Near-Optimal Approximate Dual-Failure Replacement Paths

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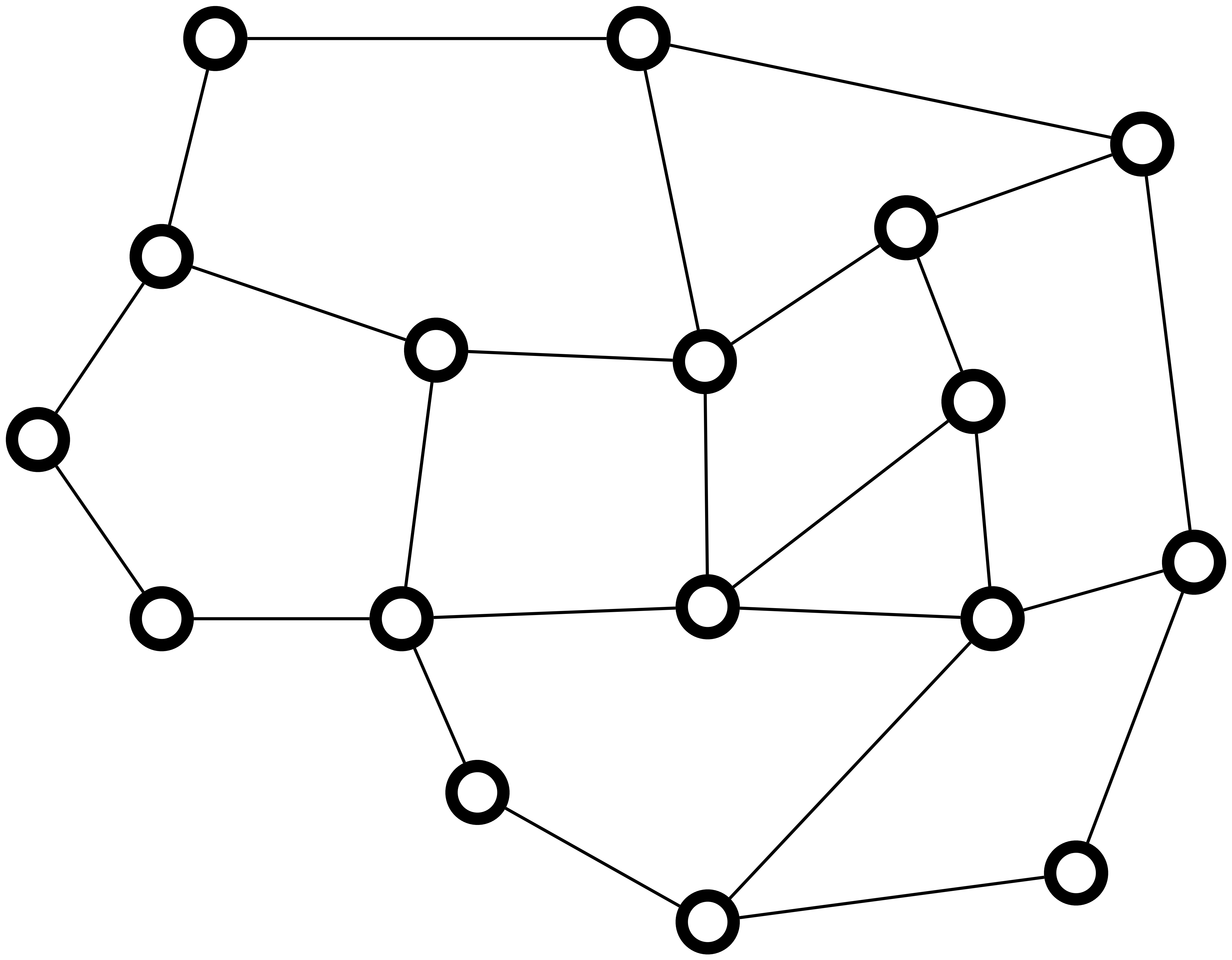
Emergency Planning

Two-phase problem:

1. Preprocess the input graph
2. One/multiple links break, and recover info in the new graph

Costs:

1. Preprocessing time
2. Recovery time



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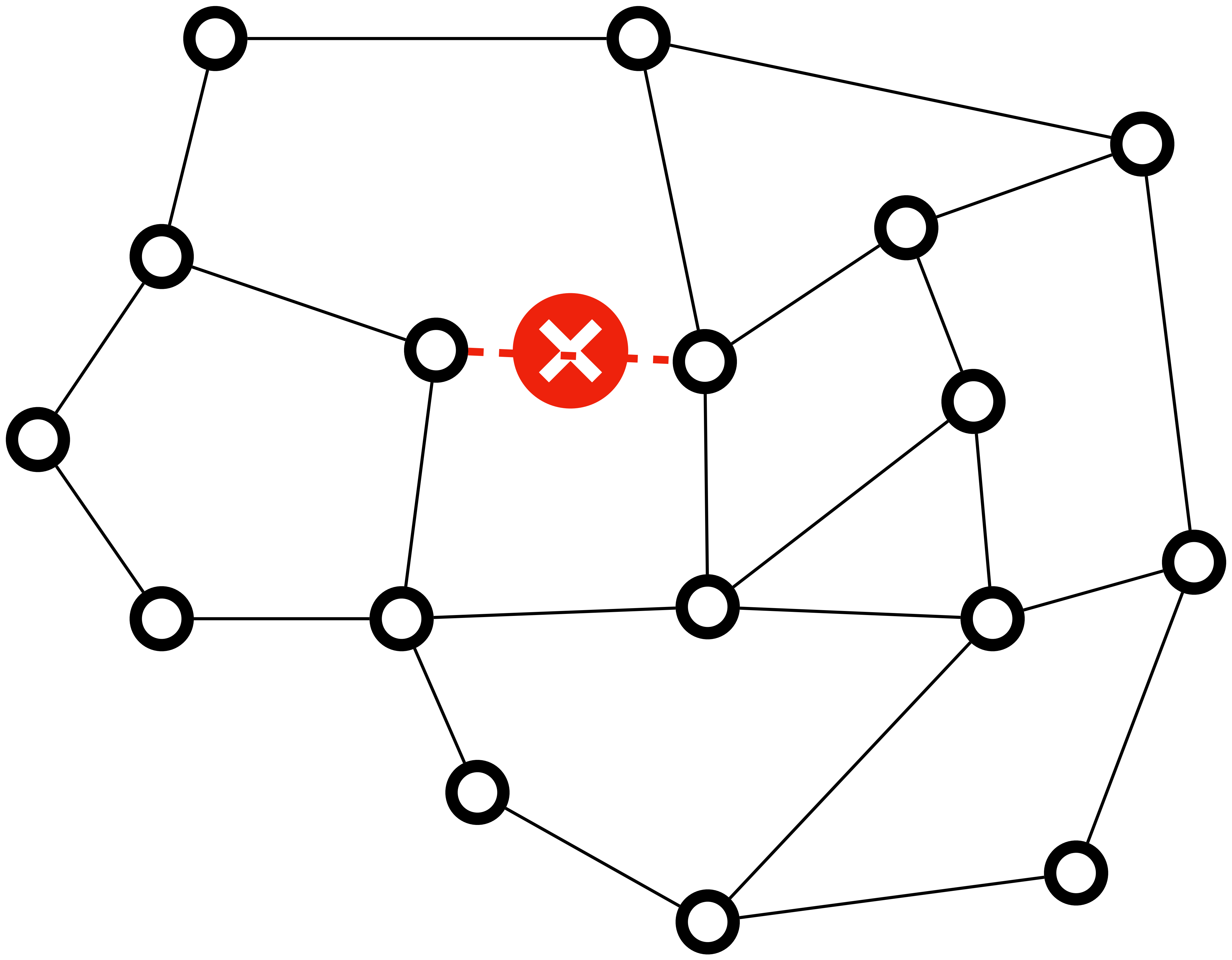

