Constant-Round Near-Optimal Spanners in Congested Clique

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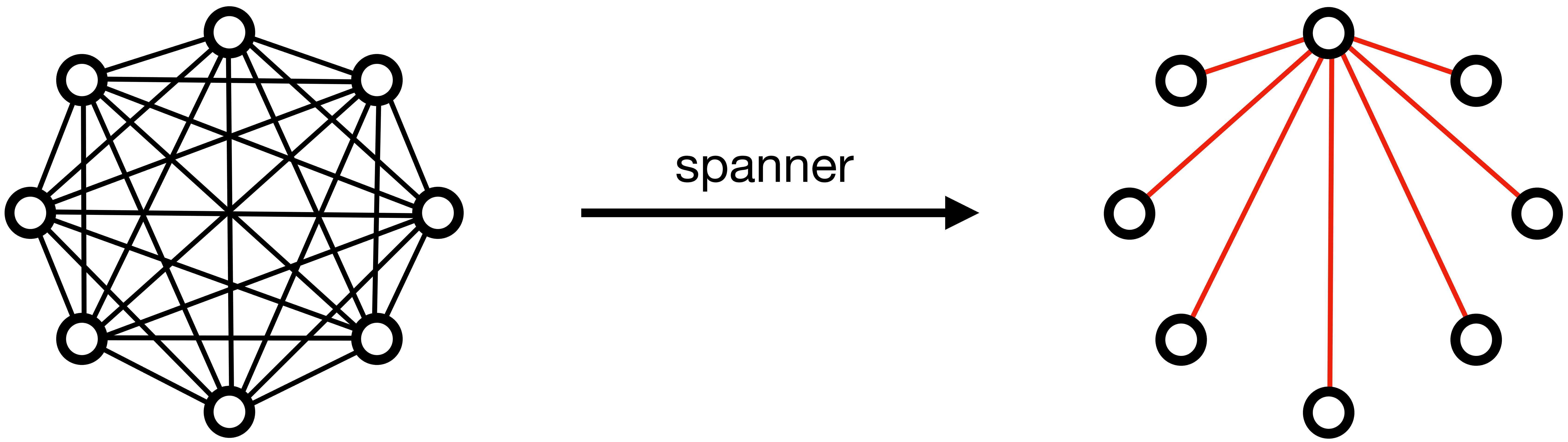
Graph Spanner

- **Input:**
  $G = (V, E, \omega)$ an undirected weighted graph

- **Output:**
  $H \subseteq G$ a **subgraph** of small size that approximately preserves distances



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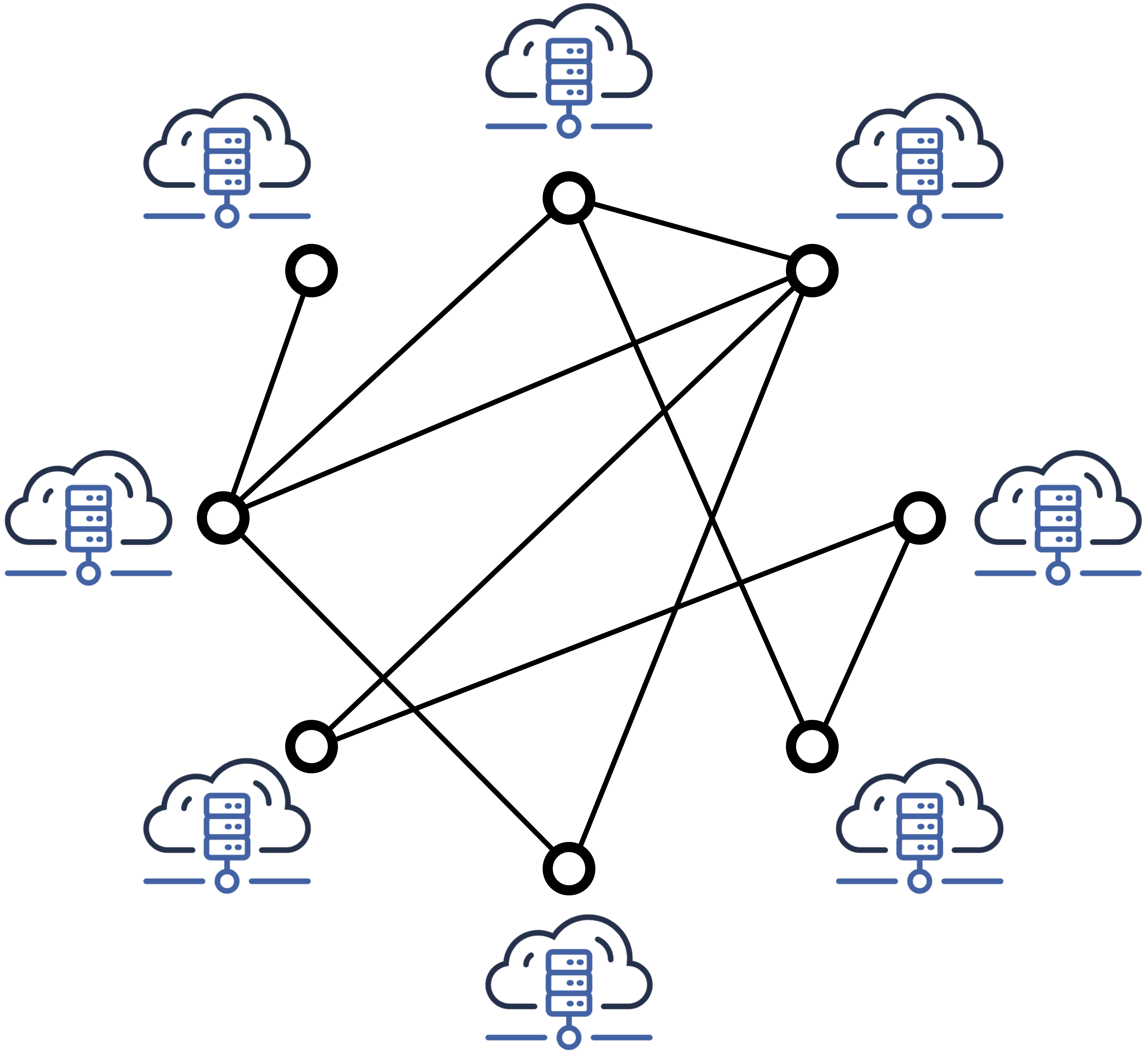
Multiplicative stretch:

- **Size:** $|H| = O(n^{1+1/k})$
- **Approximation:** $\forall s, t \in V, \text{dist}_G(s, t) \leq \text{dist}_H(u, v) \leq (2k-1)\text{dist}_G(s, t)$
- Optimal under the girth conjecture

Congested Clique

- Input graph is stored distributively
- Synchronous, all-to-all communication
- Message size = $O(\log n)$ bits
- Runtime = #communication rounds



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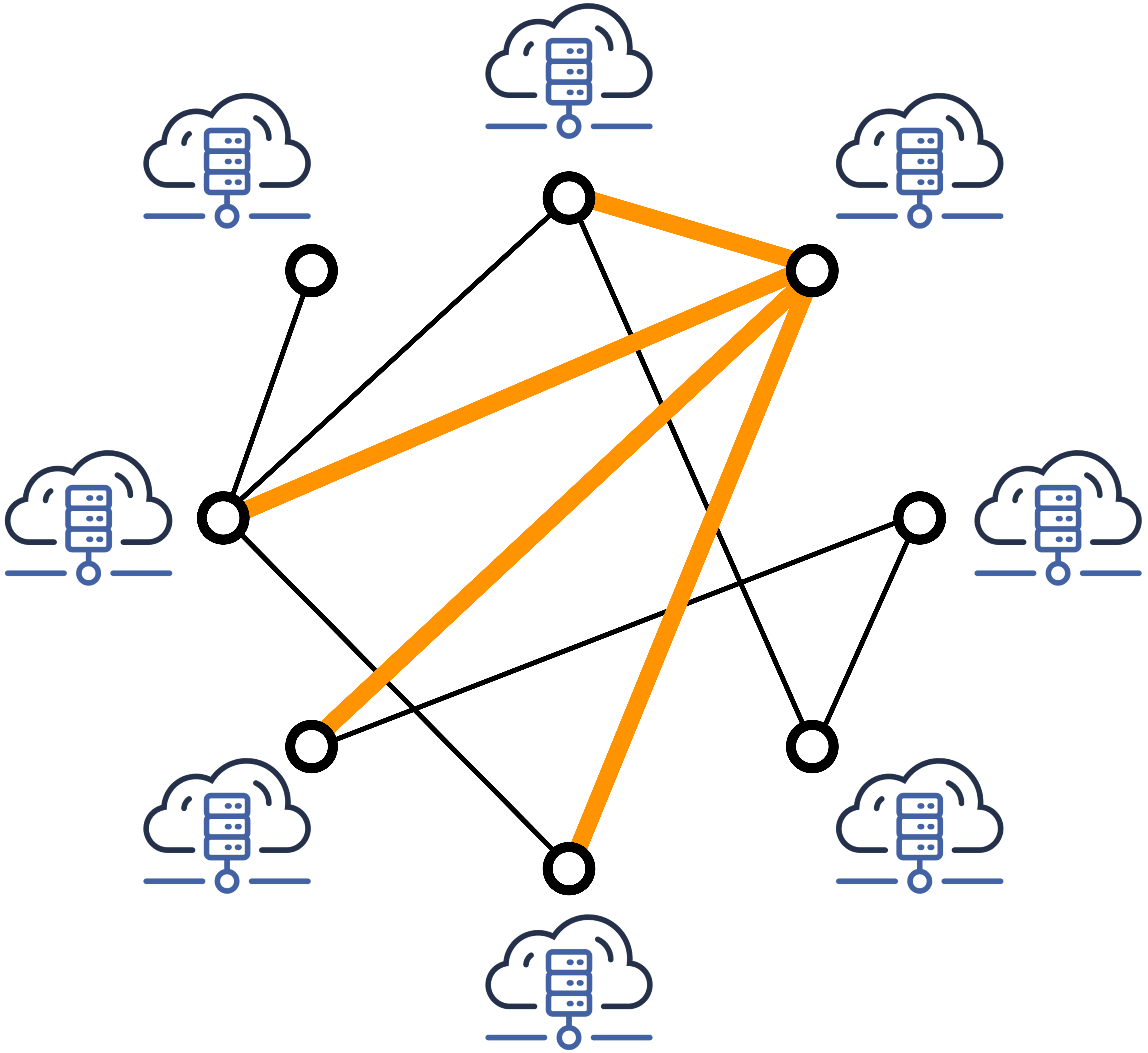

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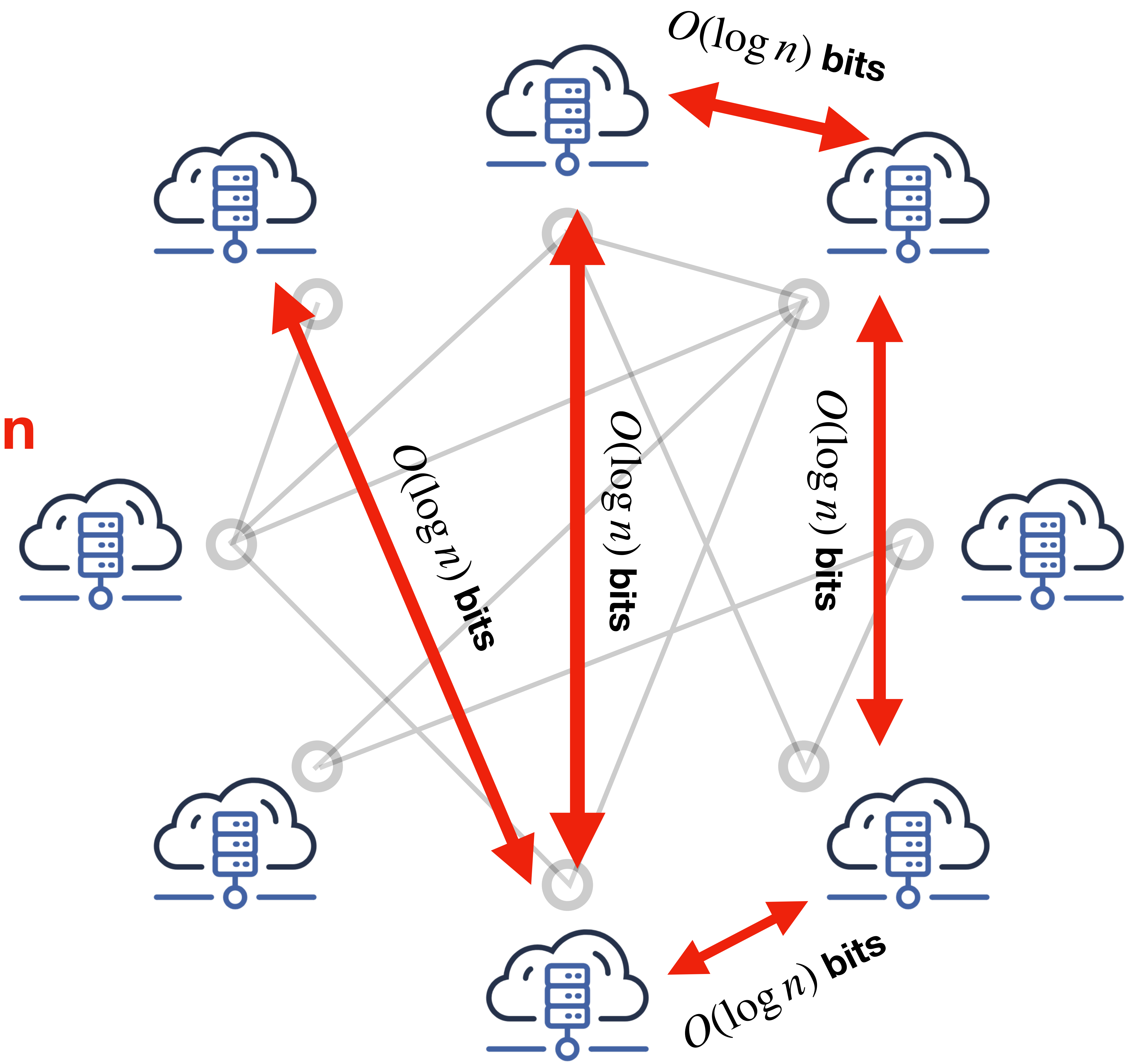
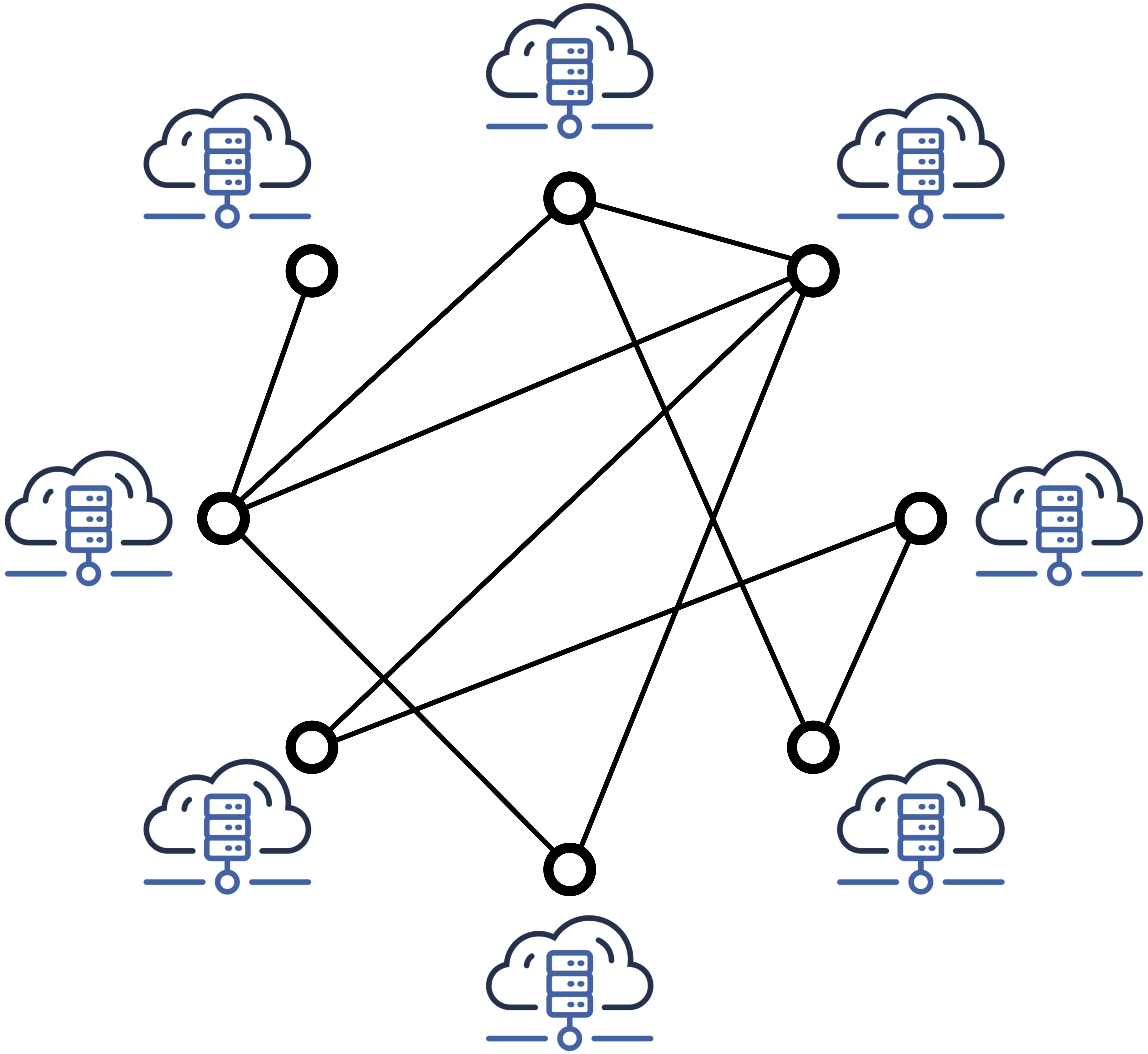


Congested Clique

- Input graph is stored distributively
- Synchronous, all-to-all communication
- Message size = $O(\log n)$ bits
- **Runtime = #communication rounds**
- **What is the runtime for spanners?**



History

reference	stretch	#edges	#rounds	input type
Baswana, Sen 2007	$2k-1$	$O(n^{1+1/k}+kn)$	$O(k)$	unweighted
Baswana, Sen 2007	$2k-1$	$O(kn^{1+1/k})$	$O(k)$	weighted
Parter, Yogev 2018	$2k-1$	$O(n^{1+1/k}\log^2 n)$	$O(\log k)$	unweighted
Biswas et al 2021	$k^{1+o(1)}$	$O(n^{1+1/k}\log k)$	$\log^{O(1)} k$	weighted

large space high runtime

History

reference	stretch	#edges	#rounds	input type
Dory et al 2021	$O(k)$	$O(n^{1+1/k})$	O(1)	unweighted
Dory et al 2021	$O(k)$	$O(n^{1+1/k} \log n)$	O(1)	weighted
Dory et al 2021	$O(k \log n)$	$O(n^{1+1/k})$	O(1)	weighted

large stretch **sub-opt space** **opt**

Our results

reference	stretch	#edges	#rounds	input type
new	$2k-1$	$O(n^{1+1/k})$	O(1)	unweighted
new	$(1+\epsilon)(2k-1)$	$O(n^{1+1/k})$	O(1)	weighted
new	$2k-1$	$O(kn^{1+1/k})$	O(1)	weighted
	near-opt	**near-opt**	**opt**	

