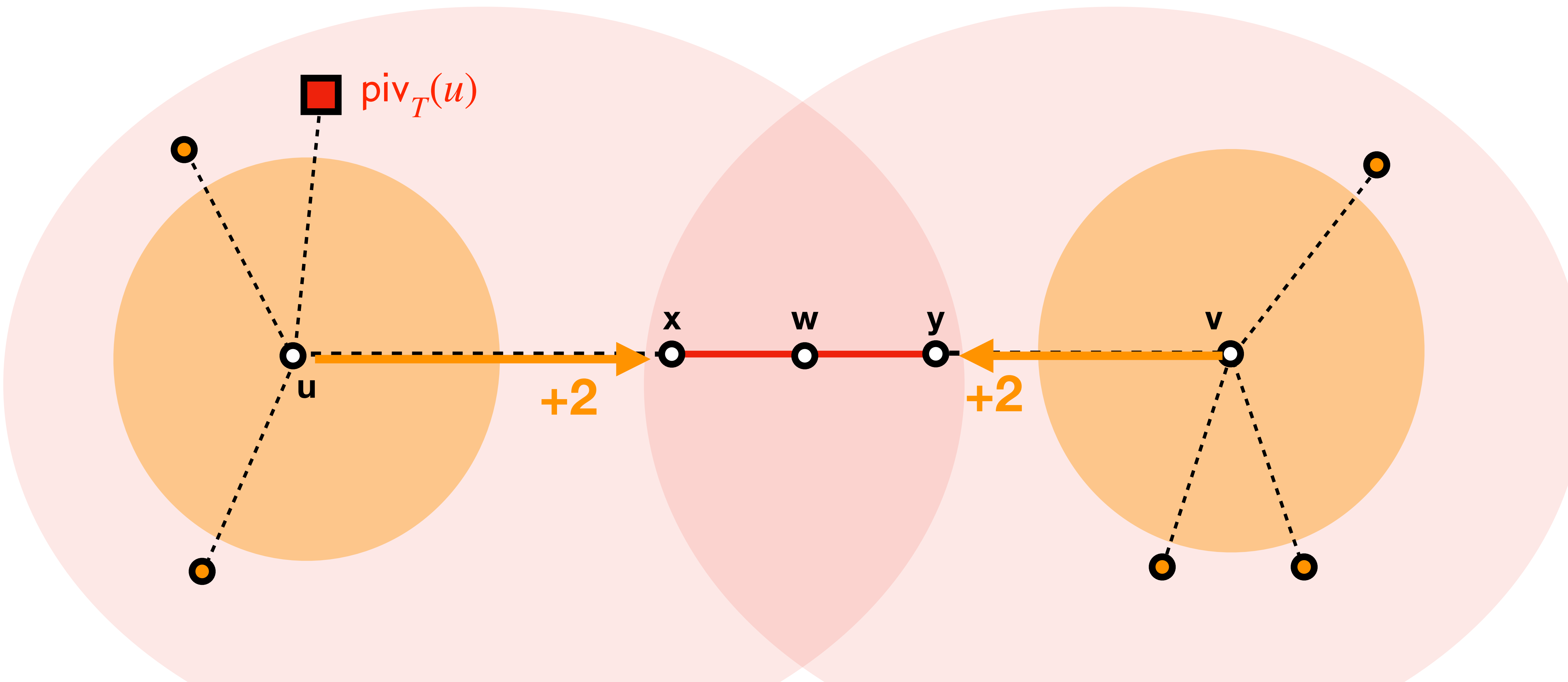
- The emulator only covers sub-paths u—x & v—y, not **x—w—y**