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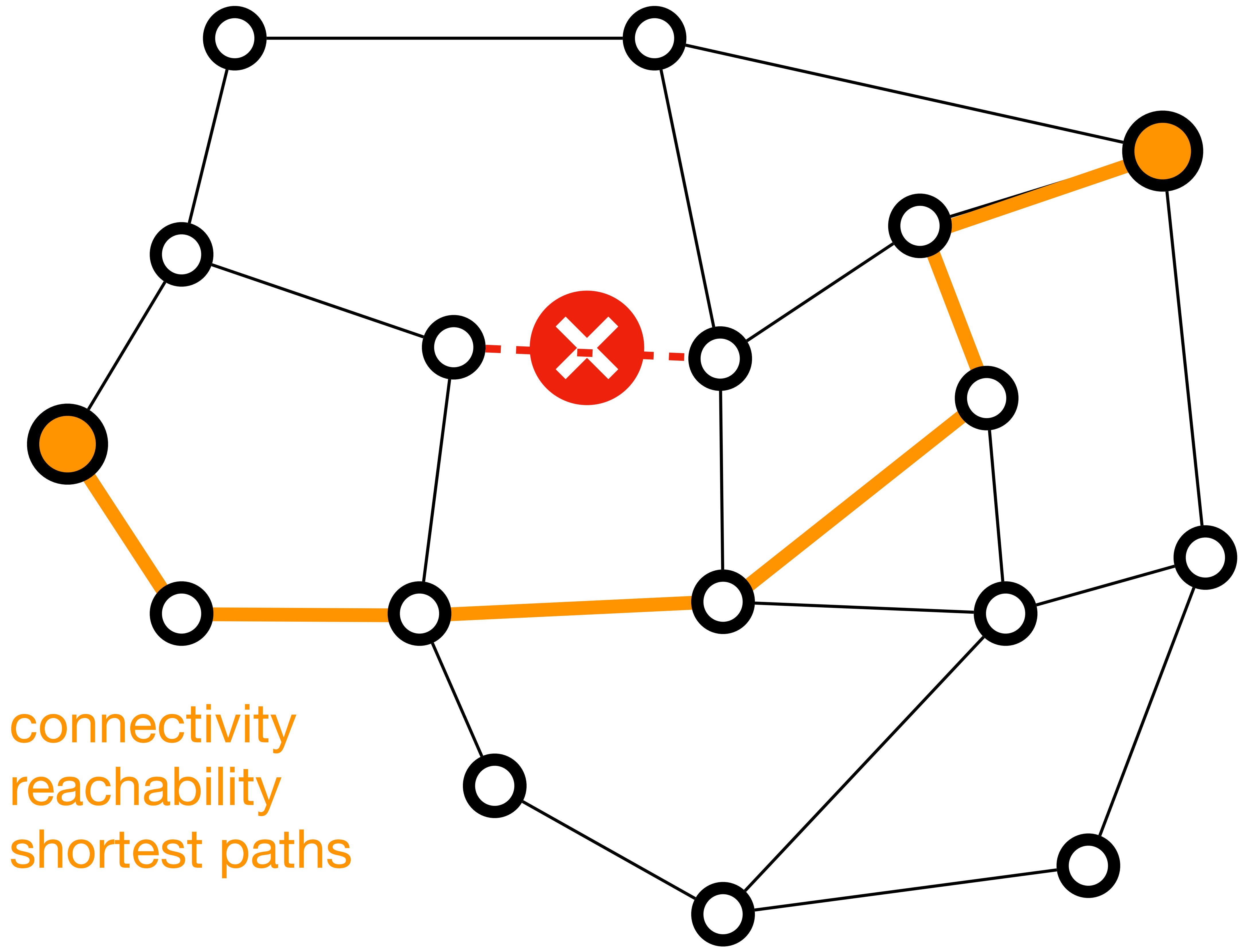

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Today's focus:

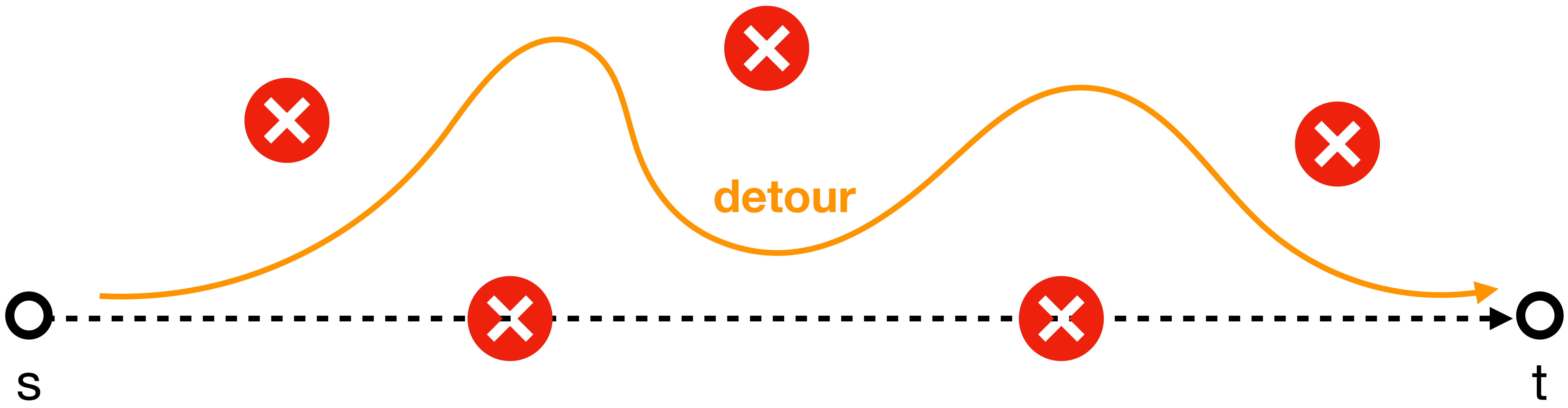
1. Recover **shortest paths info**
2. **Precompute** all answers, so recovery time is poly-log

Other settings:

1. Connectivity / reachability
2. Preprocessing vs. recovery

Replacement Path

- Weighted directed graph $G = (V, E, \omega)$, source & terminal vertices
- For all $\leq f$ edges $F \subseteq E$, compute dist$(s, t, G \backslash F)$



Replacement Path

- Weighted directed graph $G = (V, E, \omega)$, source & terminal vertices
- For all $\leq f$ edges $F \subseteq E$, compute dist$(s, t, G \backslash F)$
- Total size of output $\leq n^f$ (**exercise:** why not m^f)
- Trivial algorithm takes time $mn^f \leq n^{f+2}$
- **Main question:** How to save the **quadratic overhead**?

Single-failure replacement paths

- Trivial algorithm takes cubic runtime $mn \leq n^3$
- [VW, 2010] showed this is the best possible under a widely believed conjecture
- $(1+\epsilon)$-approximations in runtime $m \leq n^2$ [Bernstein, 2010]
- **Corollary:** $(1+\epsilon)$-approximations for f-failures in $mn^{f-1} \leq n^{f+1}$ time

Dual-failure replacement paths

- Optimal exact algorithm in runtime n^3 [VWX, 2022]
- **Corollary:** exact solutions for f-failures in n^{f+1} time when $f \geq 2$

Our result [CZ, 2024]

- $(1+\epsilon)$-approximations in runtime n^2, **optimal** runtime
- **Corollary:** $(1+\epsilon)$-approximations for f-failures in n^f time when $f \geq 2$
- **Open:** exact solutions for 3-failures in n^3 time?

Summary of results

	$f = 1$	$f = 2$	$f \geq 3$
Exact	n^3 [VW, 2010]	n^3 [VWX, 2022] [VW, 2010]	n^{f+1} [VWX, 2022]
Approximate	n^2 [Bernstein, 2010]	n^2 **New**	n^f **New**

Different variants of RP (exact)

Special cases of single-failure RP

- Undirected RP in linear time [NPW, 2001]
- Unweighted RP in $m\sqrt{n}$ time [RZ, 2012]
- Small edge weights RP in Wn^{ω} time [CN, 2020]

Single-failure single-source RP

- Unweighted single-source RP in $m\sqrt{n}$ time [CM, 2020]
- Small edge weights single-source RP in $W^{0.805}n^{2.496}$ time [GPWX, 2021]

Single-failure all-pairs RP

- All-pairs RP in $Wn^{2.58}$ time [GR, 2021]

Today’s plan

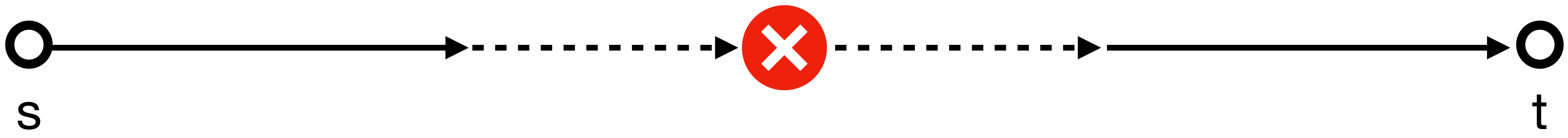
- Review of single-failure approximate st-RP [Bernstein, 2010]
- Two main cases for dual-failure approximate st-RP
  - Only one failure is on the st-path
  - Both failures are on the st-path

Single-Failure Approx-RP

[Bernstein’10]