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Outline of main algorithm

Degree reduction

Neighborhood computation

Parallel spanner simulation

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Degree Reduction

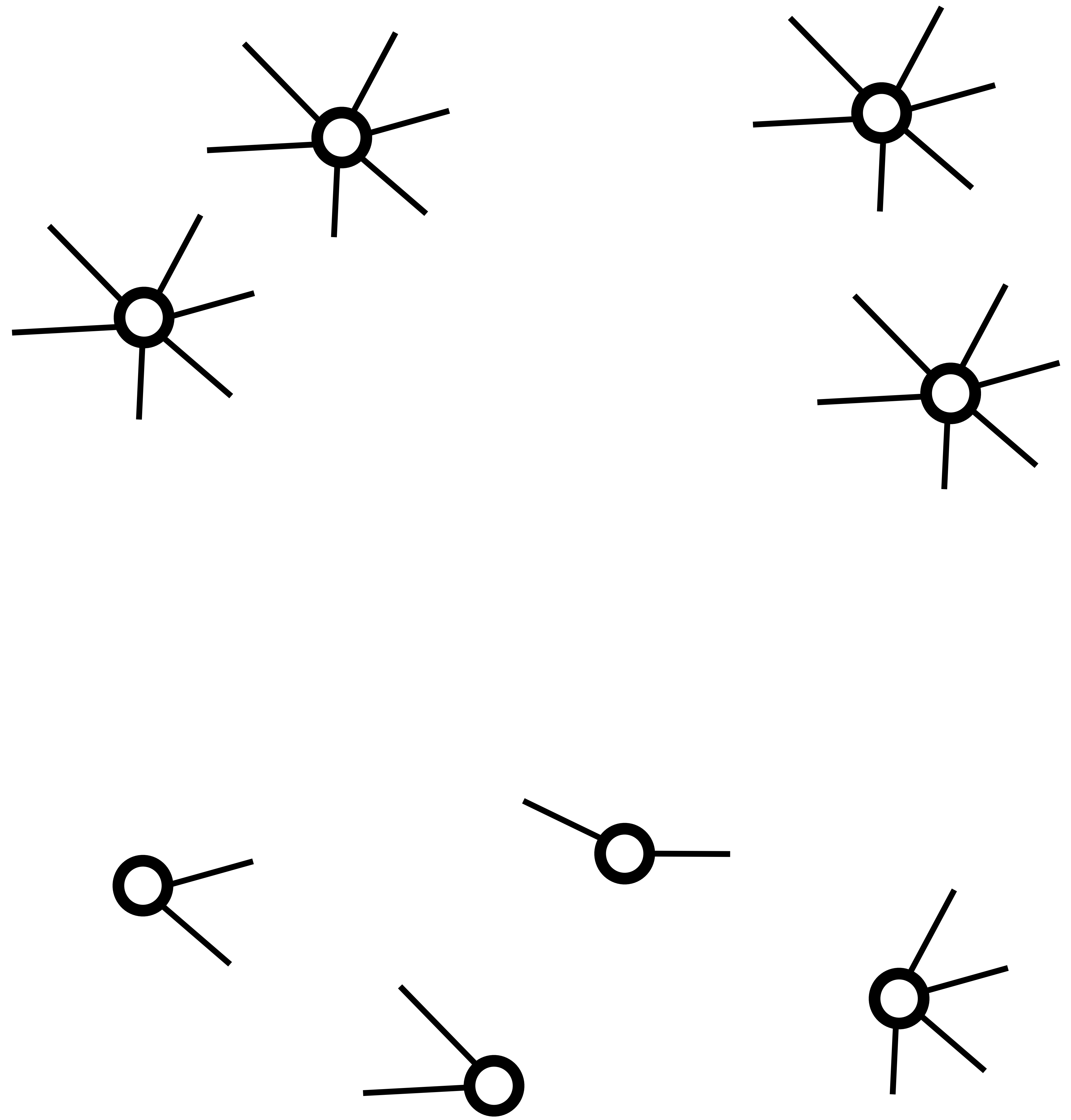
<u>Goal:</u> reduce **maximum degree** to $n^{20/k}$

1. Partition the vertex set $V = V_{\text{hi}} \cup V_{\text{lo}}$

$$V_{\text{hi}} = \{v \mid \deg(v) > n^{20/k}\}$$
$$V_{\text{lo}} = \{v \mid \deg(v) \leq n^{20/k}\}$$

2. Find a hitting set of size $n^{1-10/k}$ which dominates V_{hi} [DFKL'21]

3. Find a $0.2k$-stretch spanner [DFKL'21] on a contracted graph



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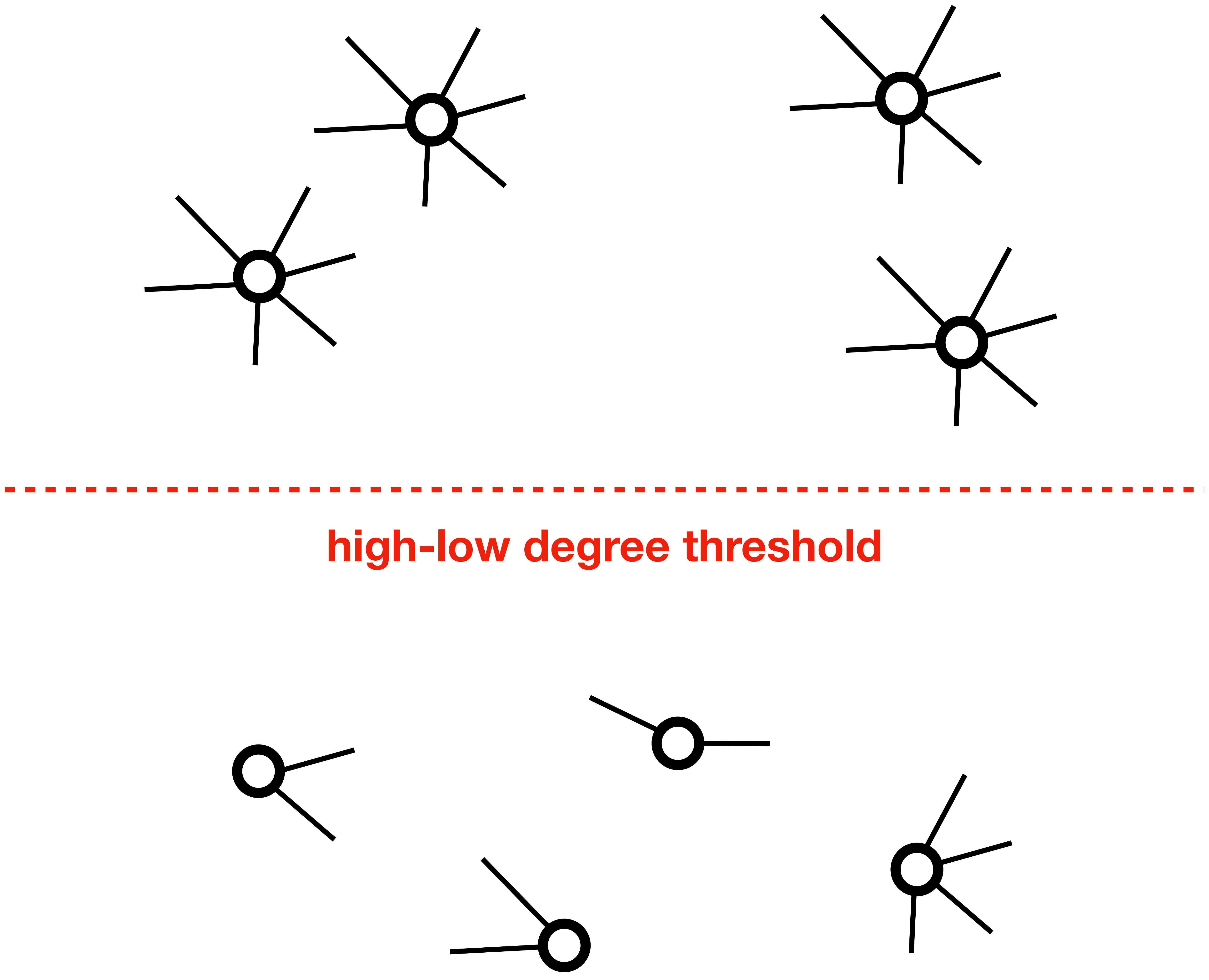

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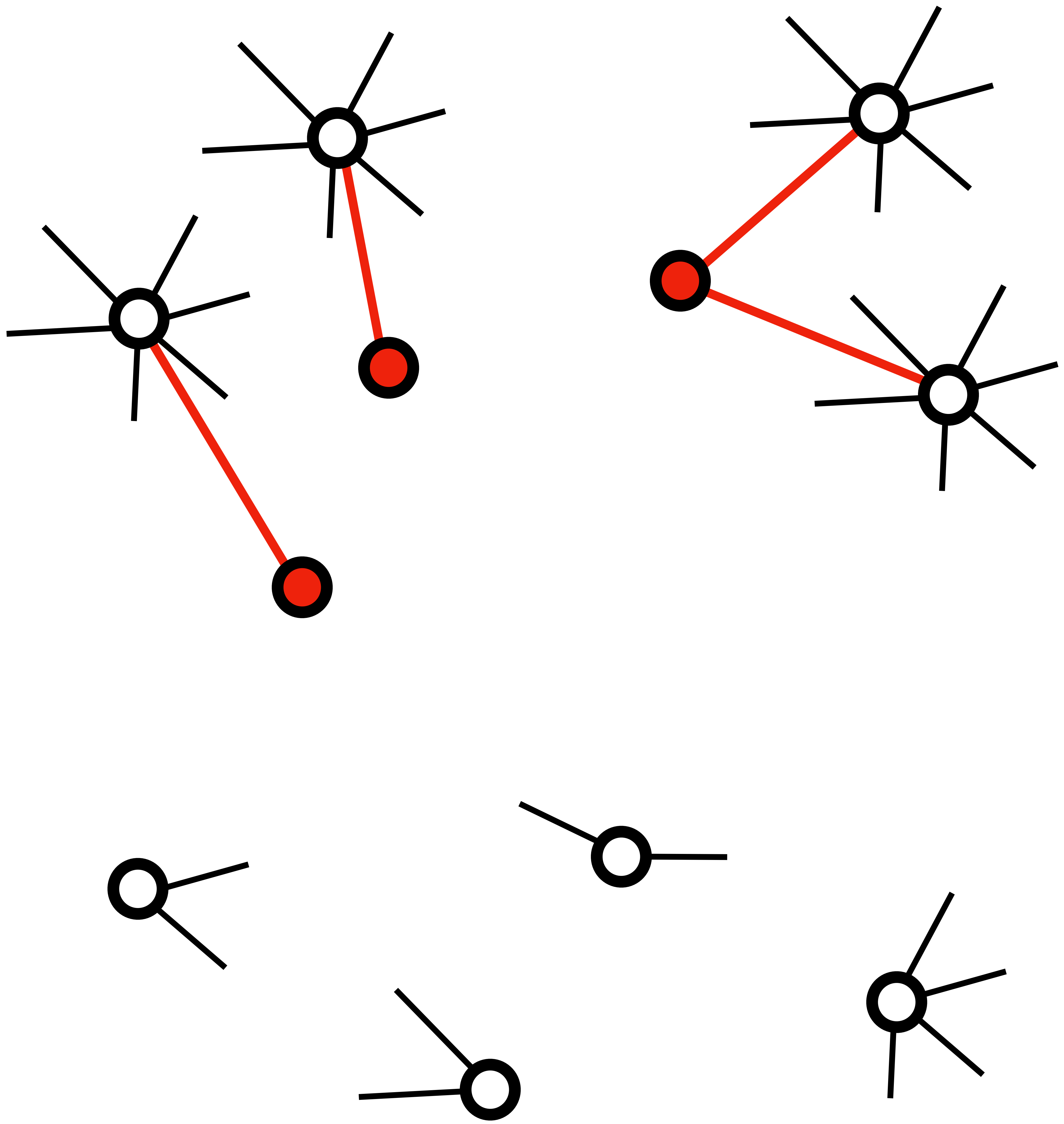

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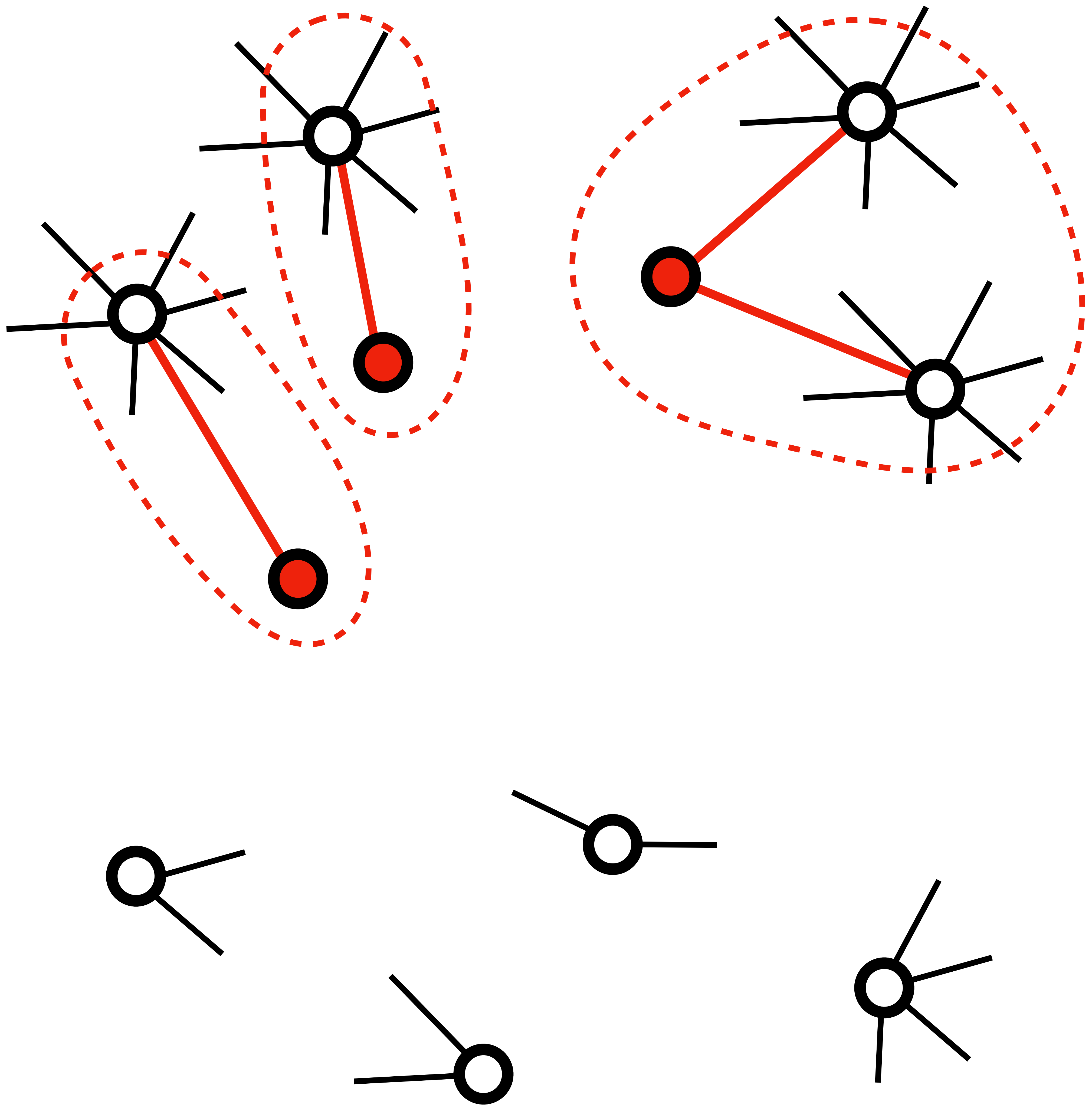

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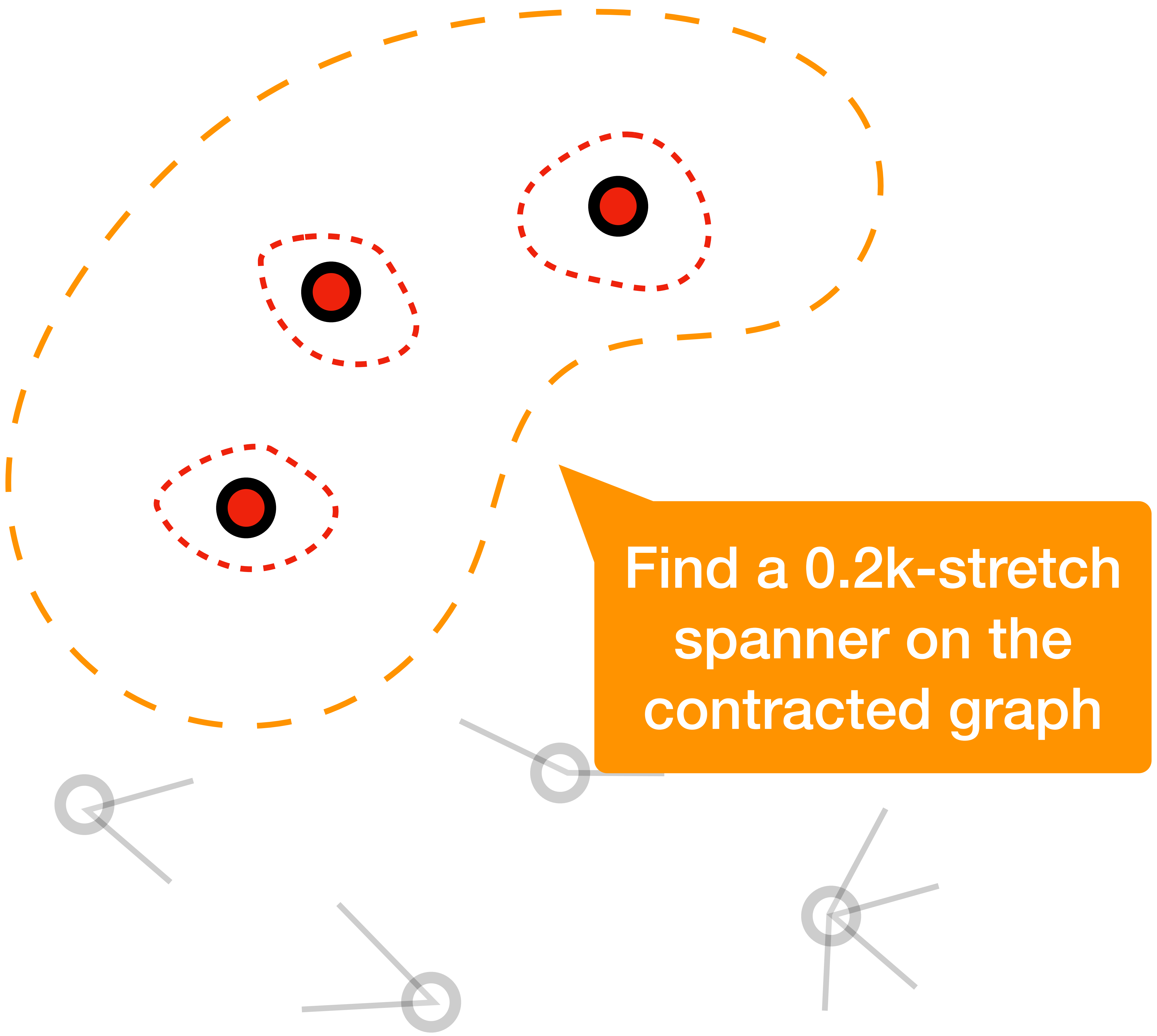