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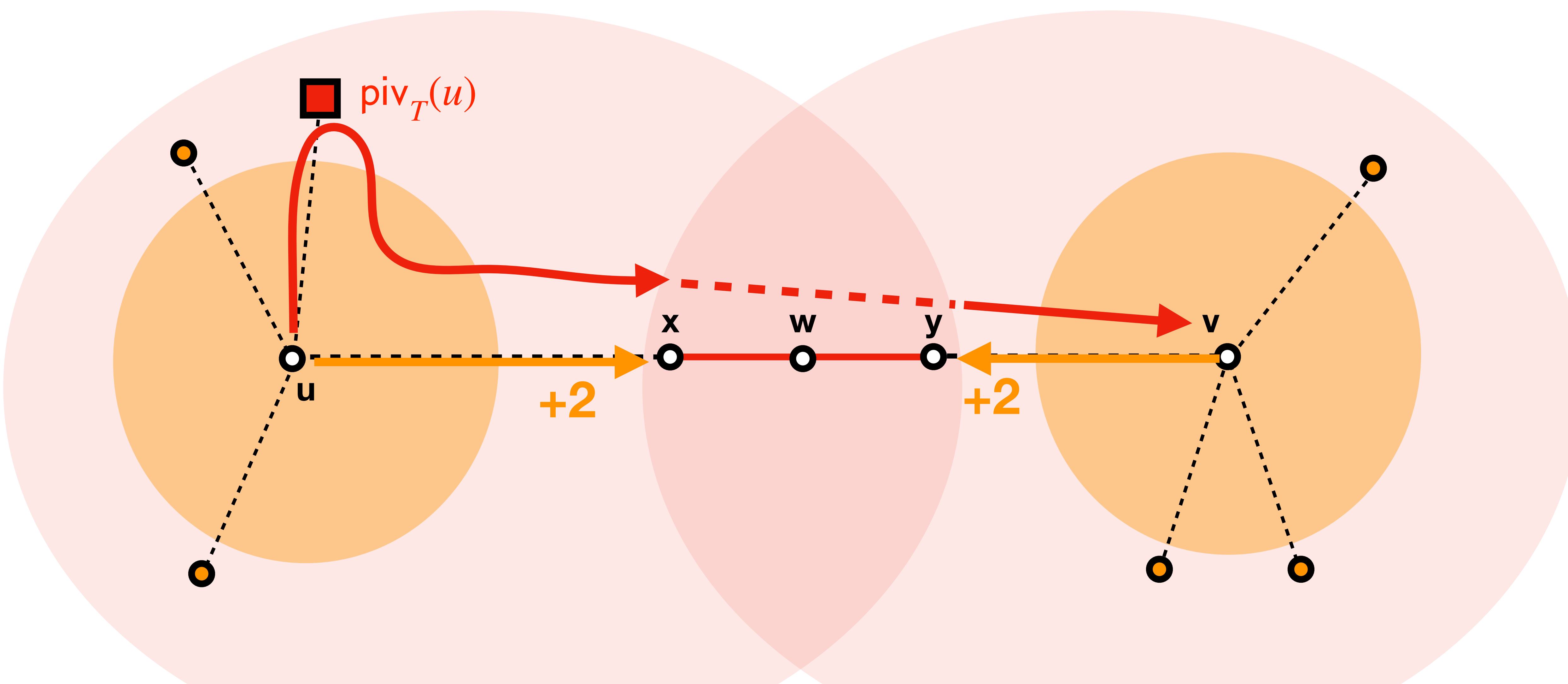