Single-Failure RP

Single-Failure RP



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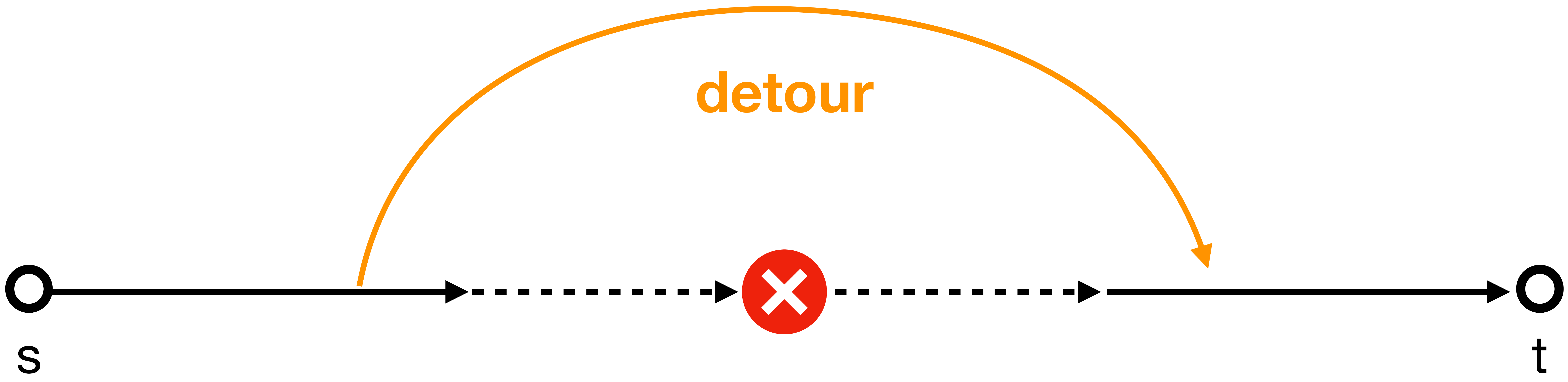

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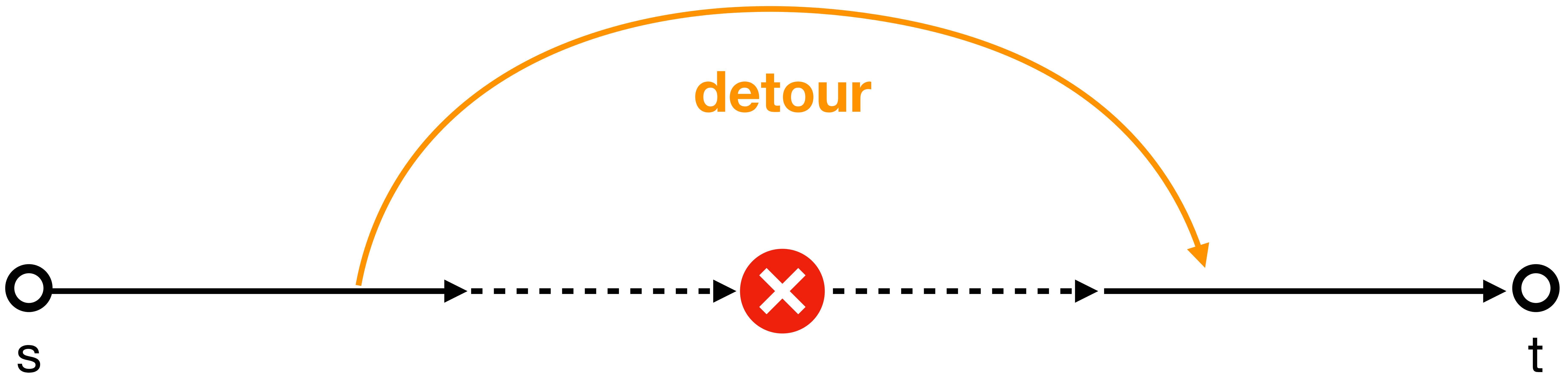
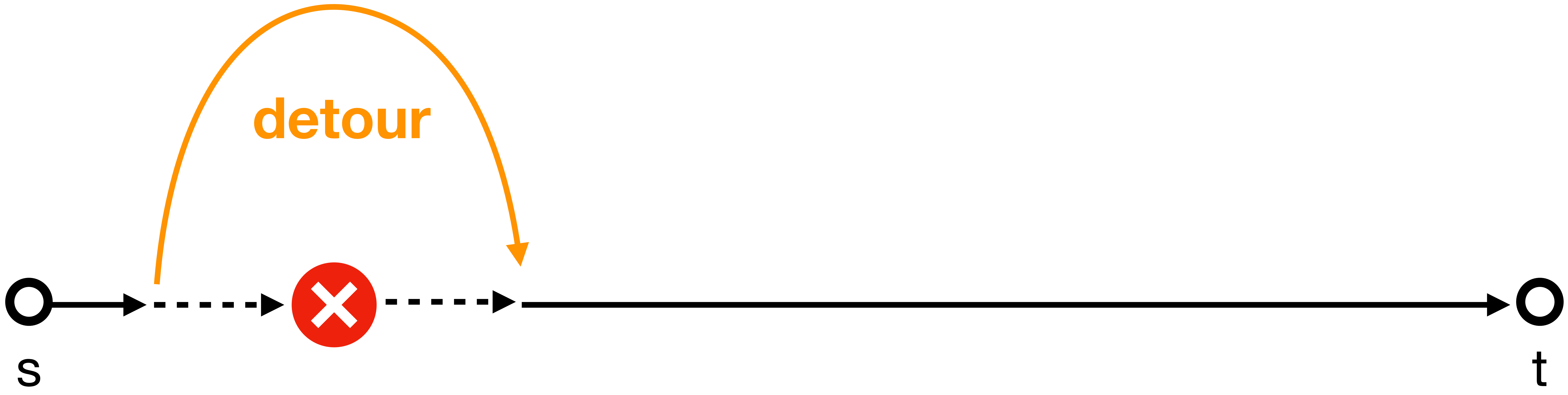


- Single-failure RP = best detour avoiding intervals
- **Compute the detours** for **all** possible failures

Single-Failure RP



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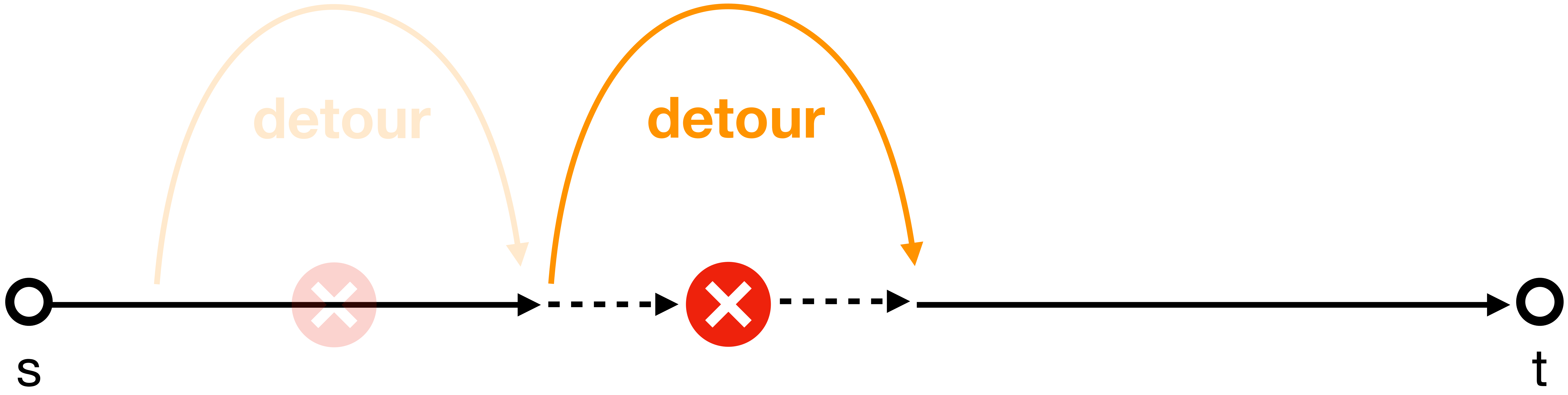

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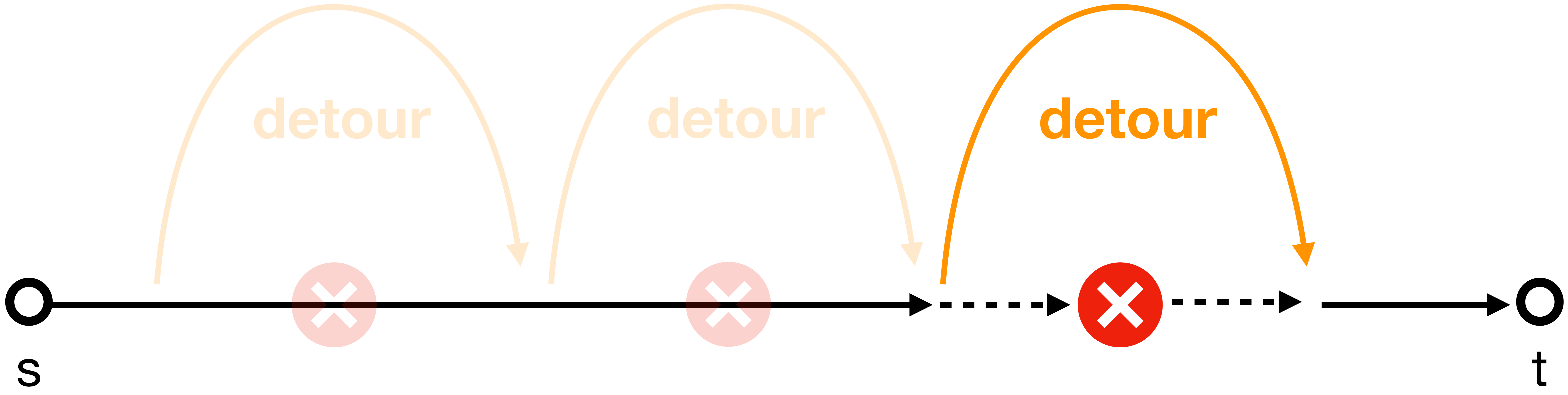