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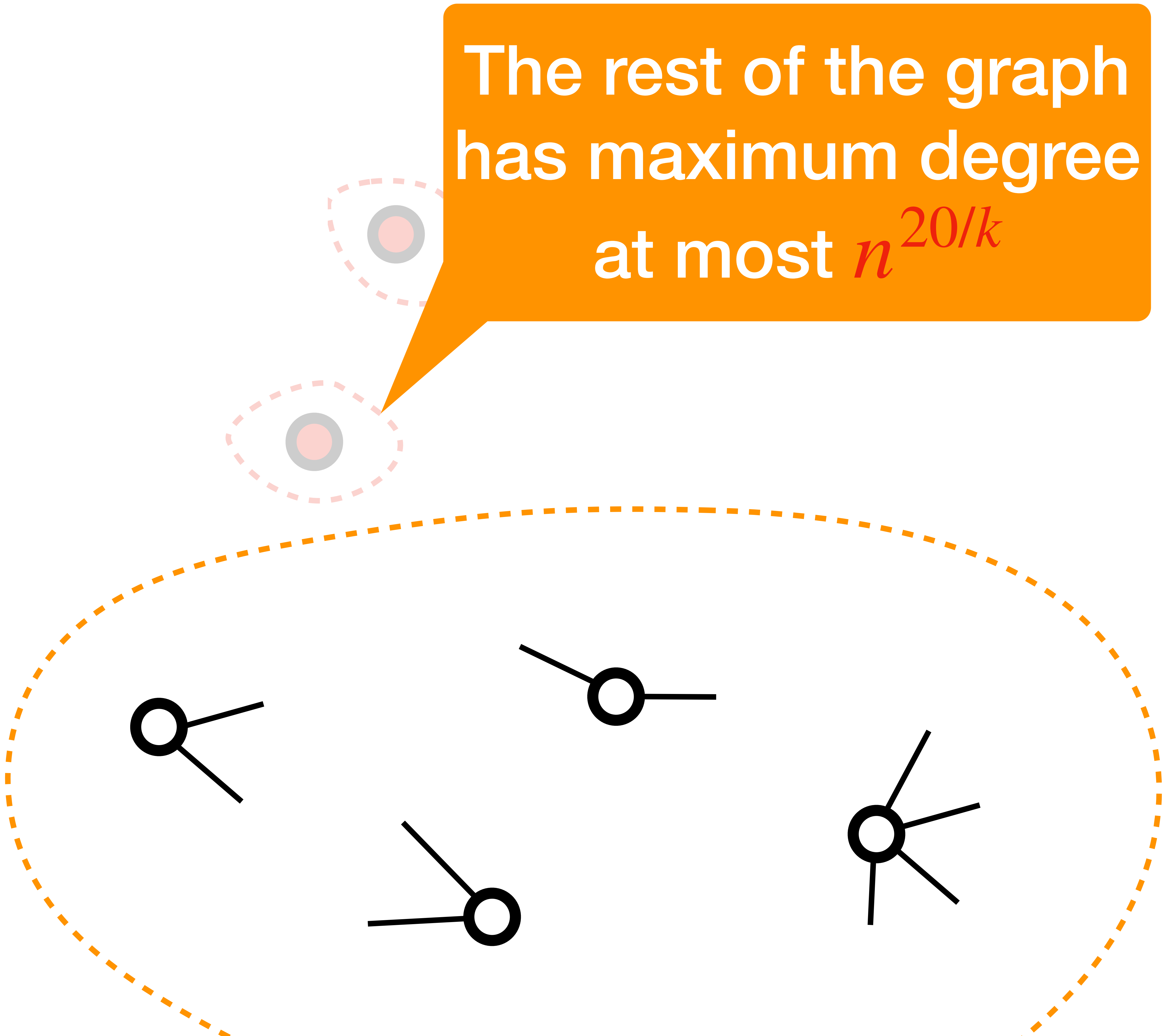
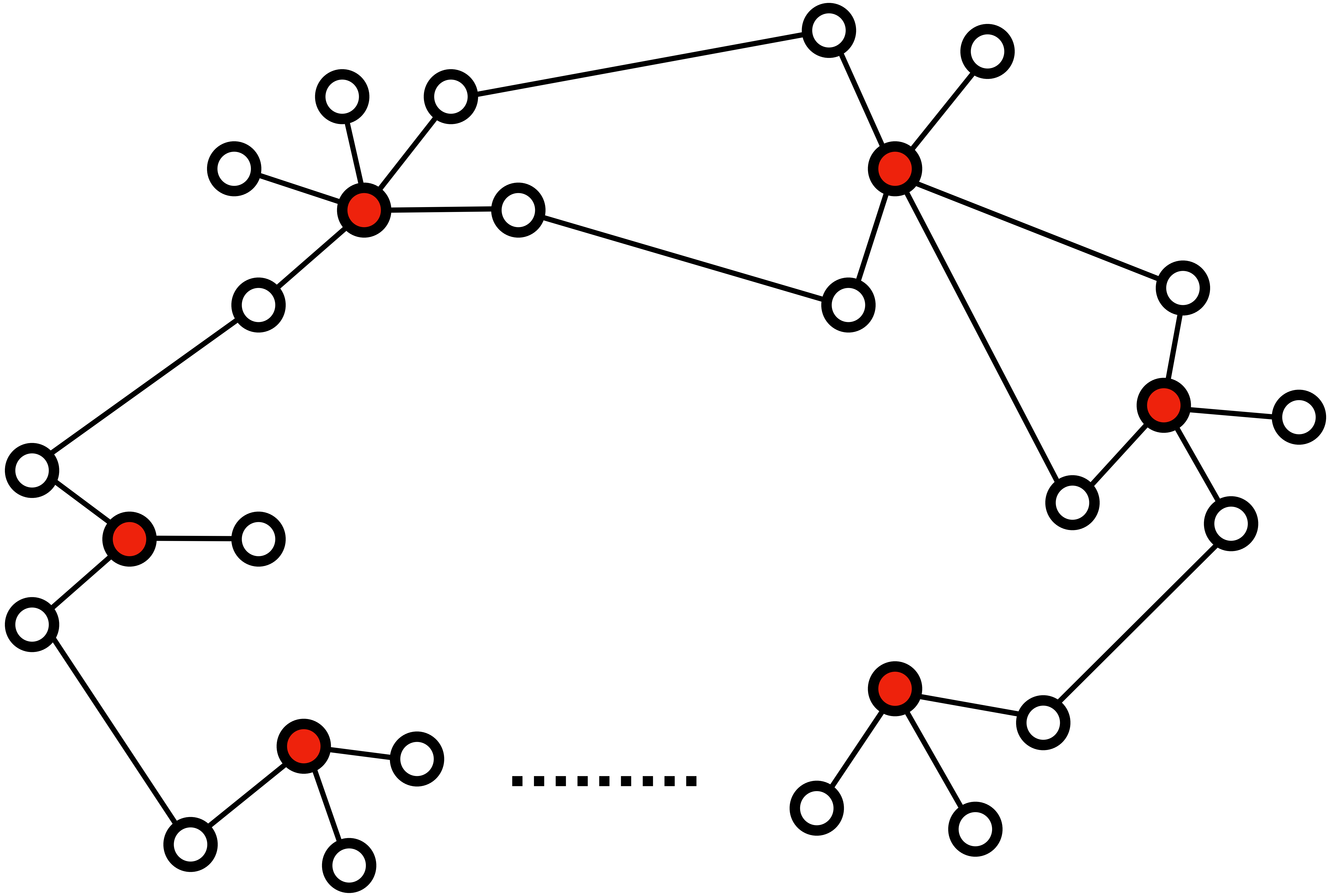
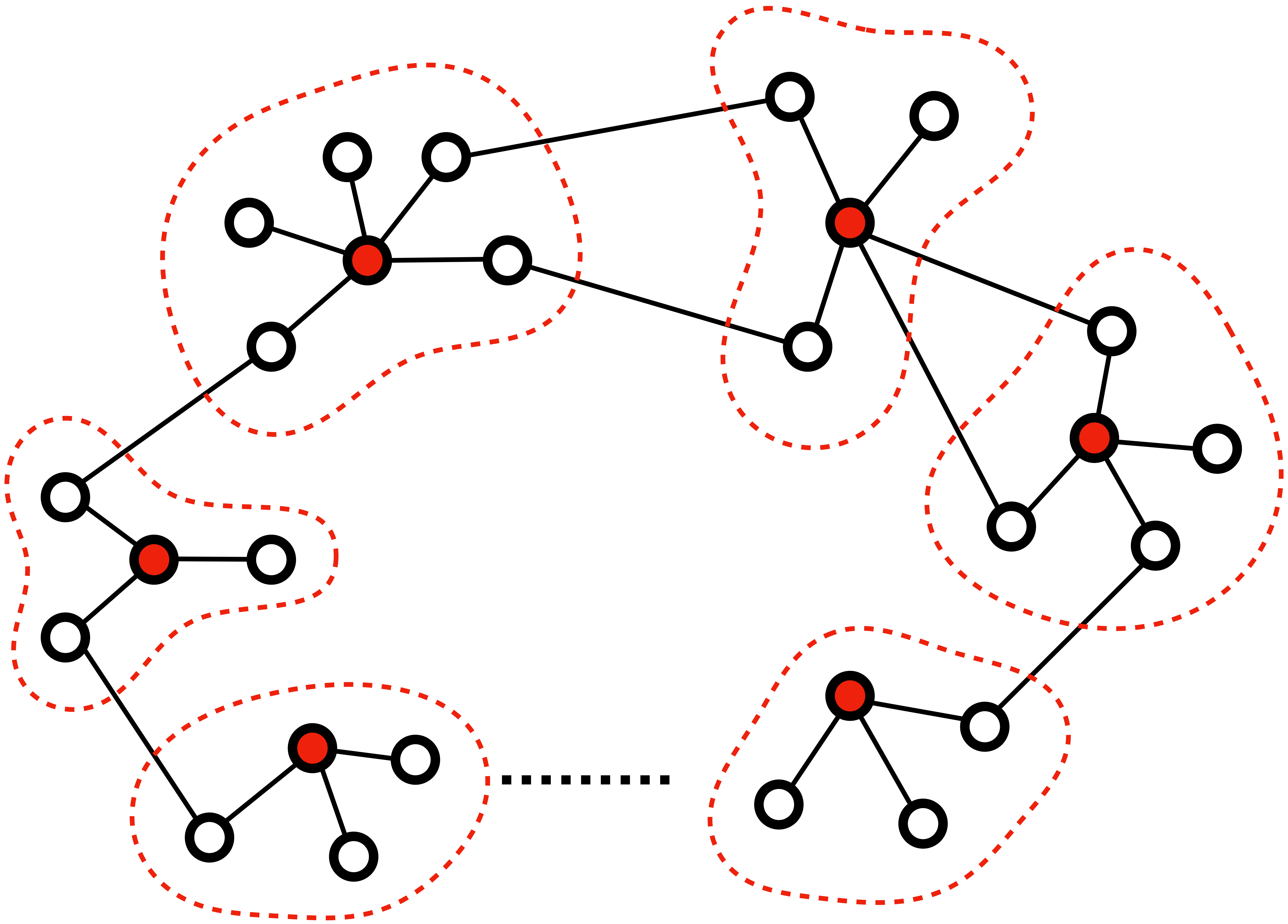


Analysis of size & stretch



Spanner on the **hitting set A**

Analysis of size & stretch



Spanner on the **hitting set A**

1. Assign high-deg vertices to neighbor hitting set vertices

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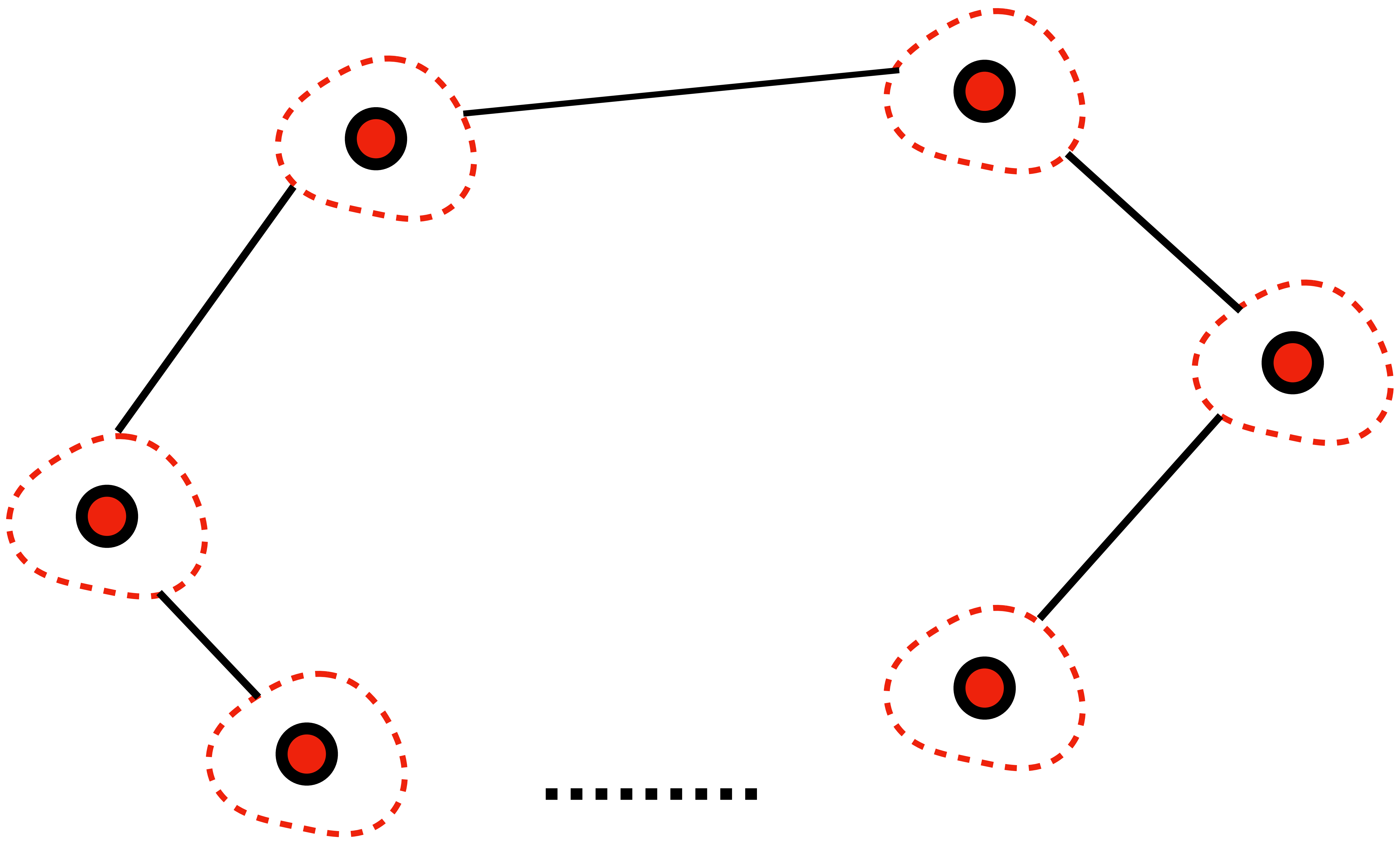
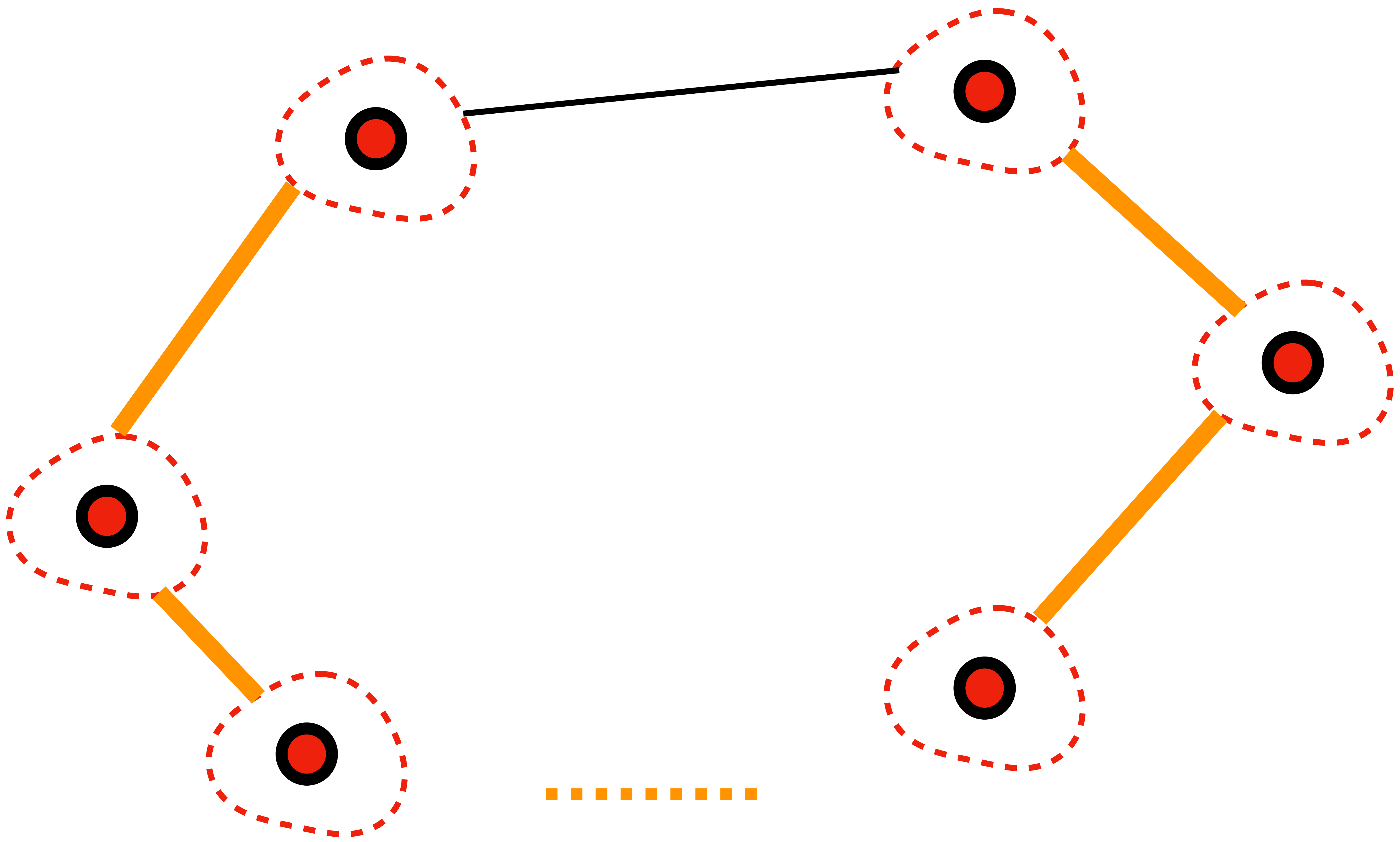