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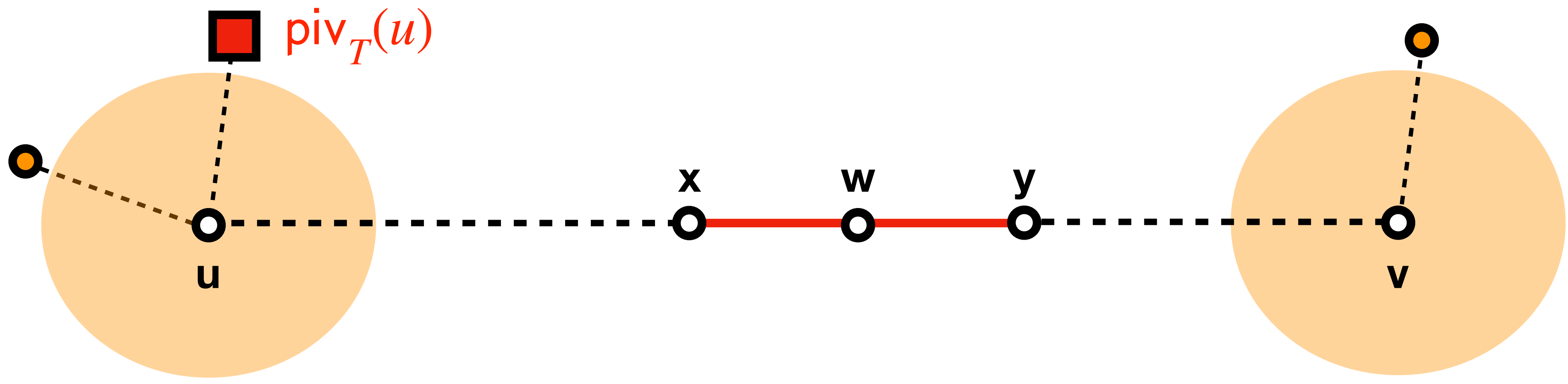
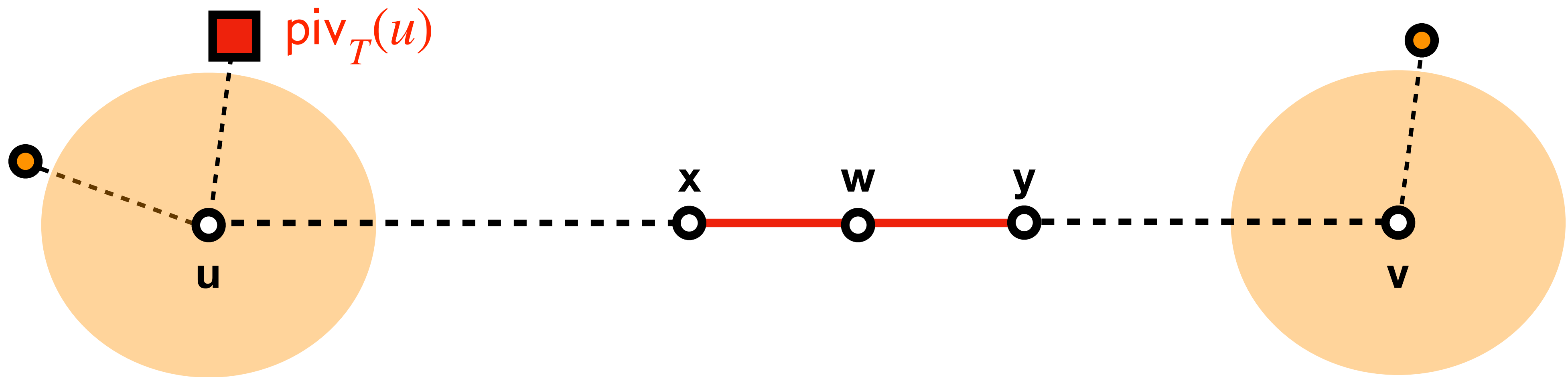