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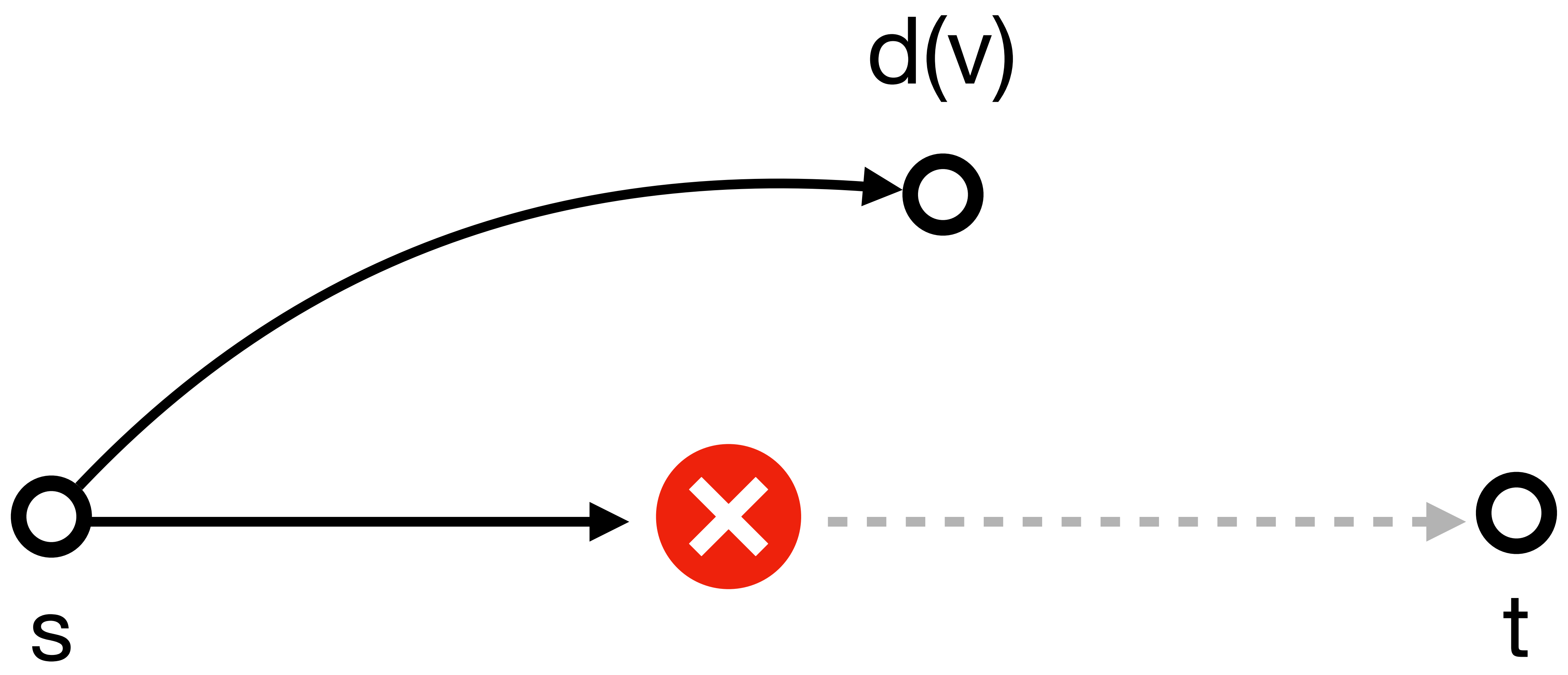
- **Compute the detours** for **all** possible failures

Progressive Dijkstra [Bern'10]

First idea:

Incre maintain all Dijkstra labels

1. Start with $G\backslash\pi$
2. add back π edge by edge
3. Update $d(v)$, but scan out-edges of v iff $d(v)$ has decreased by $1-\epsilon$



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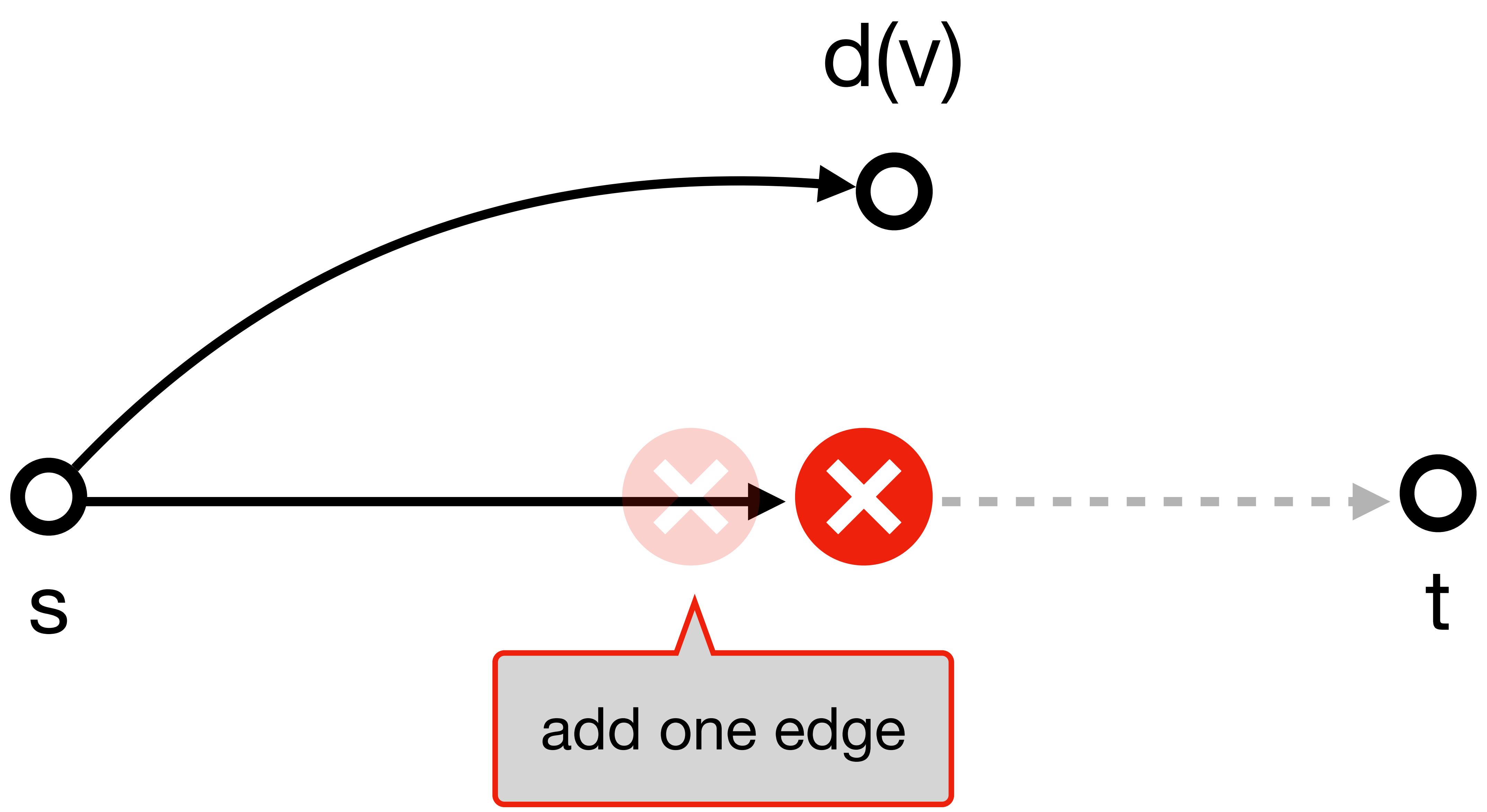