Spanner on the **hitting set A**

1. Assign high-deg vertices to neighbor hitting set vertices

2. Contract stars around hitting set vertices to single nodes

Analysis of size & stretch



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1. Assign high-deg vertices to neighbor hitting set vertices

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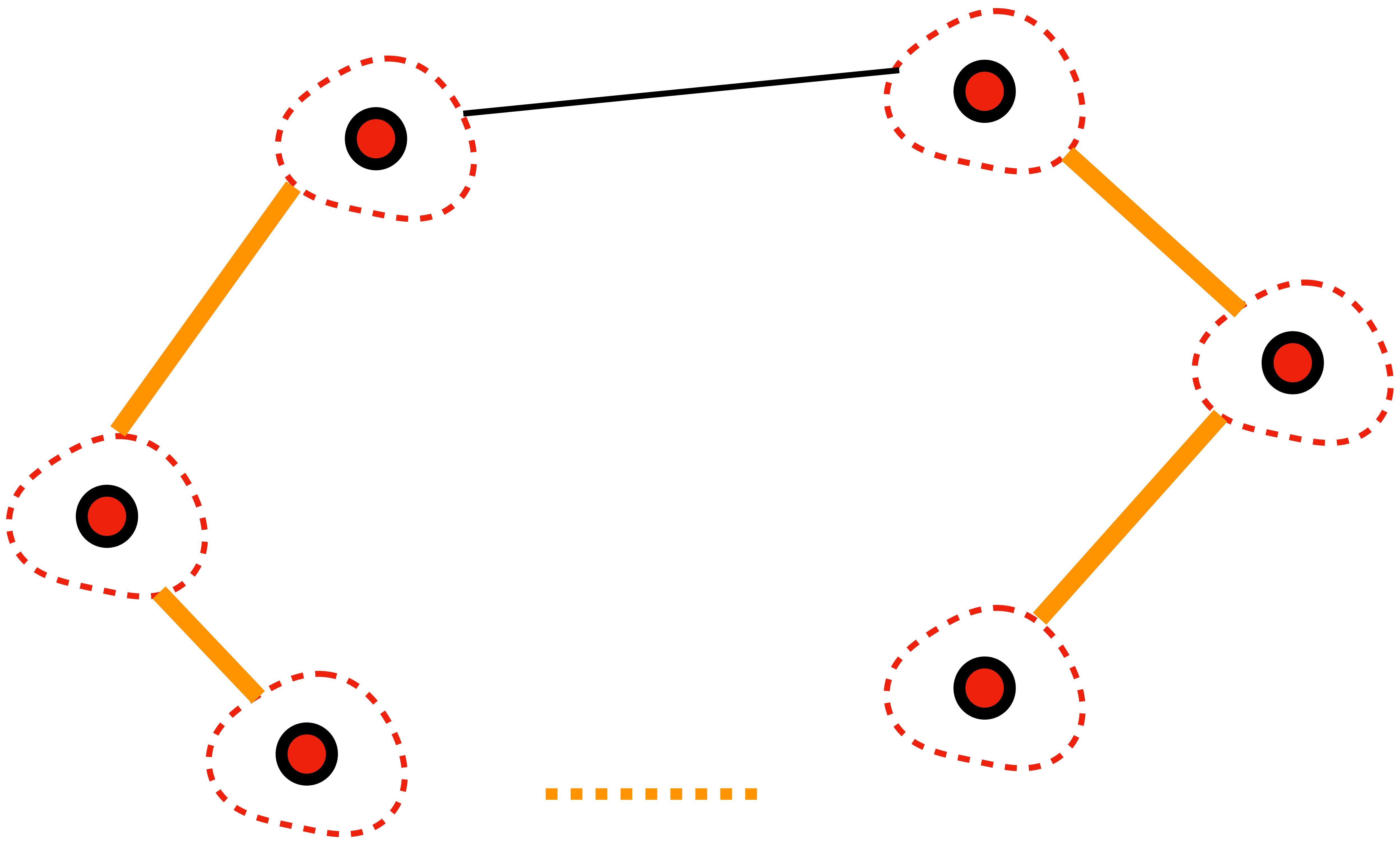

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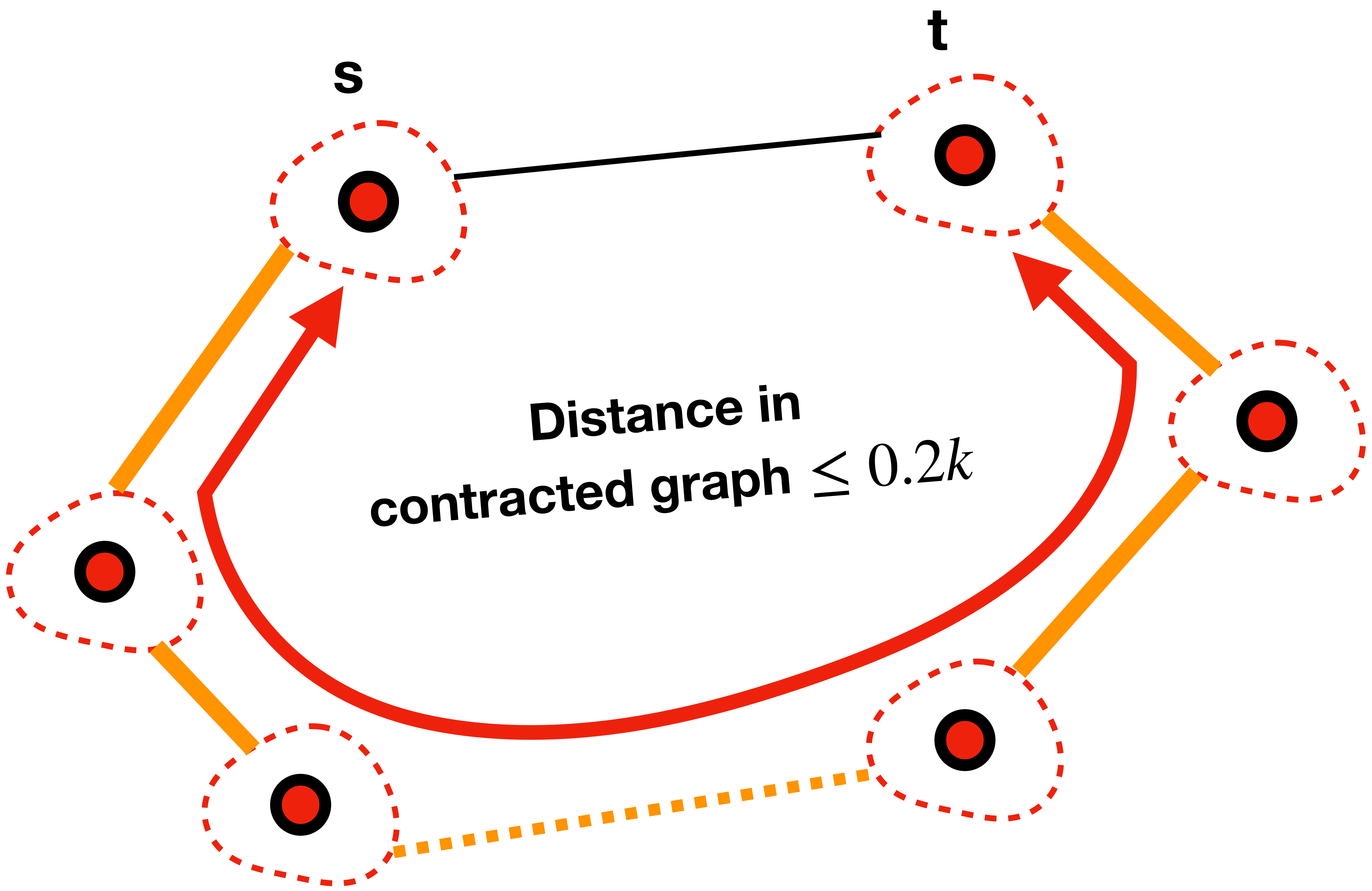
Size analysis:

$$|A|^{1+O(1/k)} = n^{(1-10/k)\cdot(1+O(1/k))} = O(n)$$

Analysis of size & stretch



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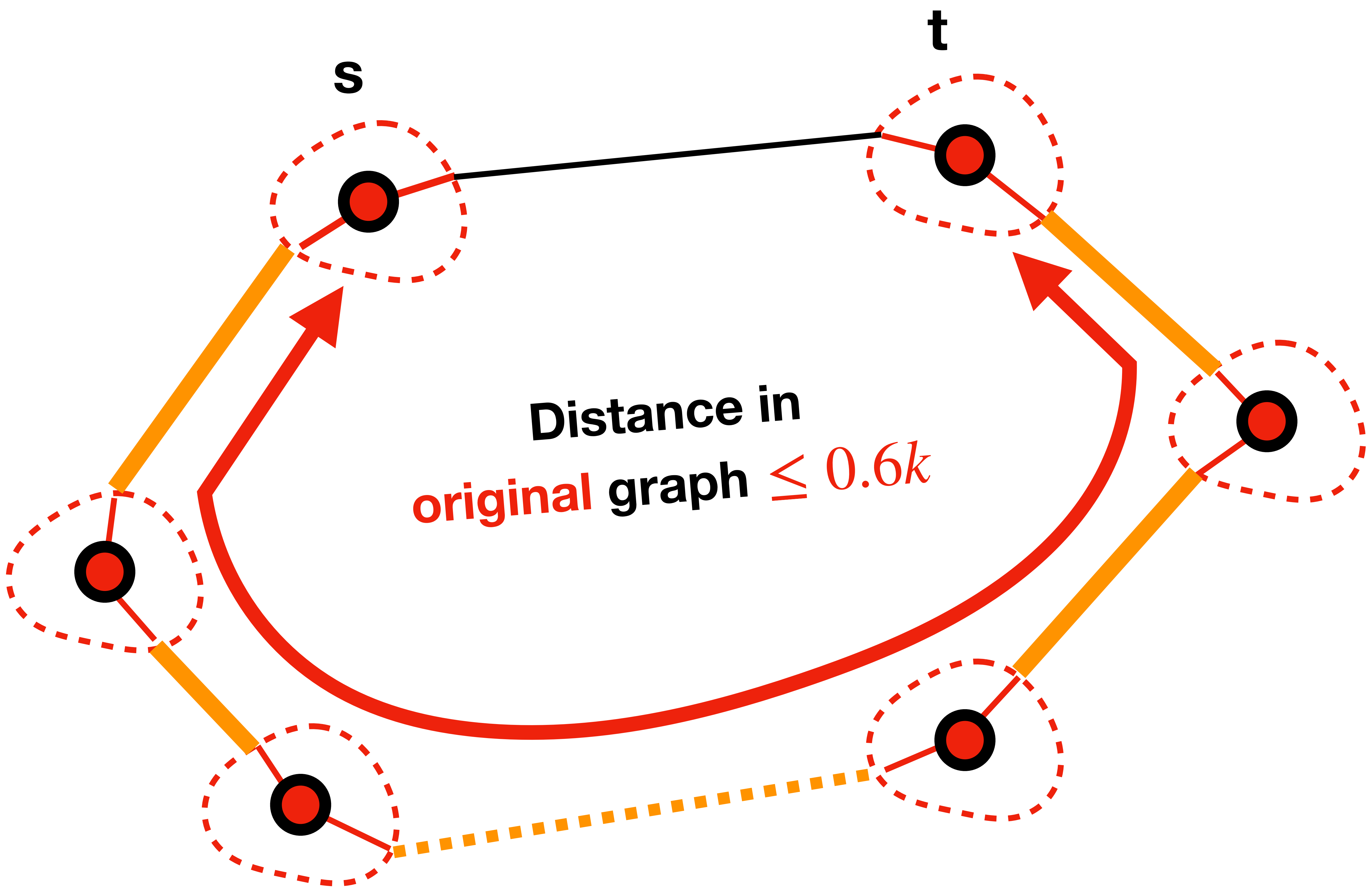
Stretch analysis:

$$\mathrm{dist}(s, t) \leq 0.2k \cdot 3 = 0.6k$$

Analysis of size & stretch



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Stretch analysis:

$\mathrm{dist}(s, t) \leq 0.2k \cdot 3 = 0.6k$

Outline of main algorithm

Degree reduction

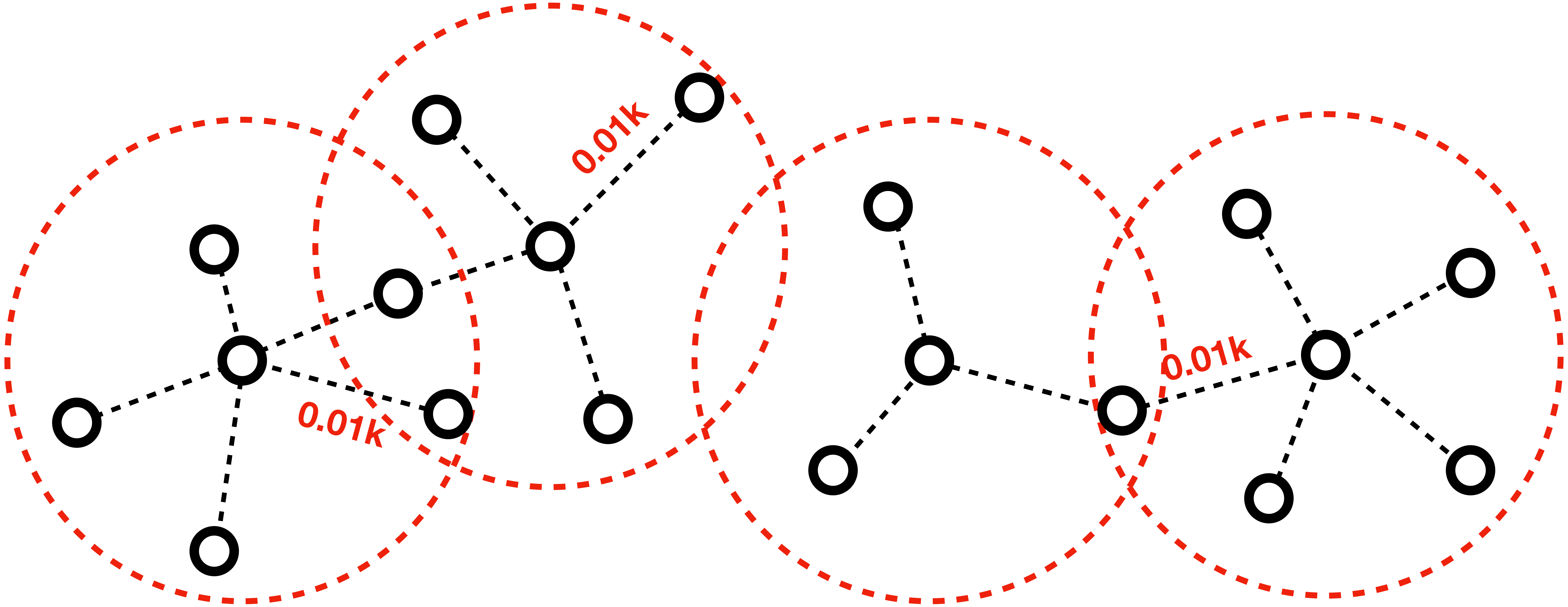
Neighborhood computation

Parallel spanner simulation

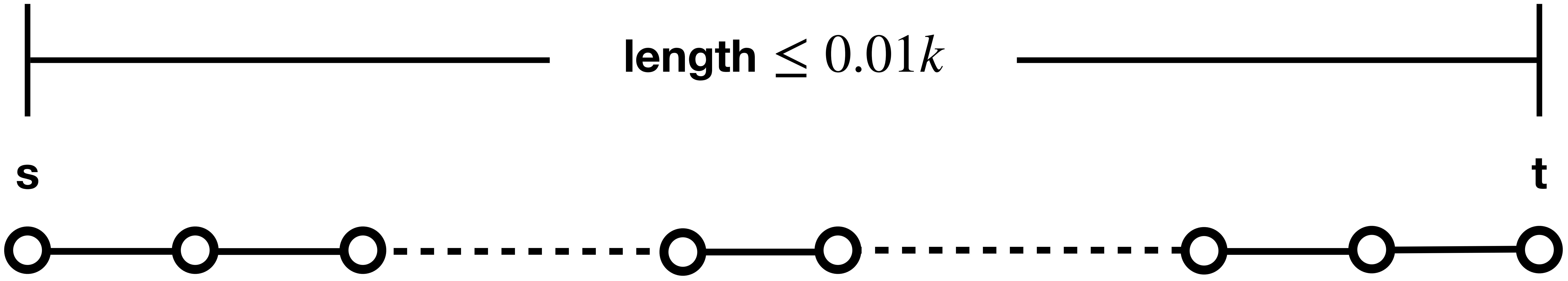
Neighborhood Computation

Input: a graph with max-deg $\leq n^{20/k}$

Output: compute the $0.01k$-neighborhood of each vertex



Neighborhood Computation



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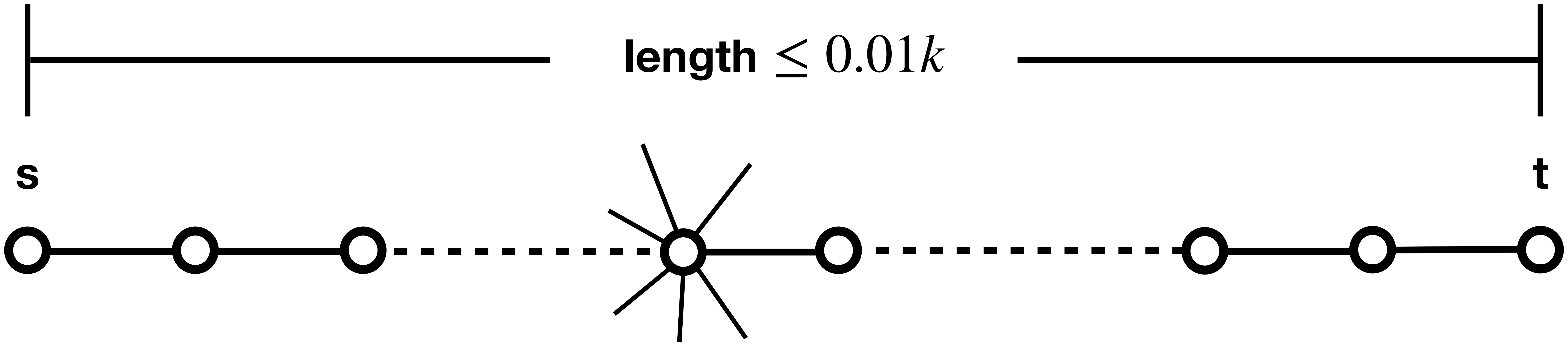