How to deal with **red** edges during SSSP at $\mathsf{piv}_T(u)$

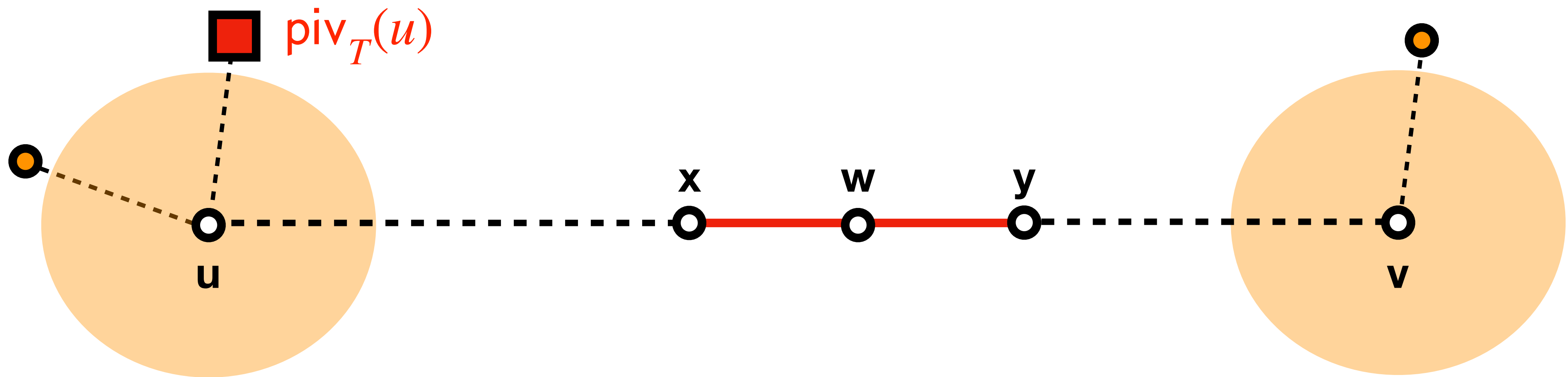
Shaving additive errors



How to deal with **red** edges during SSSP at $\text{piv}_T(u)$

- SSSP in G takes time $\tilde{O}(m)$, total time = $mn^{1-\alpha}$ ✘

Shaving additive errors



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- SSSP in G takes time $\tilde{O}(m)$, total time = $mn^{1-\alpha}$ ✗
- Overlay a (2, 1)-spanner? Total time = $n^{2.5-\alpha}$ ✗

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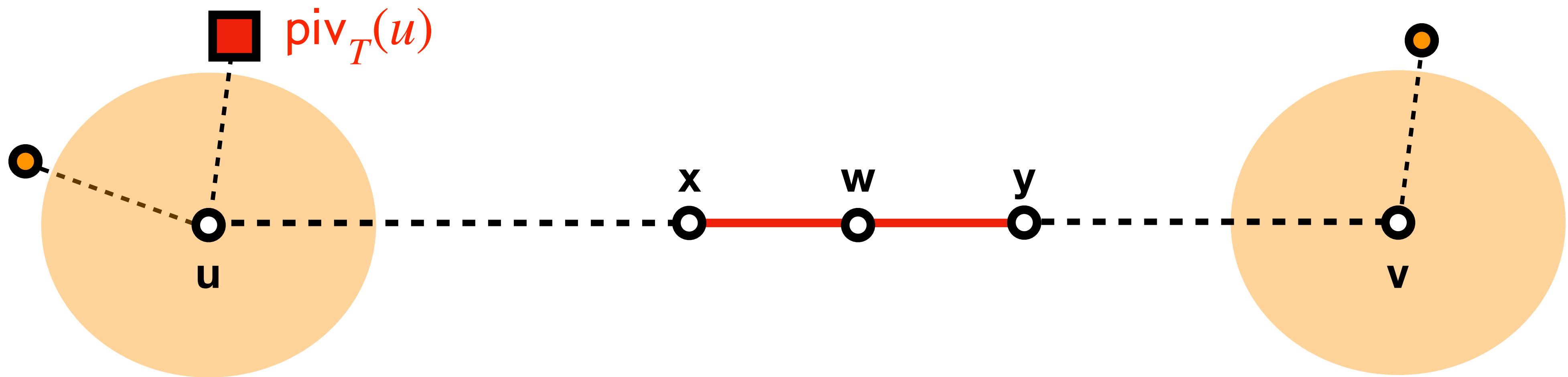

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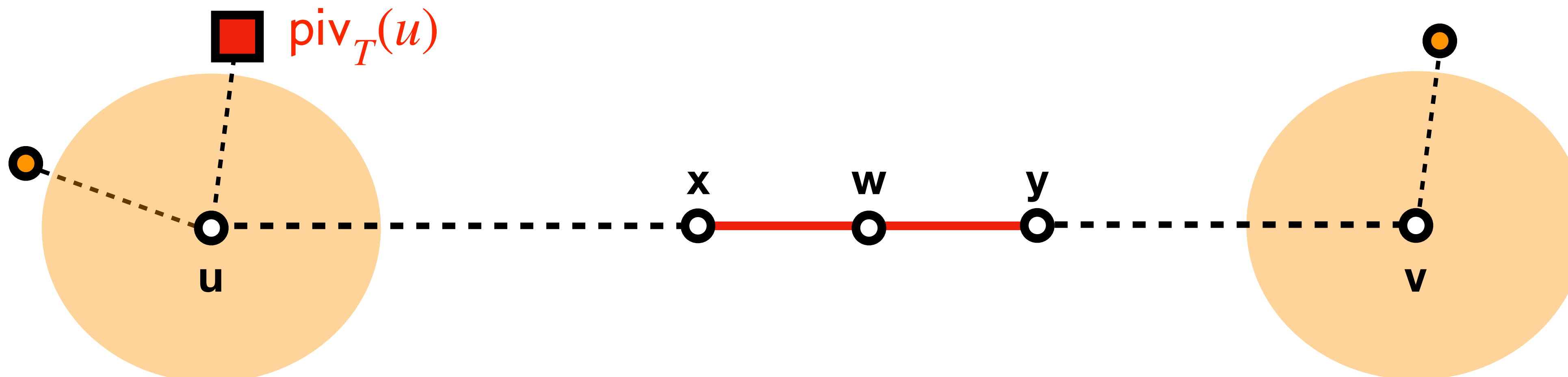
- Overlay a (3, 2)-spanner? Total time = $n^{7/3-\alpha}$ ✓