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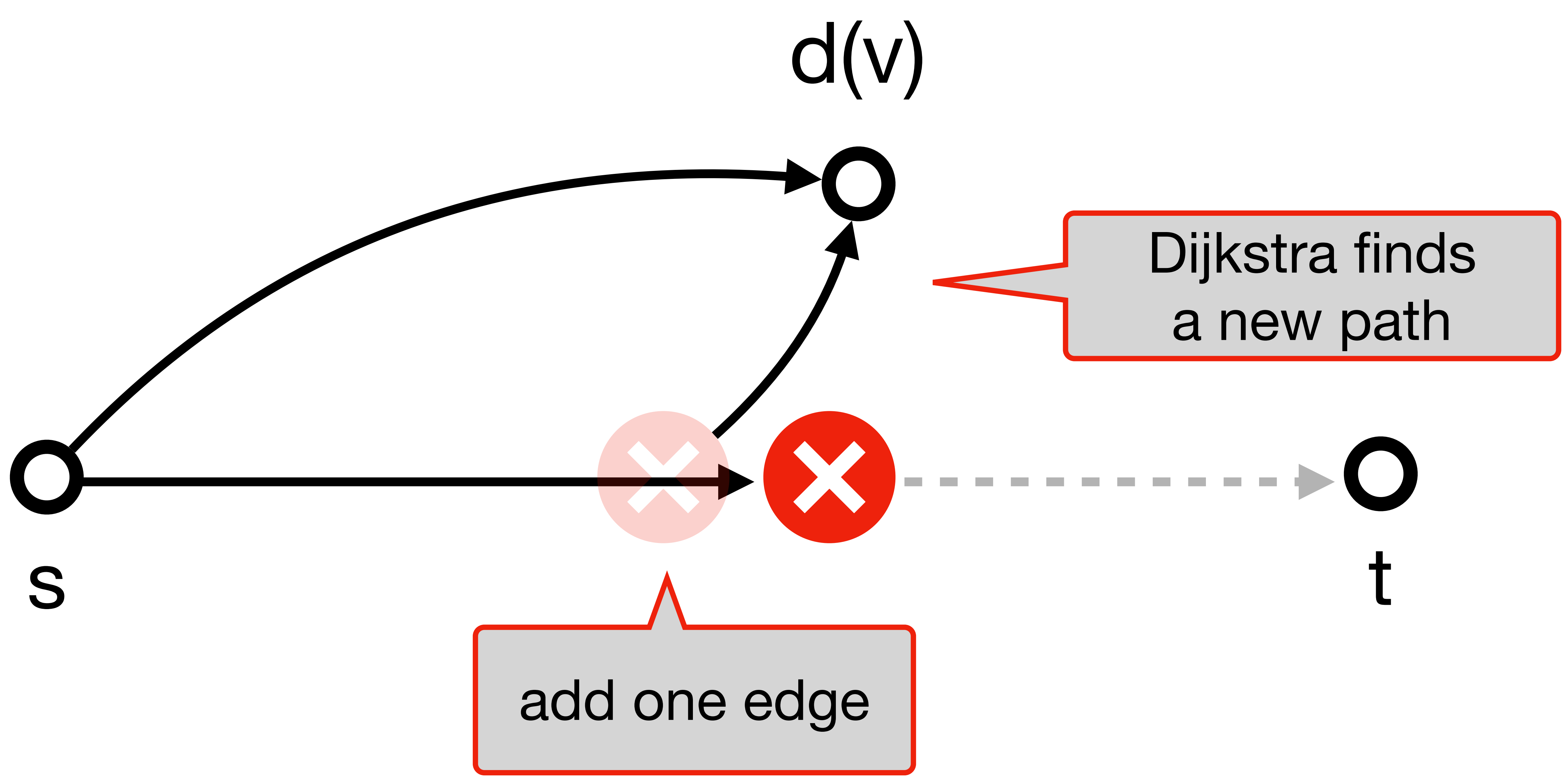

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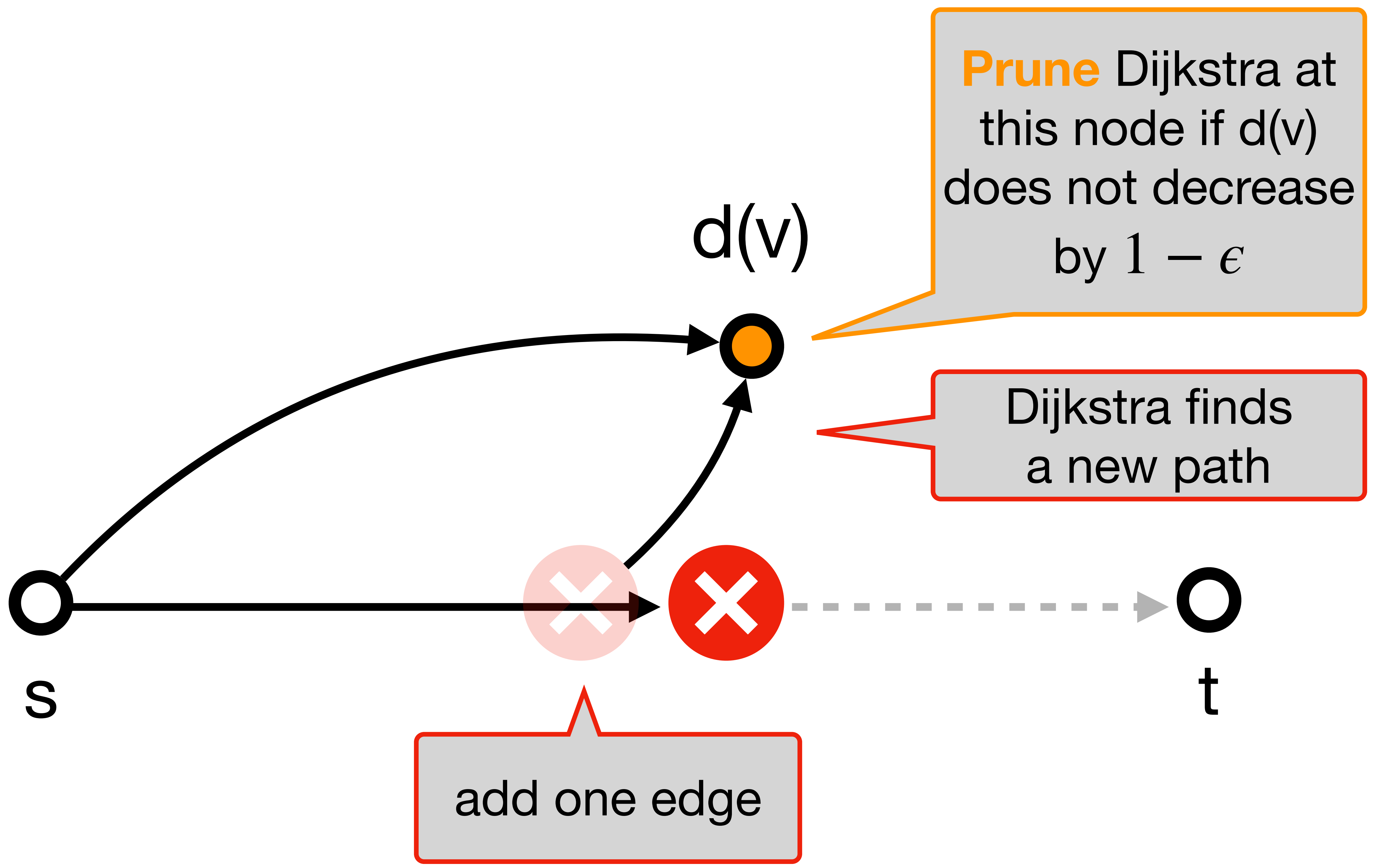

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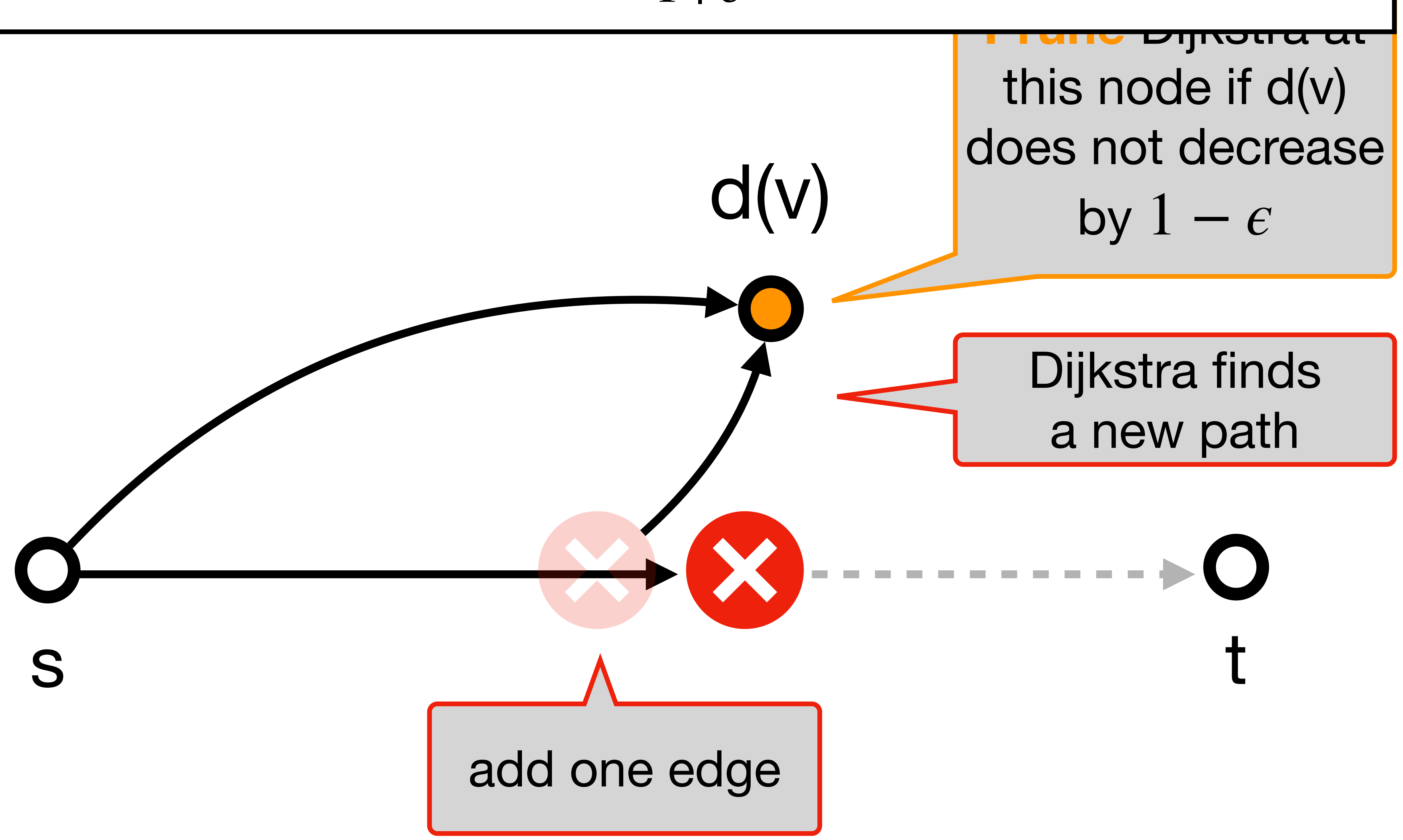
Runtime:

- Each out-neighbor scanned $\log_{1+\epsilon}(nW)$ times
- Total runtime = $n^2 \log_{1+\epsilon}(nW)$



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Approximation error:

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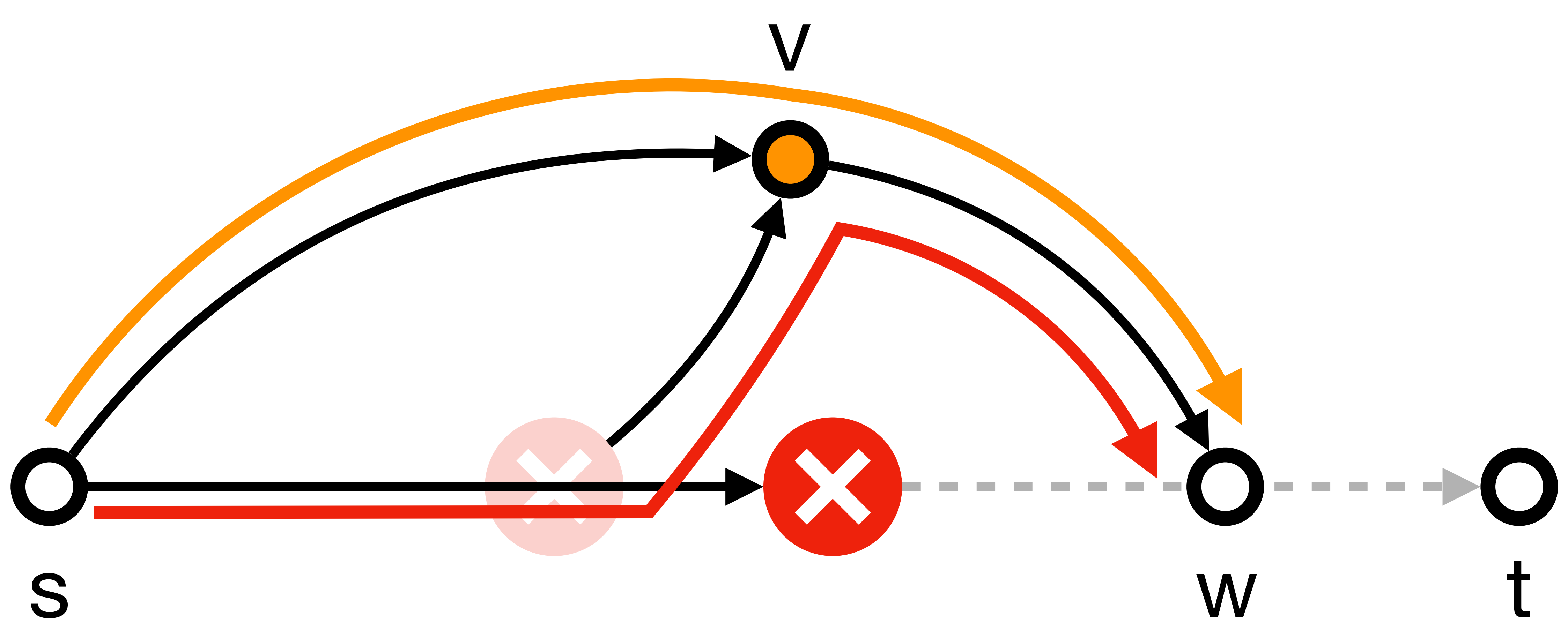

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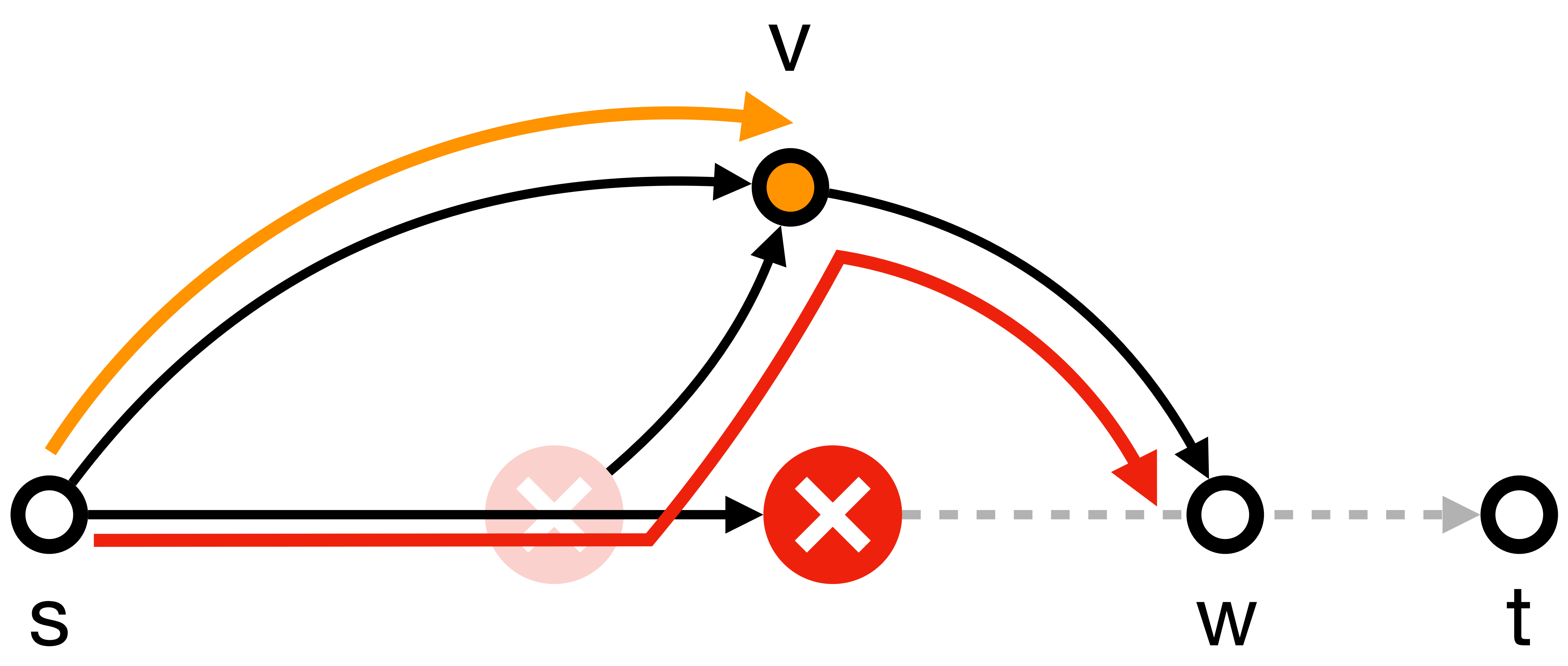