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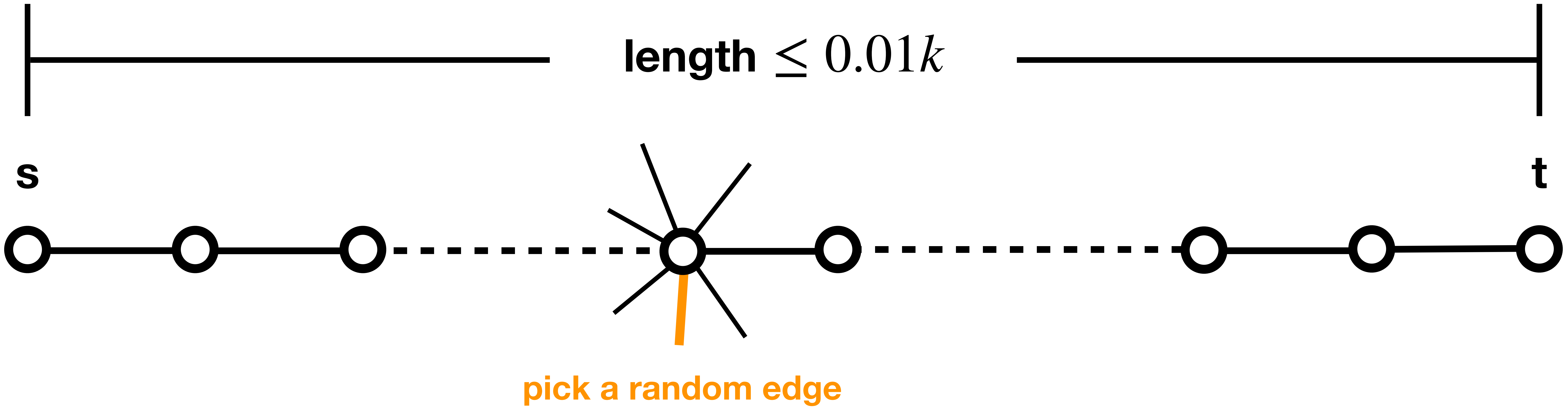

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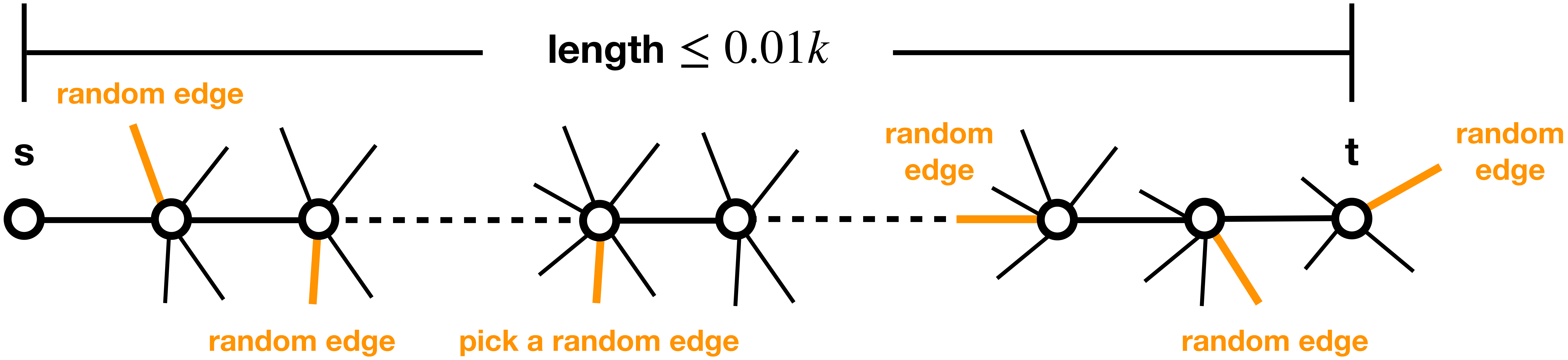

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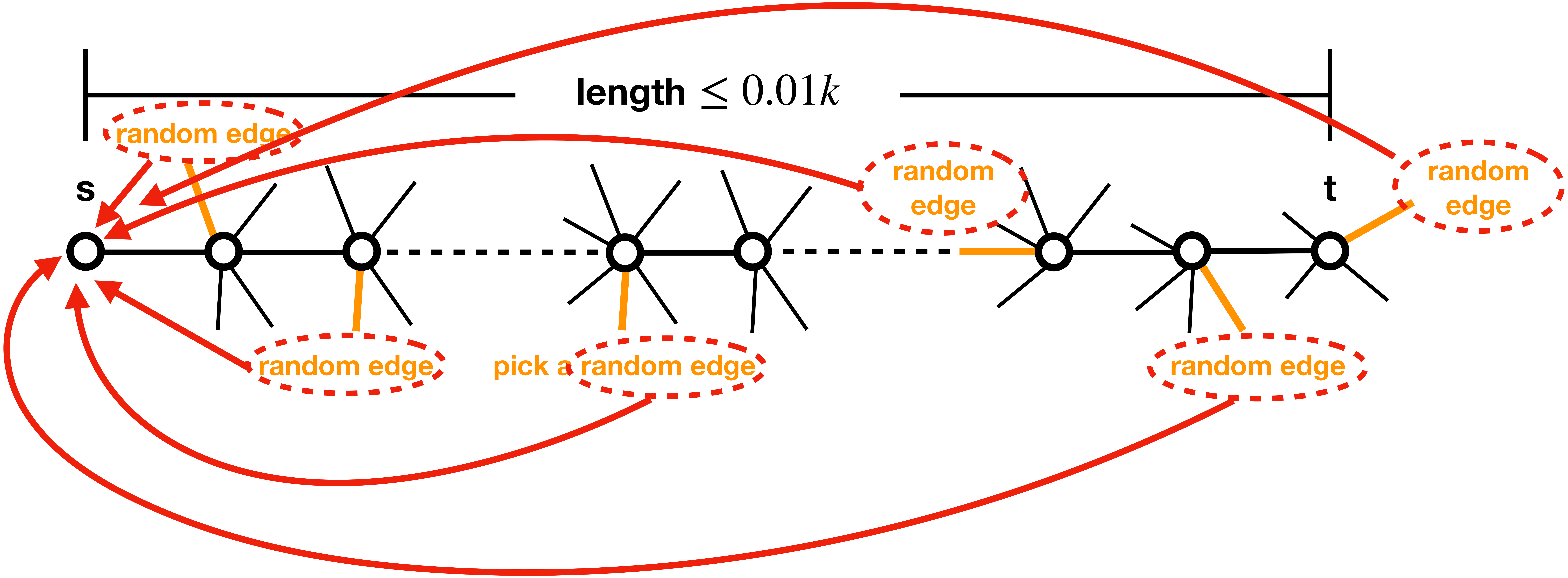
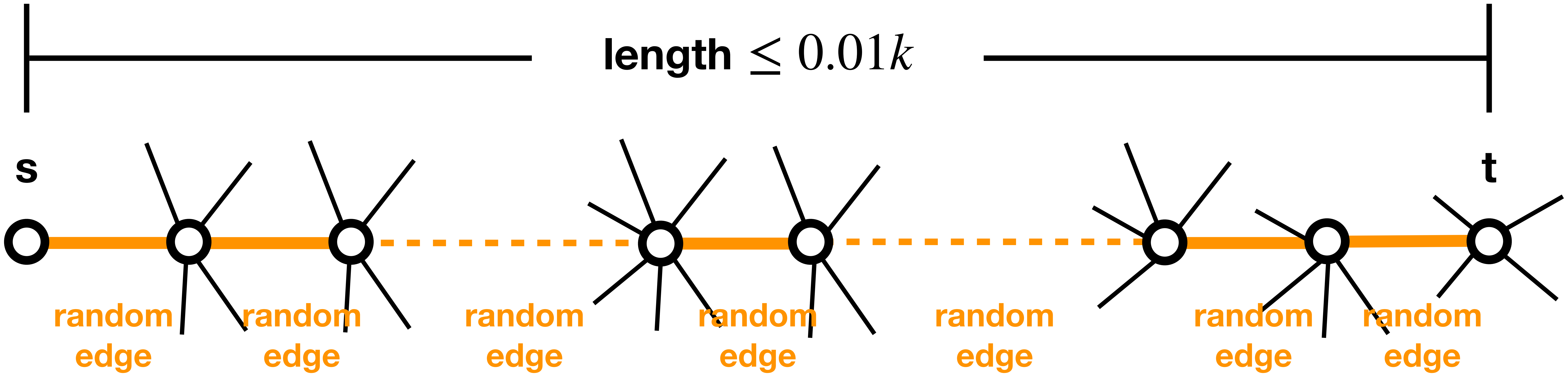


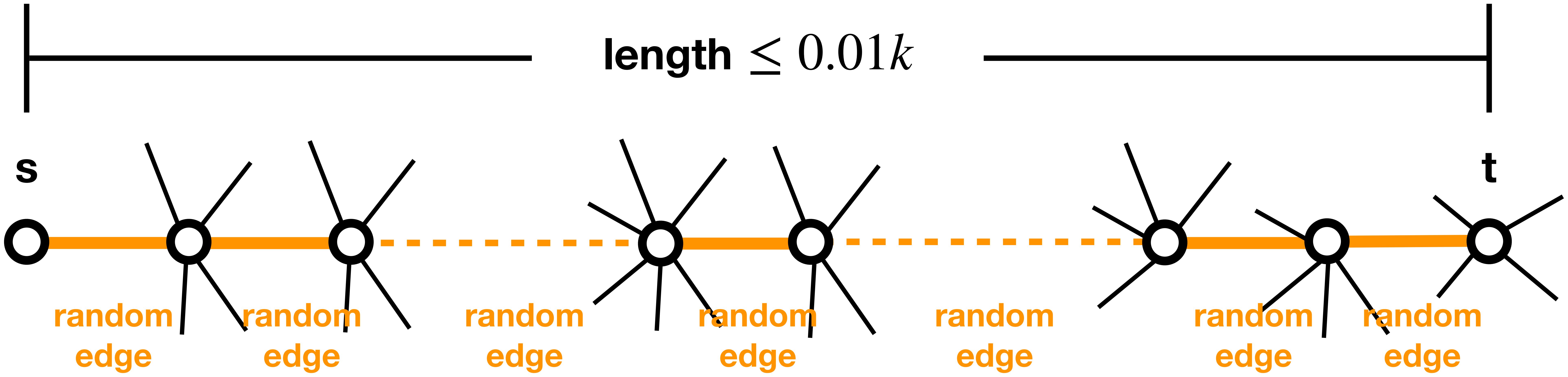
Send all random edges to source vertex s in 1 round

Neighborhood Computation



What is the **probability** that the **s-t path is picked**?

Neighborhood Computation

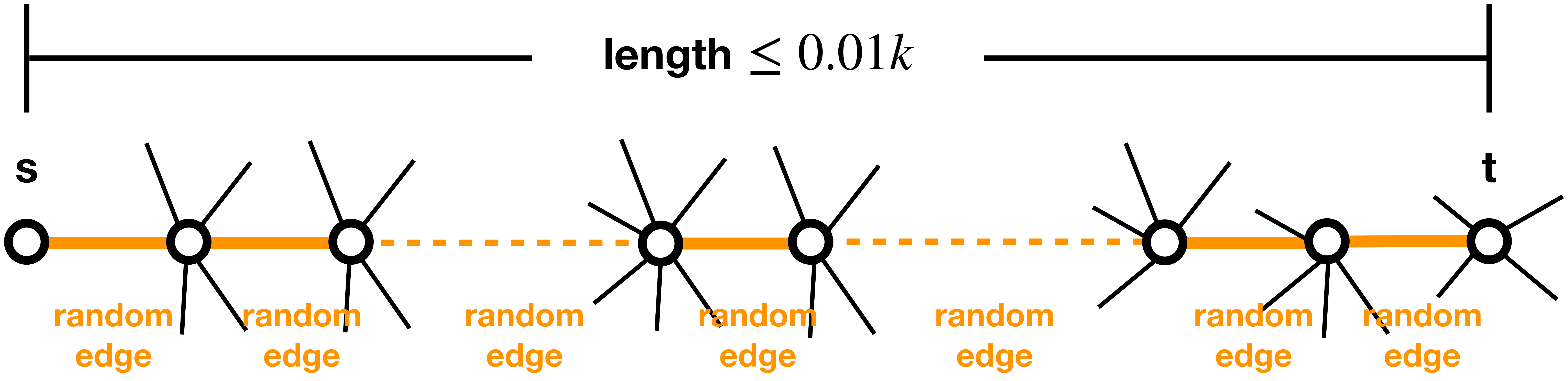


What is the **probability** that the **s-t path is picked**?

- $\Pr[\text{st-path is picked}] \geq (1/\deg)^{0.01k} \geq n^{-20/k \cdot 0.01k} = n^{-0.2}$

(distributed) random walk

Neighborhood Computation



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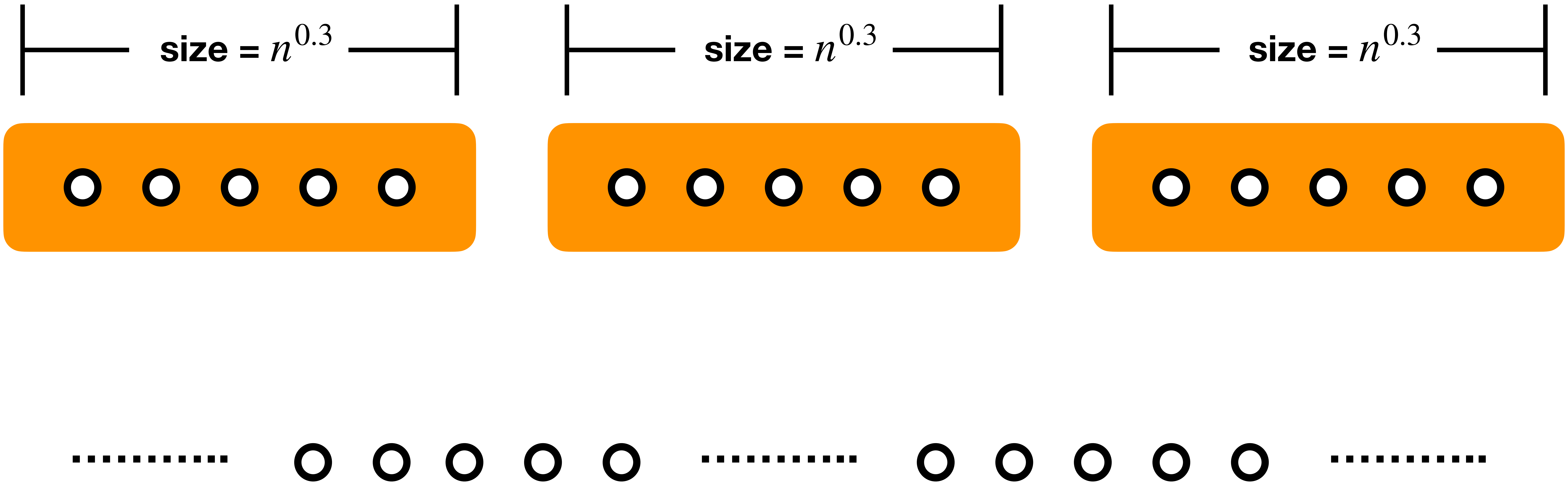
- How to boost success probability?

Boosting by replication



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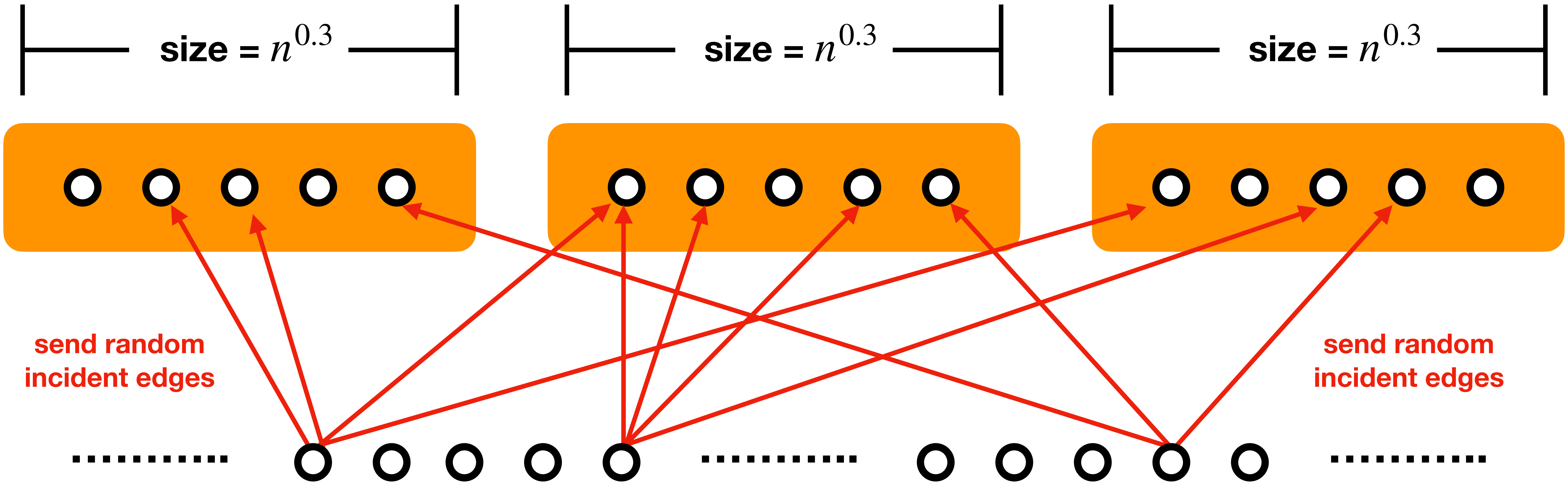