Shaving additive errors



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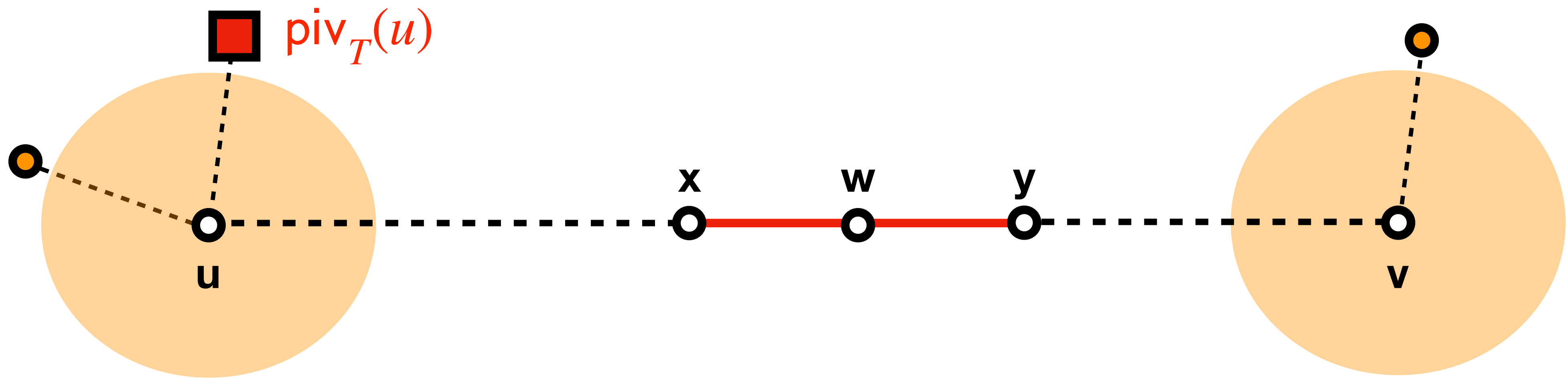
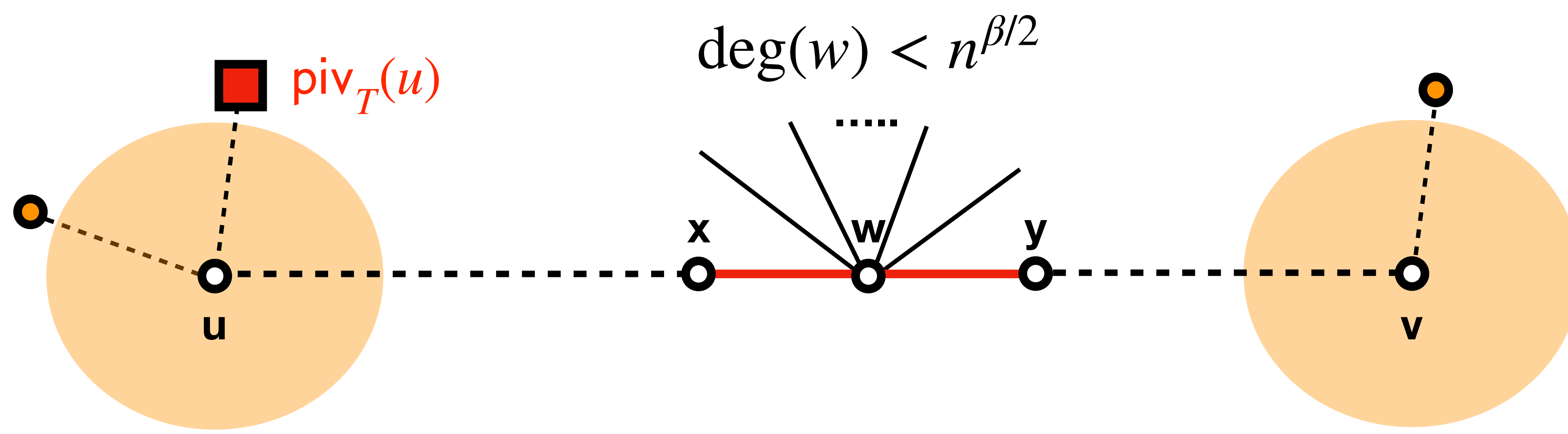
- Overlay a (2, 1)-spanner? Total time = $n^{2.5-\alpha}$ ✗
- Overlay a (3, 2)-spanner? Total time = $n^{7/3-\alpha}$ ✓
  But error = $(2 \times 3 + 2) - 2 = 6$ ✗