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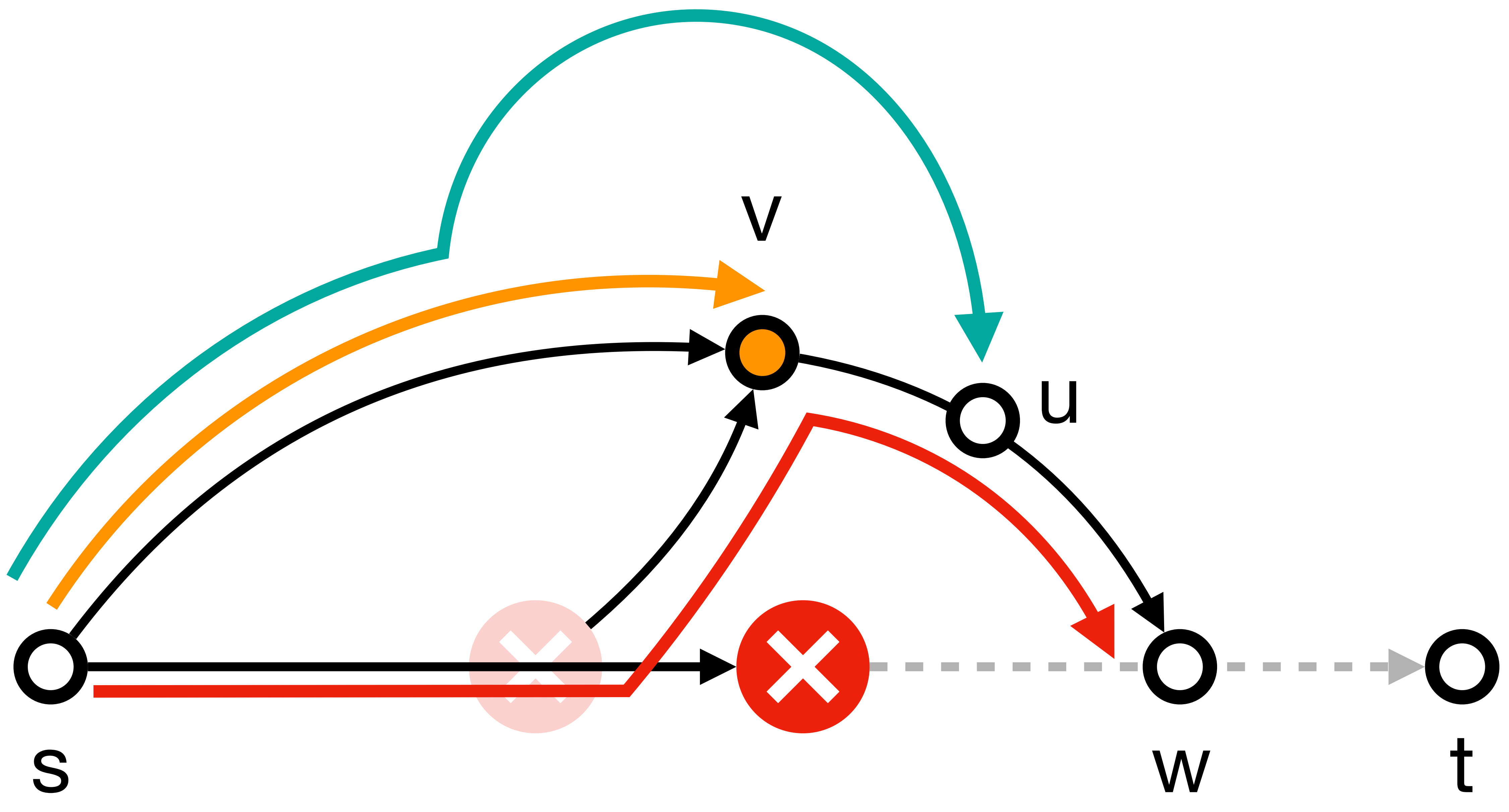

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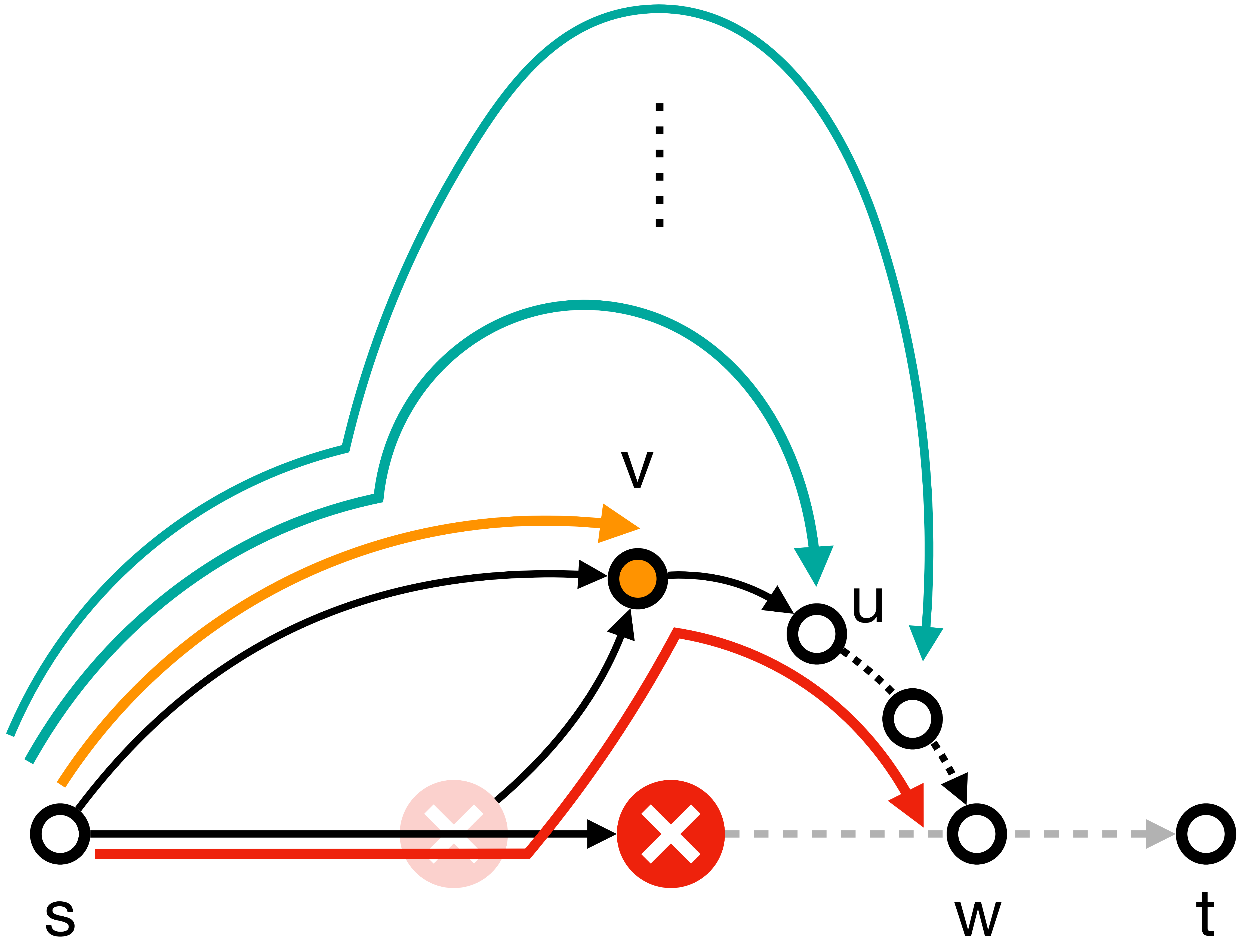
- $(1 + \epsilon)$ factors could accumulate

i-th earlier **blue** $< (1 + \epsilon)^i \times$ **red**



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Main issue:

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Second idea:

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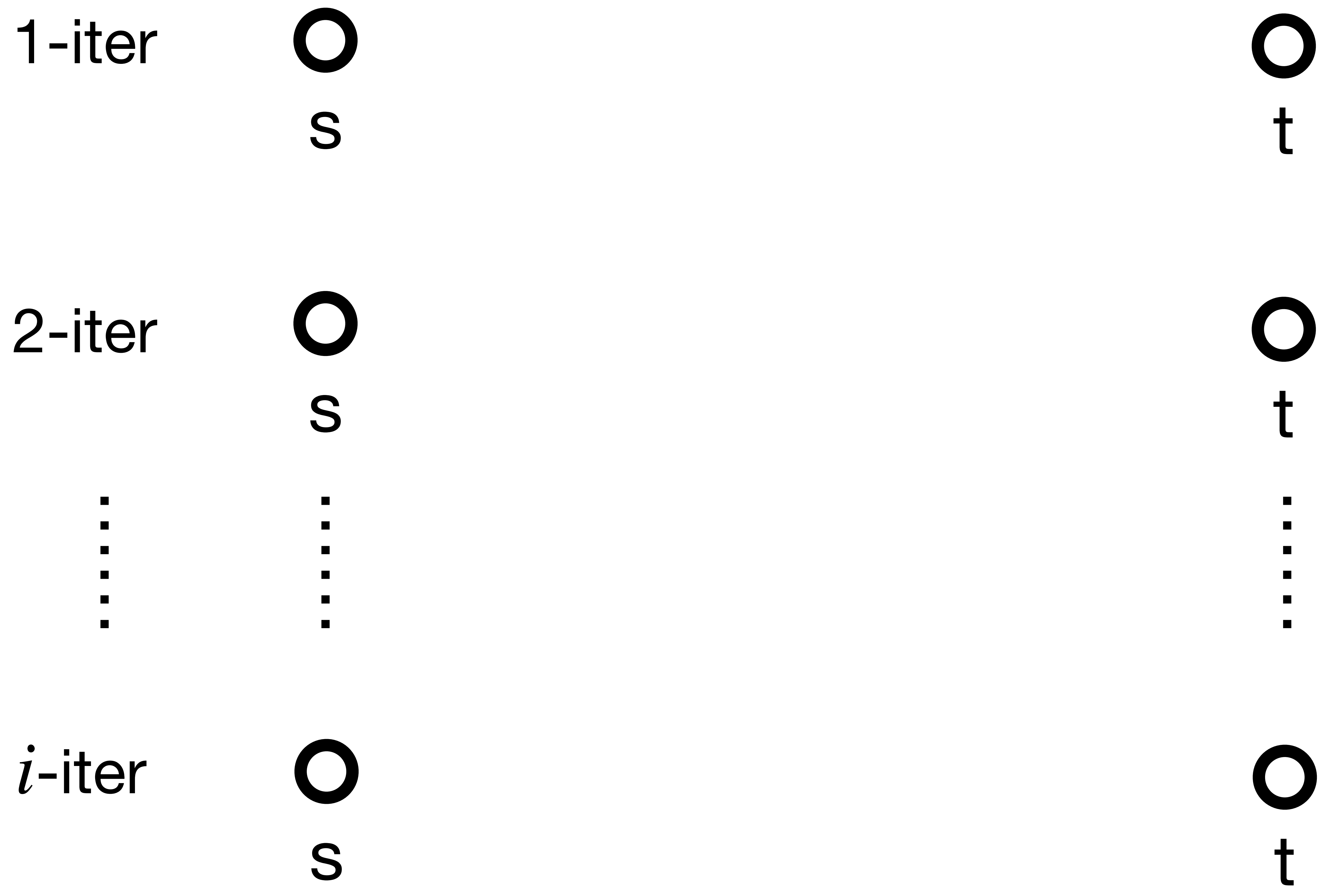