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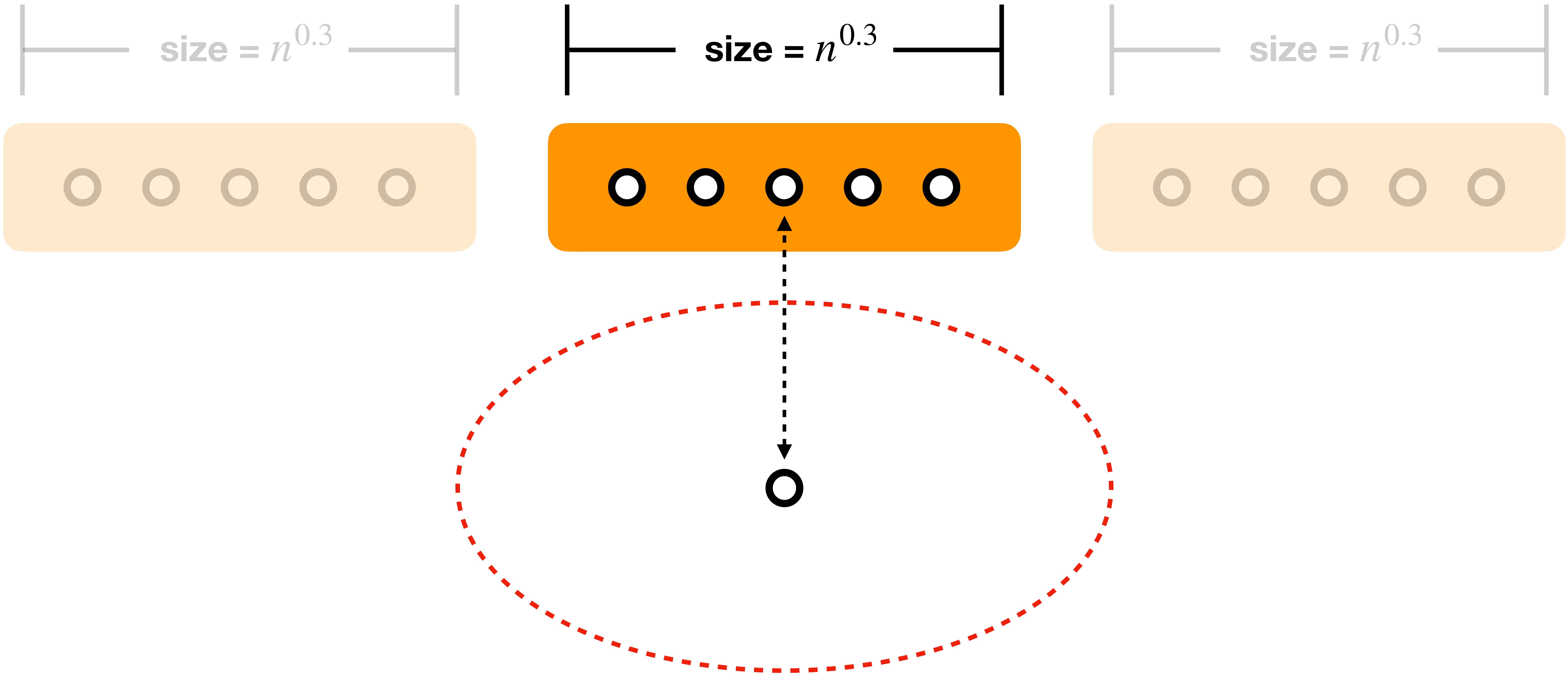

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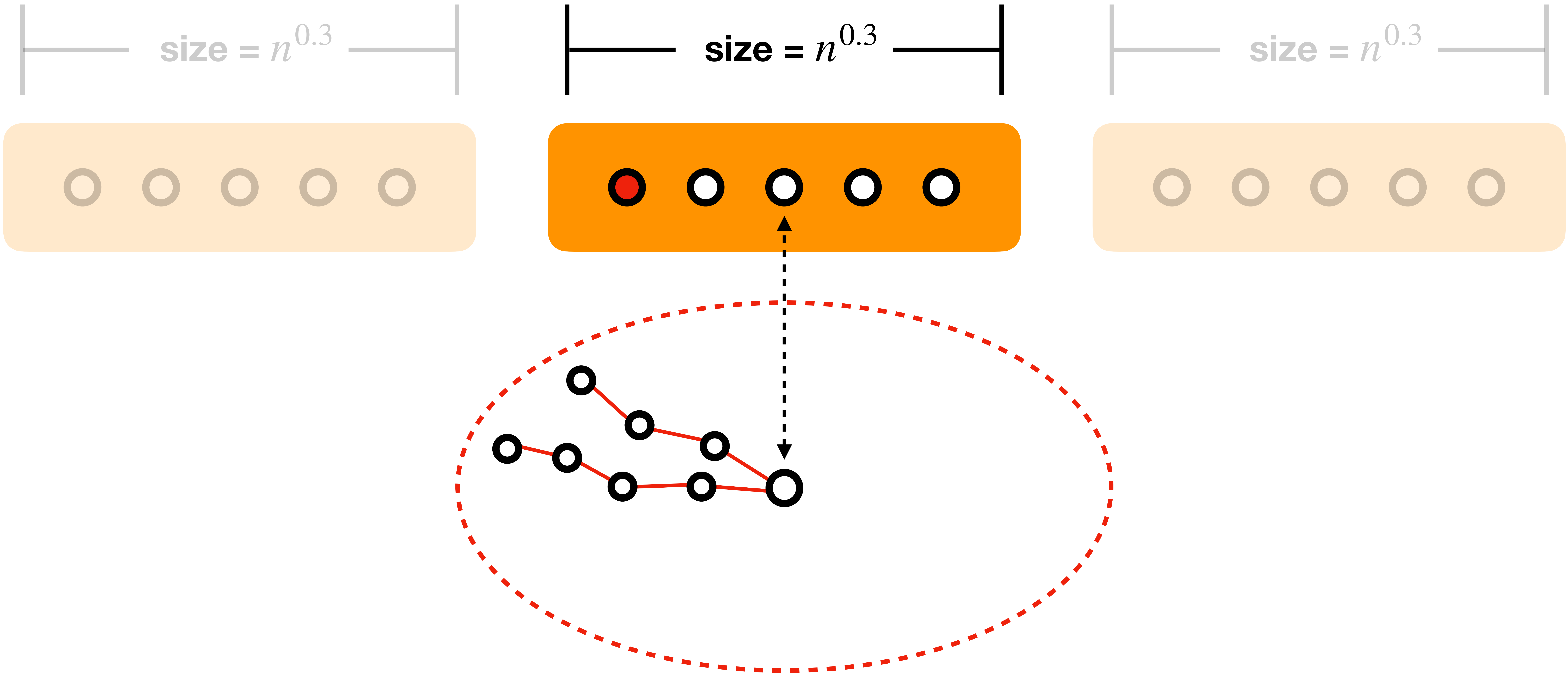

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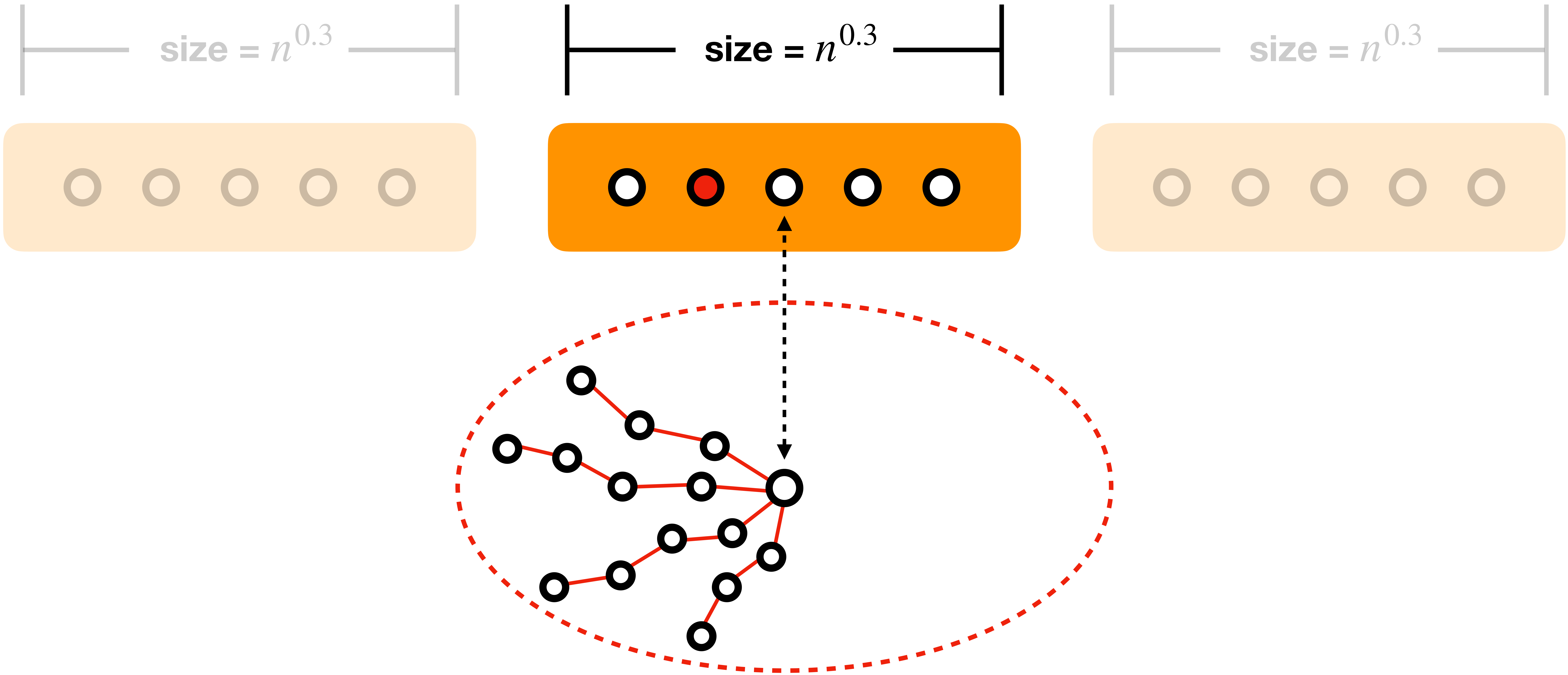

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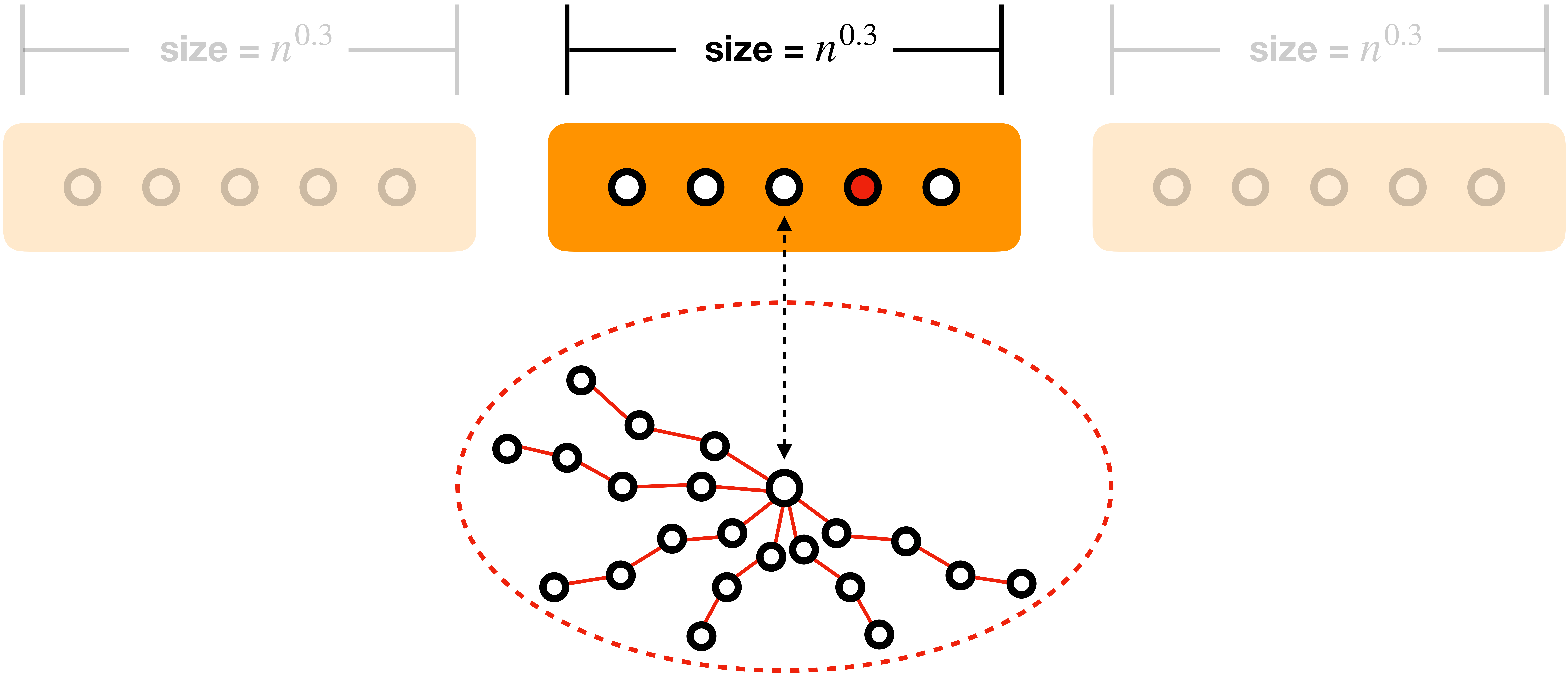

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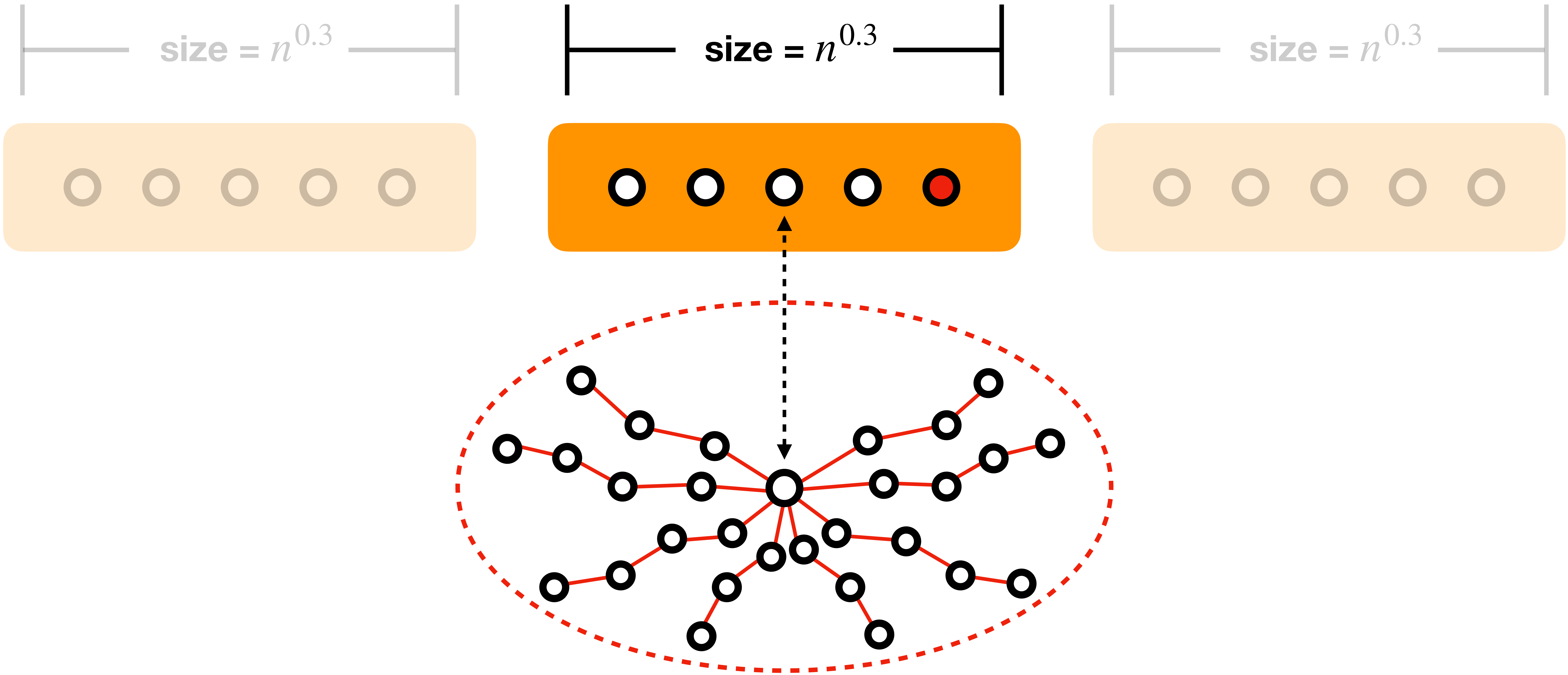

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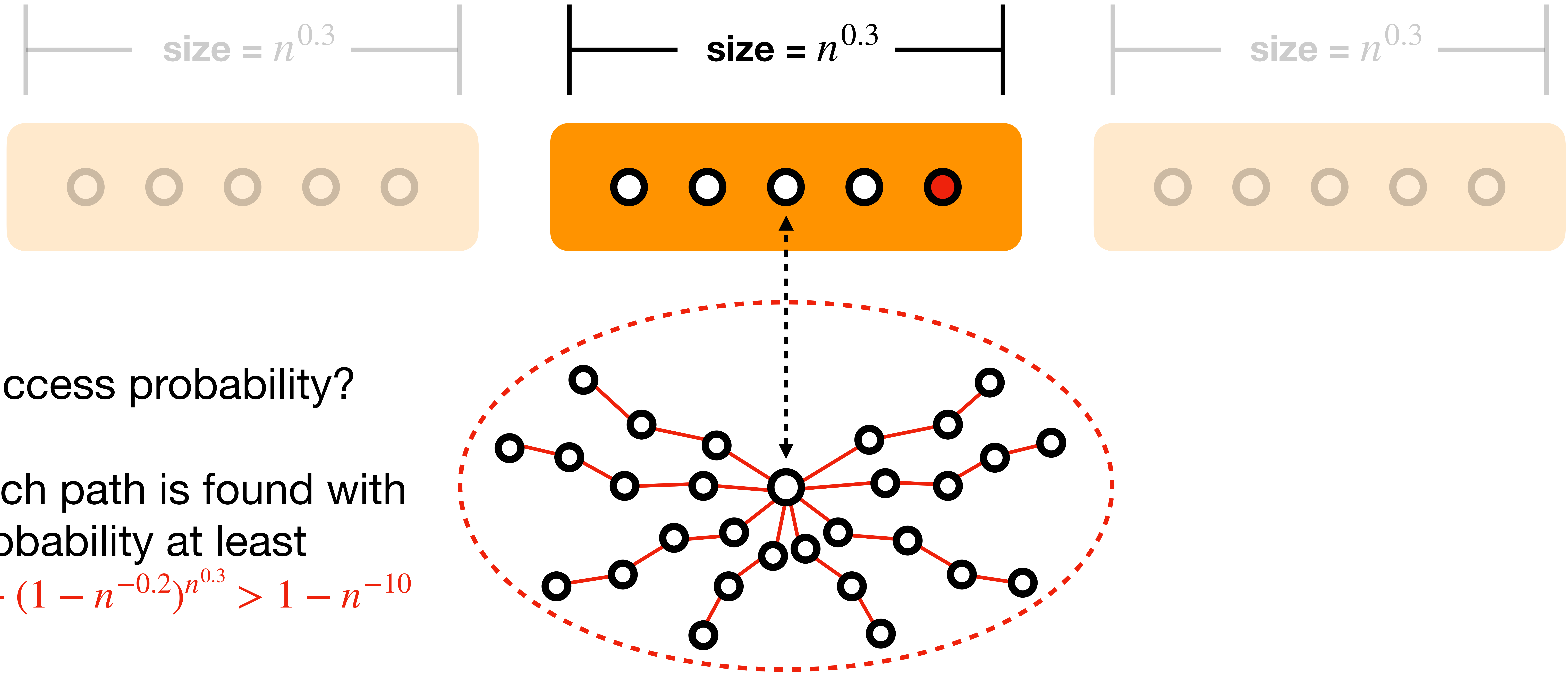

Success probability?