Adding shortcuts



Add shortcuts across x—w—y

Adding shortcuts

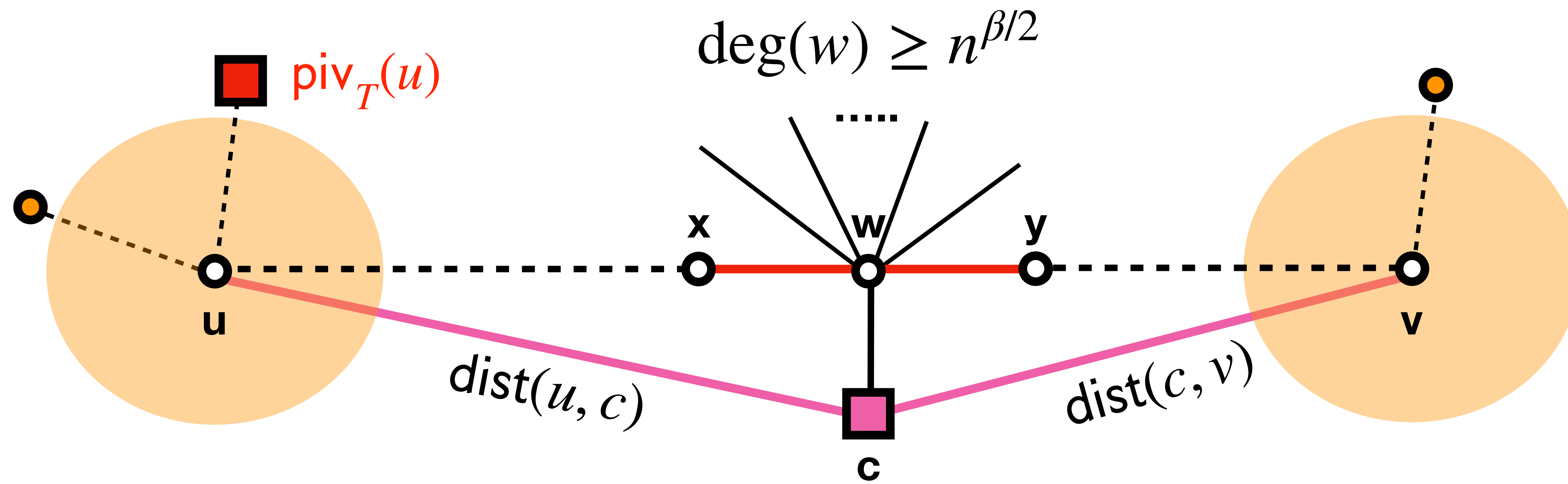


Add shortcuts across x—w—y

If $\deg(w) < n^{\beta/2}$, then
add all incident edges to SSSP

- Total #edges = $n^{1+\beta/2}$
- Total runtime = $n^{2-\alpha+\beta/2<2}$

Adding shortcuts



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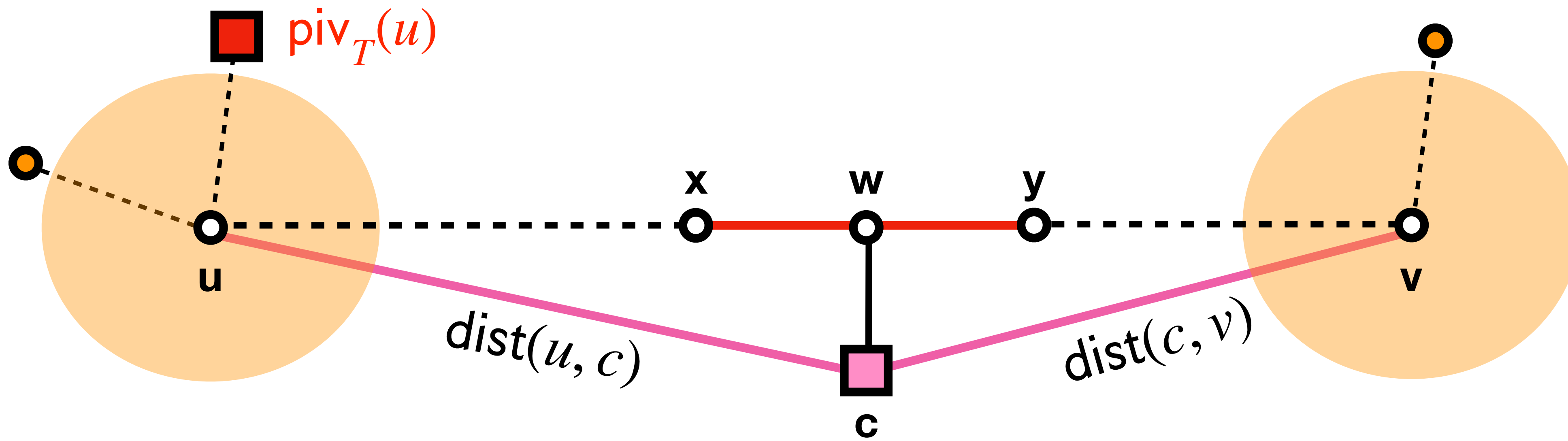
If $\deg(w) < n^{\beta/2}$, then add all incident edges to SSSP

- Total #edges = $n^{1+\beta/2}$
- Total runtime = $n^{2-\alpha+\beta/2<2}$

If $\deg(w) \geq n^{\beta/2}$, then take a rand-set $R \subset V$ with rate $n^{-\beta/2}$

- w is adjacent to $c \in R$
- Add shortcuts u—c & v—c

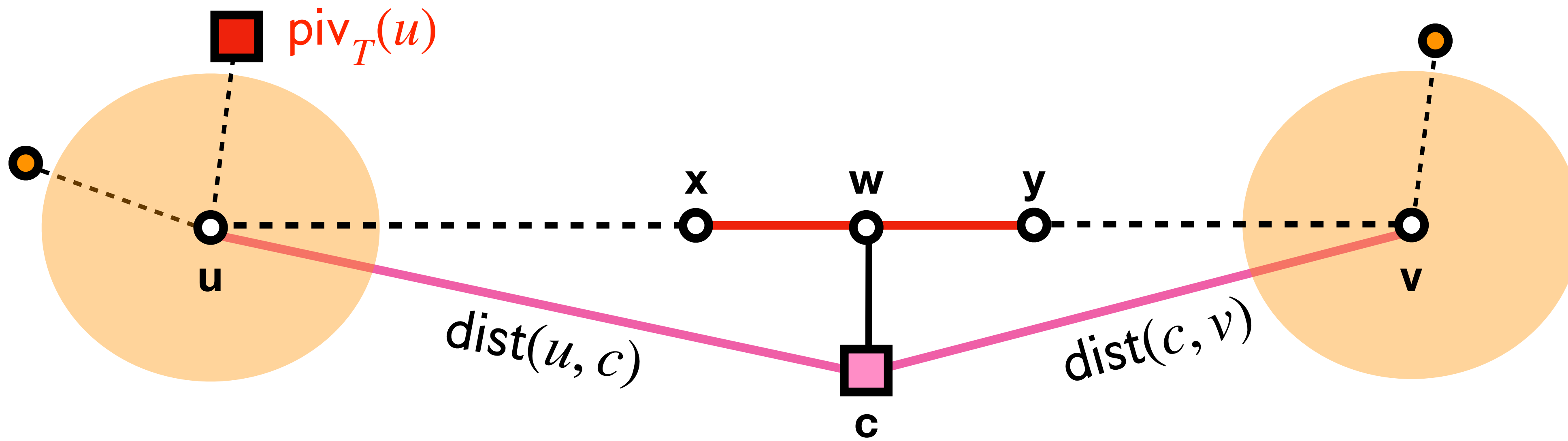
Adding shortcuts



How to compute shortcuts efficiently?