Each path is found with probability at least
$1-(1-n^{-0.2})^{n^{0.3}} > 1-n^{-10}$

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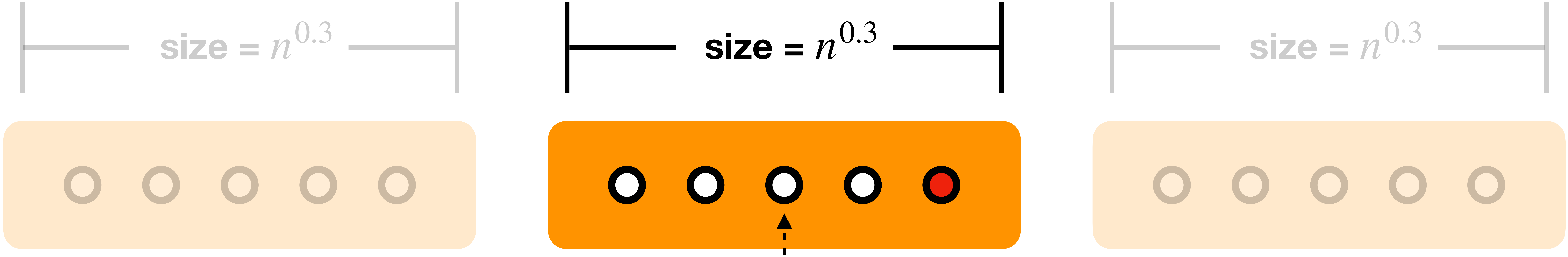

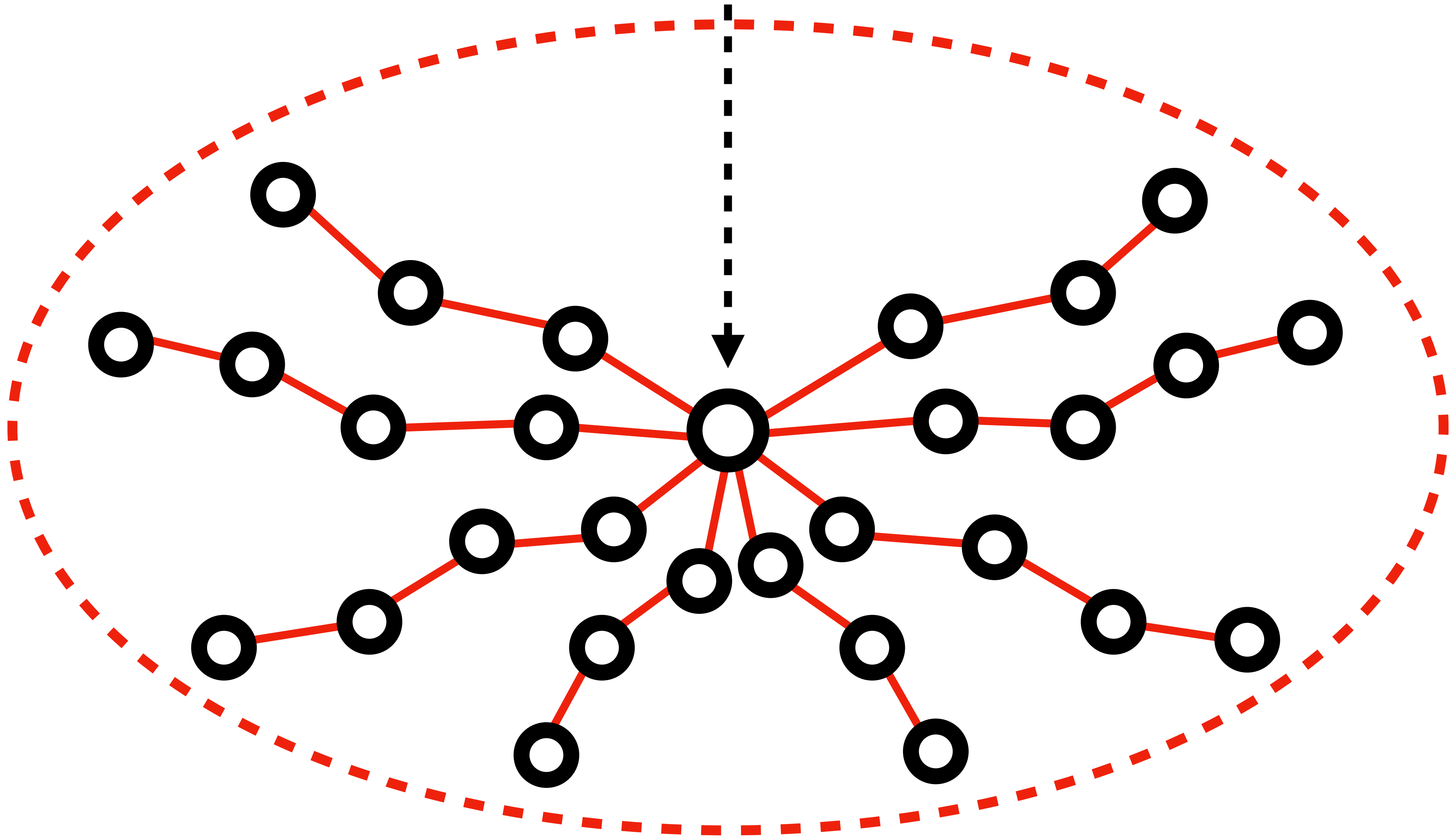

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Communication?

- Receive: $O(n)$
- Send: $n^{0.3} \cdot (\deg)^{0.01k} \leq n^{0.5}$

Outline of main algorithm

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Parallel spanner simulation

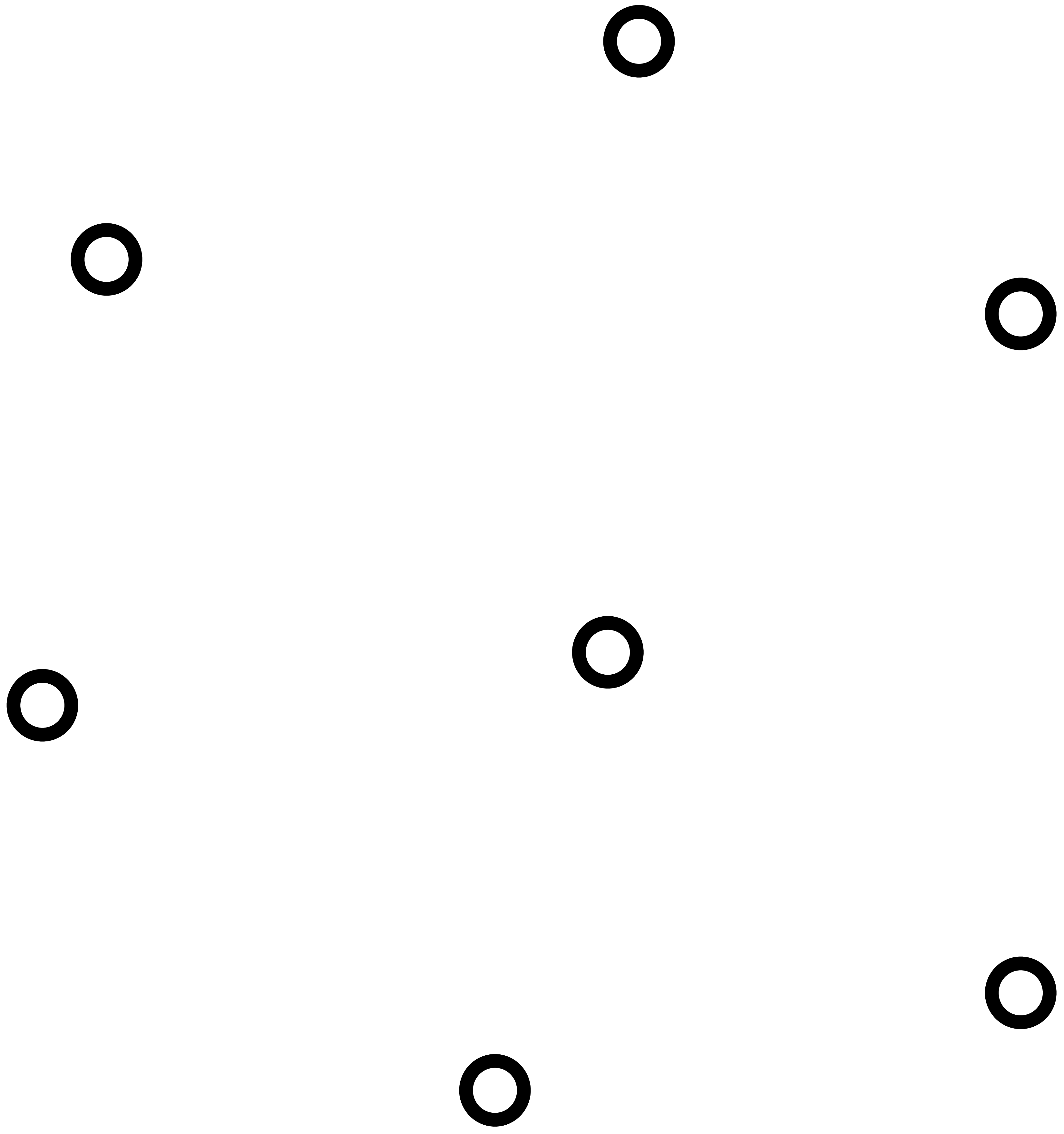
Parallel Spanner Simulation

A PRAM spanner algorithm
[Miller, Peng, Vladu, Xu, 2015]

1. Each vertex *v* takes a random value
   $r_v \sim \exp[\ln(10n)/k]$

2. Define $\text{shift}_u(v) = \text{dist}(u, v) - r_v$
   and $\text{shift}_u = \min_{v \in V}\{\text{shift}_u(v)\}$

3. Add (u, w) to spanner, if
   $\text{shift}_u(v) \leq \text{shift}_u + 1$



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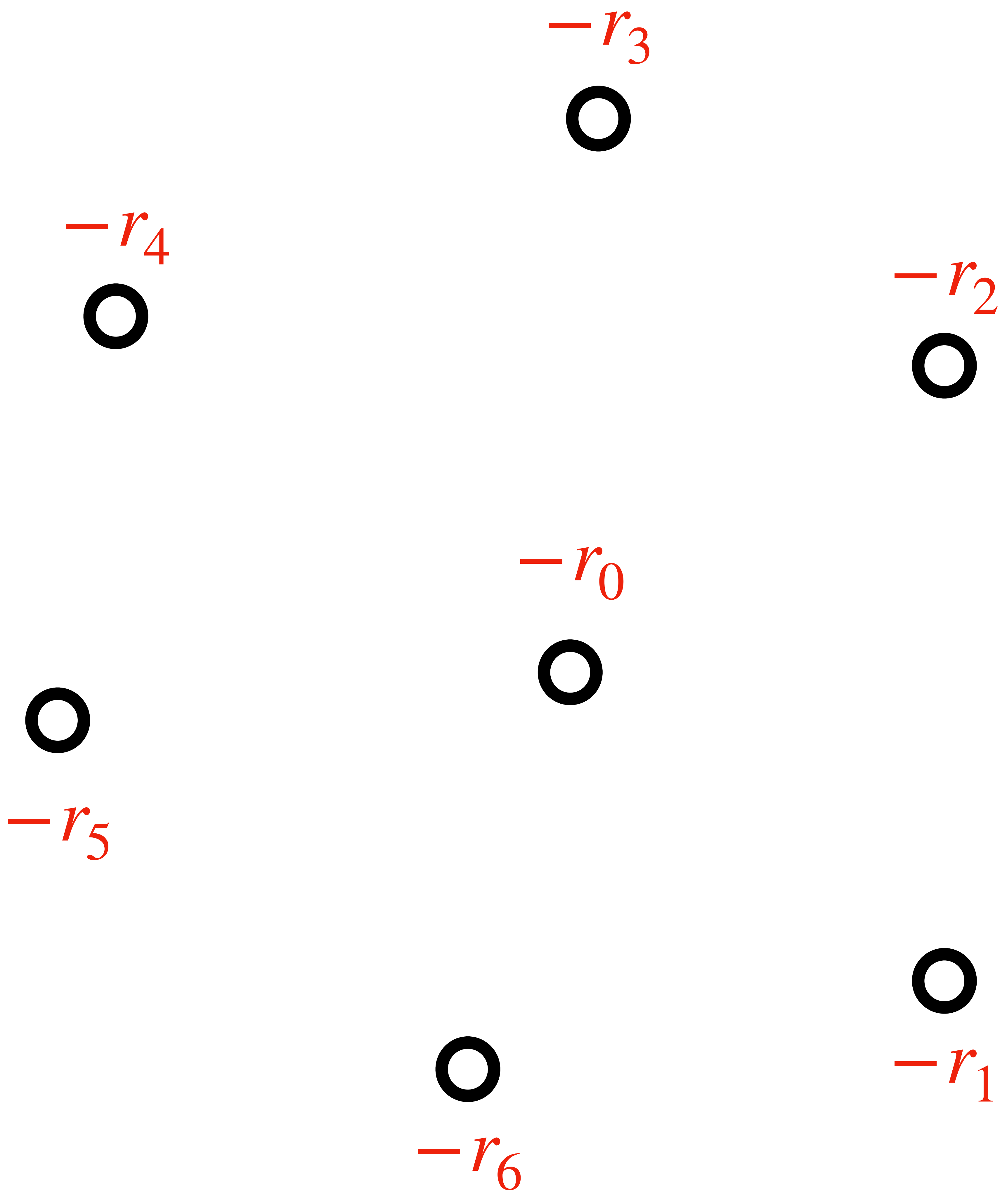

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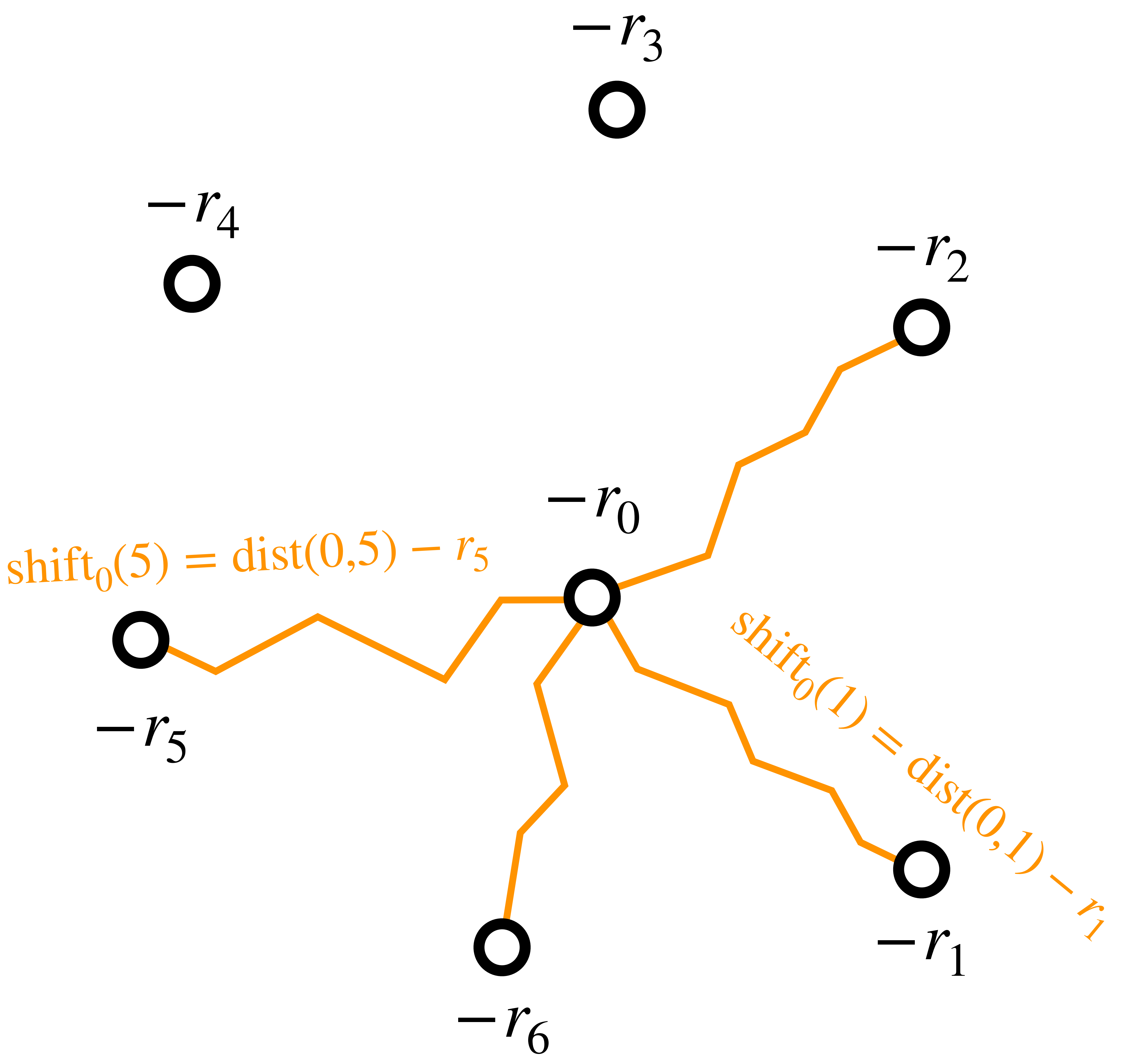

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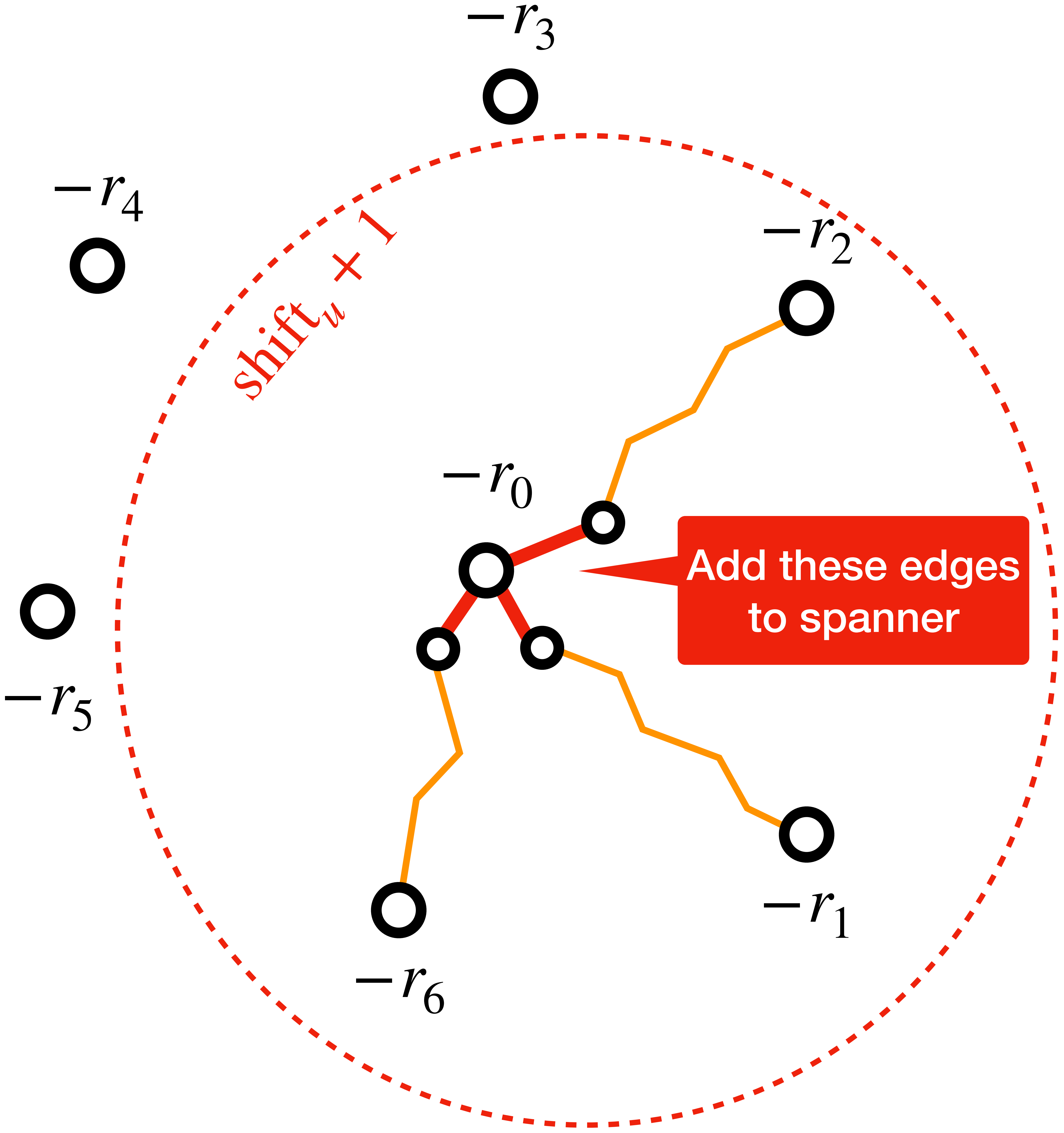
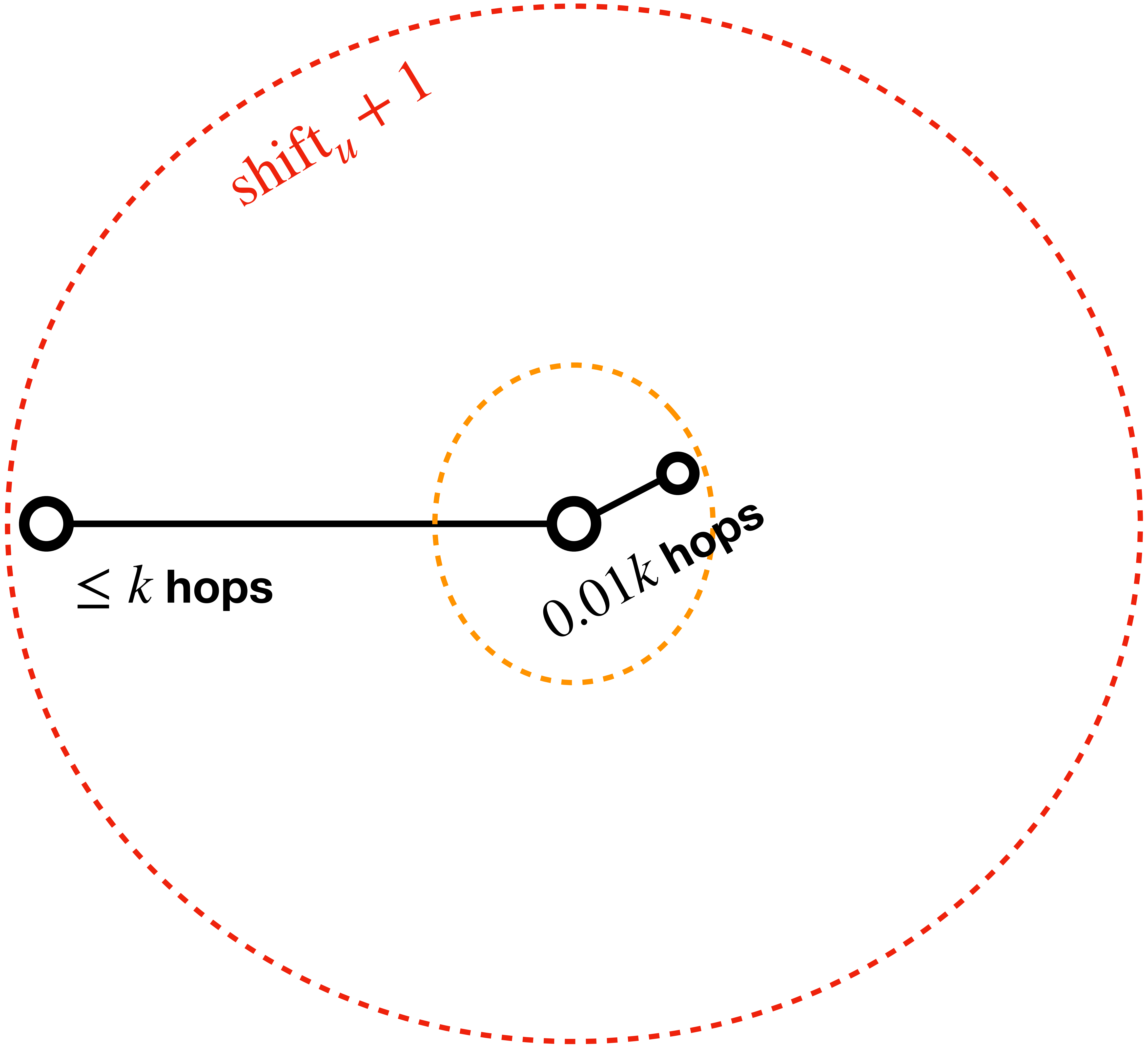


Parallel Spanner Simulation

Simulating the parallel spanner

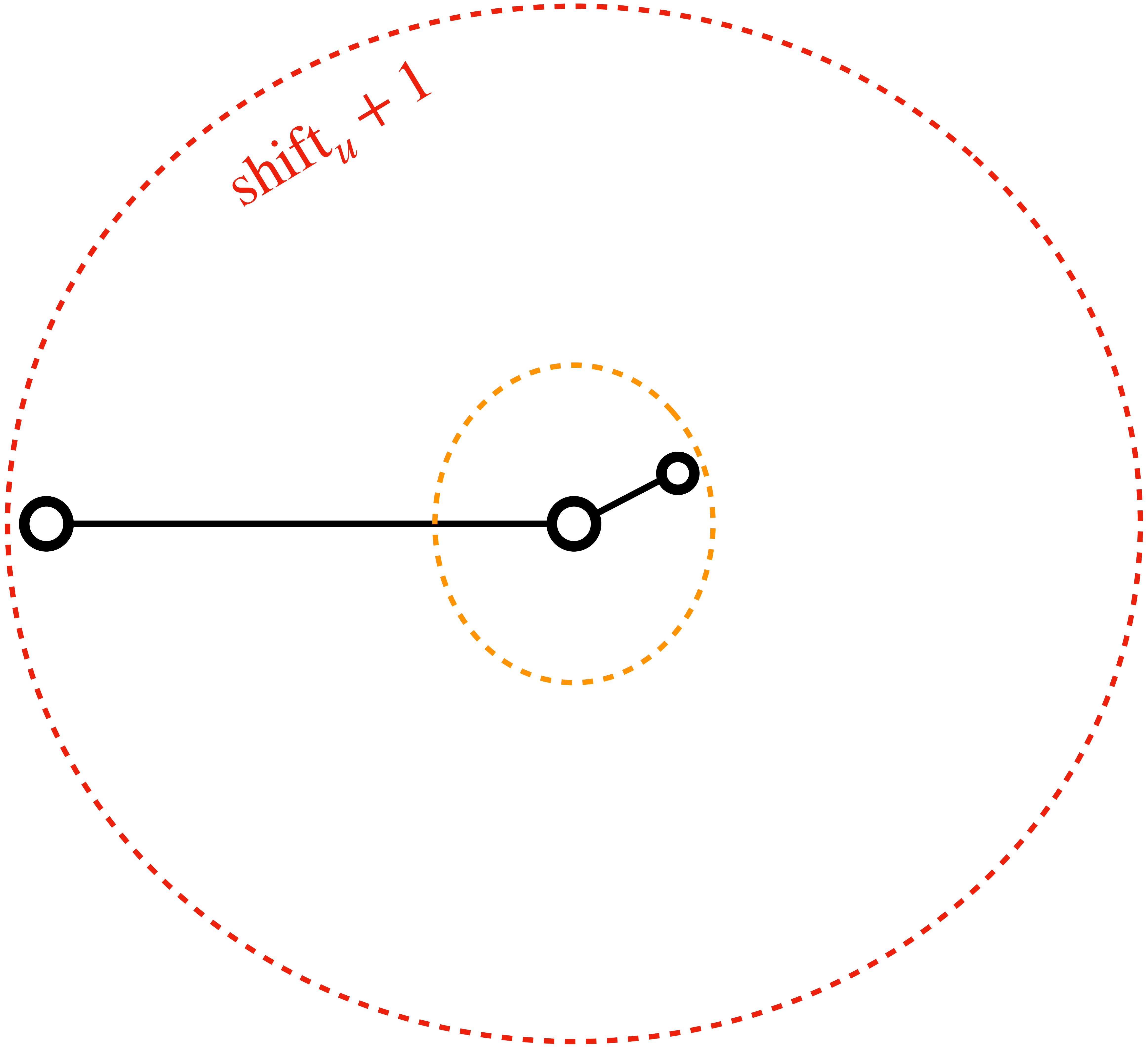
- The radius of $(\mathrm{shift}_u + 1)$**-area** is at most **k hops** [MPVX'15]
- Our neighborhood subroutine only computes **0.01k** ball



Parallel Spanner Simulation

Simulating the parallel spanner

- The radius of $(\mathrm{shift}_u + 1)$**-area** is at most **k hops** [MPVX'15]
- Our neighborhood subroutine only computes **0.01k** ball
- **100-round** of communication can collect all relevant values of $\mathrm{shift}_u(v)$ within the $(\mathrm{shift}_u + 1)$**-area**



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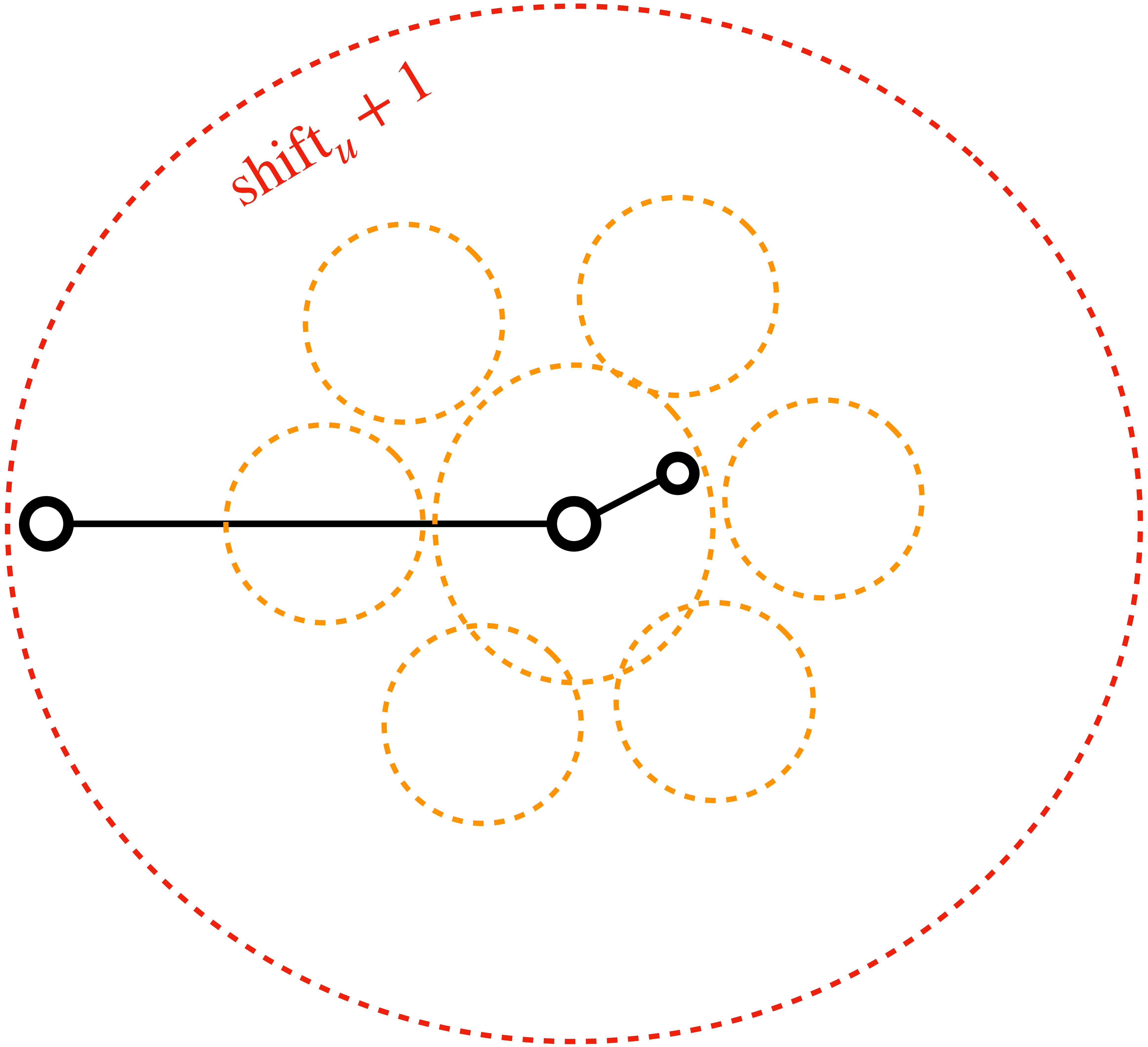

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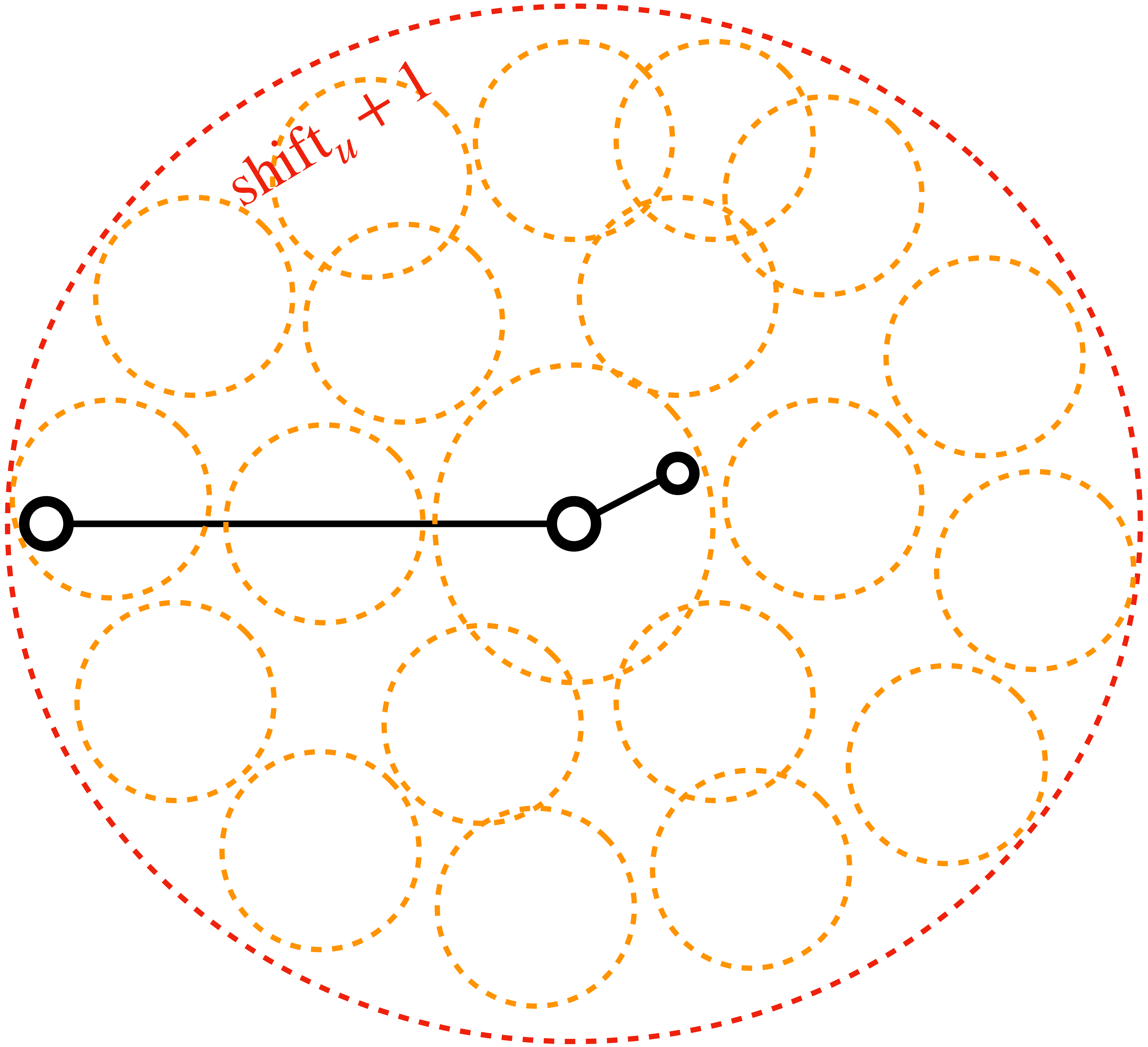

Further questions

Optimal spanners in weighted graphs in congested clique?
In our work, either $(1+\epsilon)(2k-1)$ stretch or $kn^{1+1/k}$ edges

1. Greedy spanner, highly sequential

2. $O(kn^{2+1/k})$ sequential runtime [Roditty & Zwick, 2004]