Total #shortcuts?

Adding shortcuts

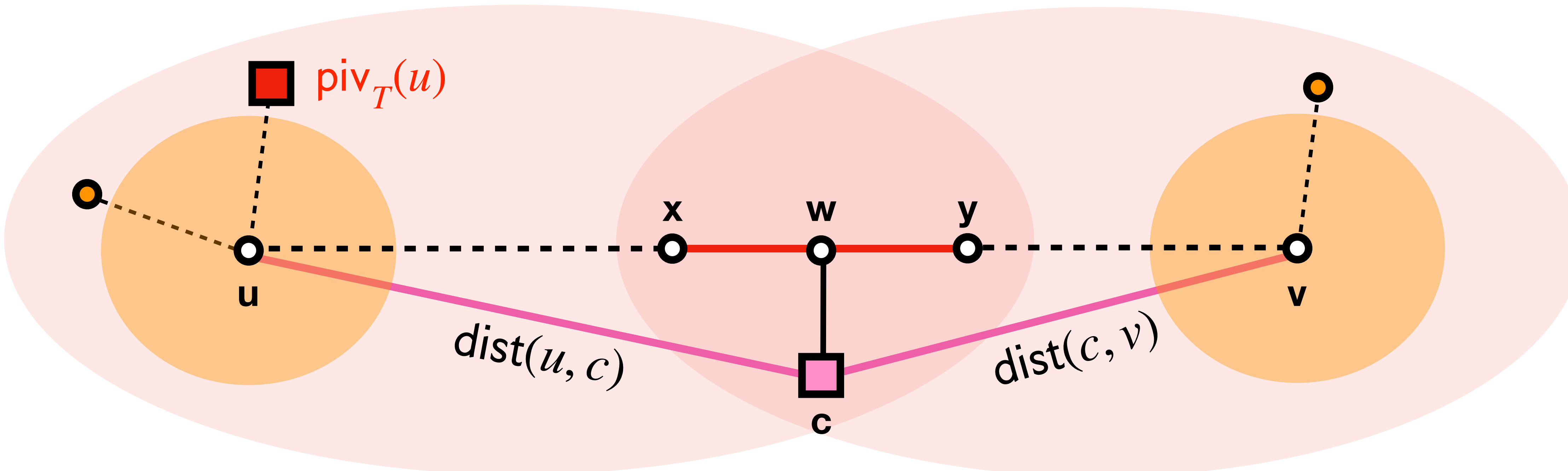


How to compute shortcuts efficiently?

Total #shortcuts?

- #pairs (u, c) could be $n^{2-\beta/2}$, total SSSP = $n^{3-\alpha-\beta/2}$ > 2 ✗

Adding shortcuts



How to compute shortcuts efficiently?

Total #shortcuts?

- #pairs (u, c) could be $n^{2-\beta/2}$, total SSSP = $n^{3-\alpha-\beta/2>2}$ ✗
- #pairs (u, c) with $c \in \mathrm{B}_S(u)$ is $n^{1+\beta/2}$, total SSSP = $n^{2-\alpha+\beta/2<2}$ ✓

Further questions

1. Weird runtime $m + n^{1.987}$ for (2, 3)-DOs
   Maybe improve to $m + n^{5/3}$, matching $\tilde{O}(n^{5/3})$ space

2. Sub-quadratic runtime for (2, 1)-DOs [Sommor, 2016]

3. Exact space complexity of (2, 1)-DOs

   Upper bound $\tilde{O}(n^{5/3})$, lower bound $\tilde{\Omega}(n^{1.5})$
   [Pătraşcu & Roditty, 2010]

   Lower bound $\tilde{\Omega}(n^{5/3})$ against (2.33, 0)-DOs
   [Pătraşcu, Roditty, Thorup, 2012]

Thank you!