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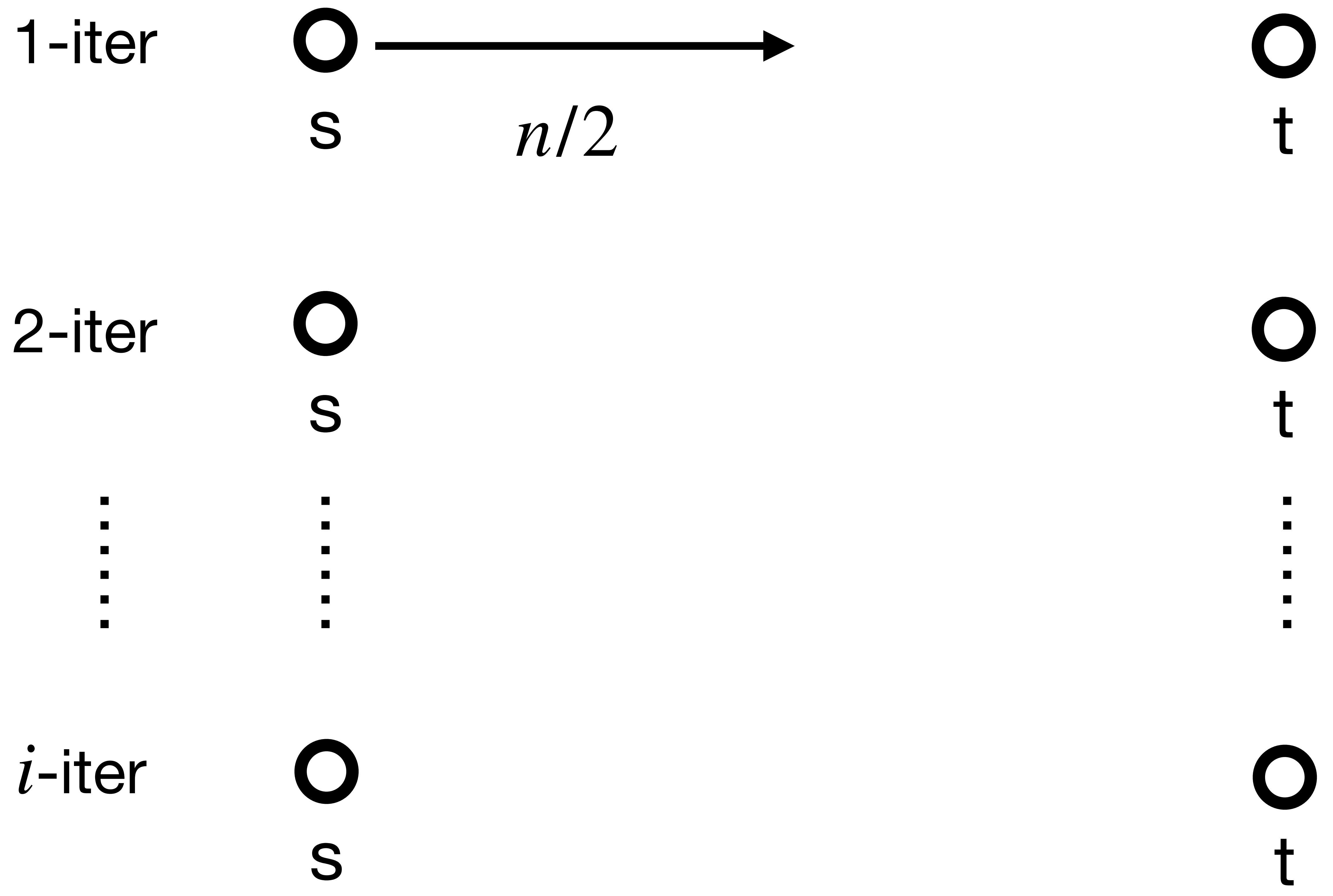

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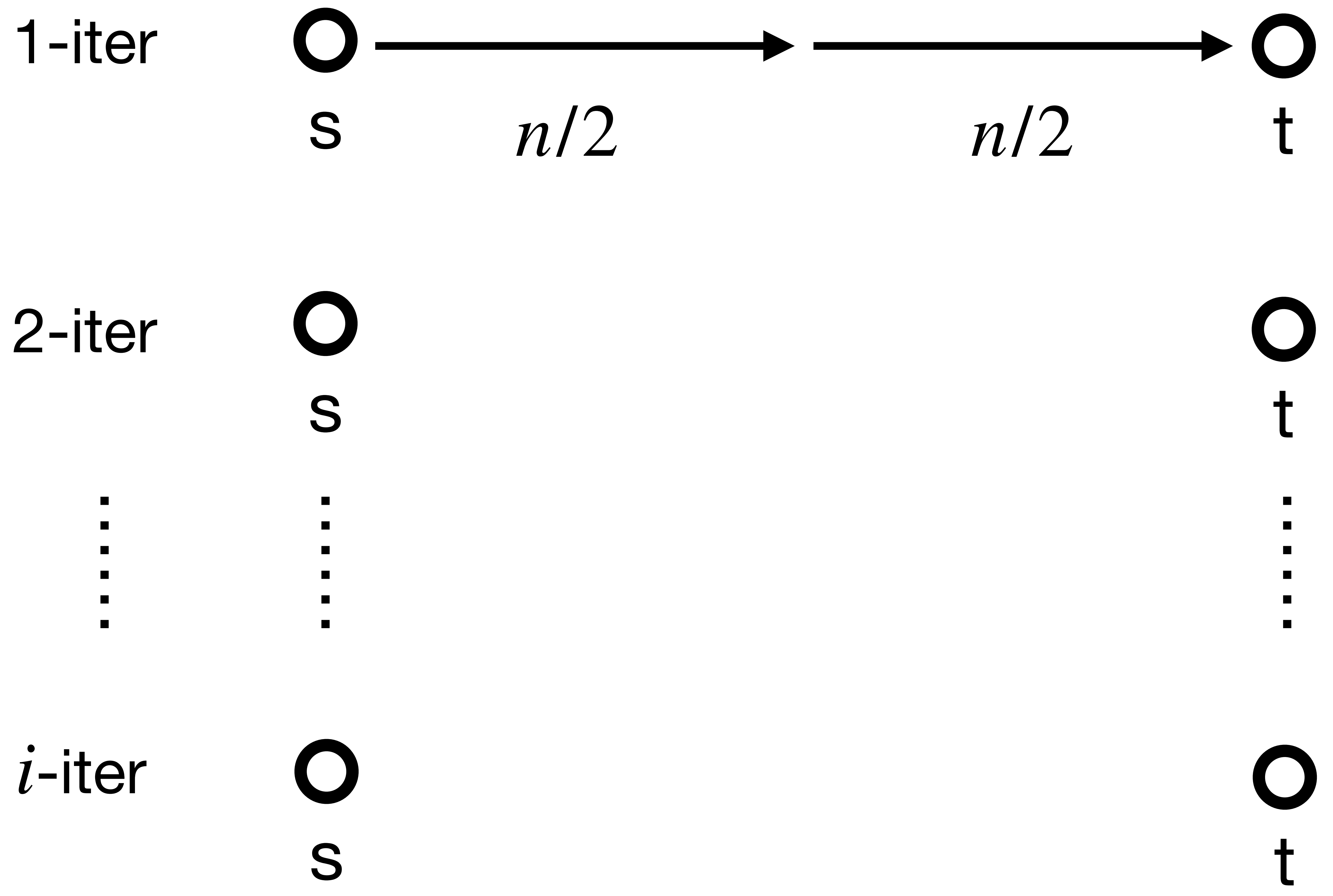

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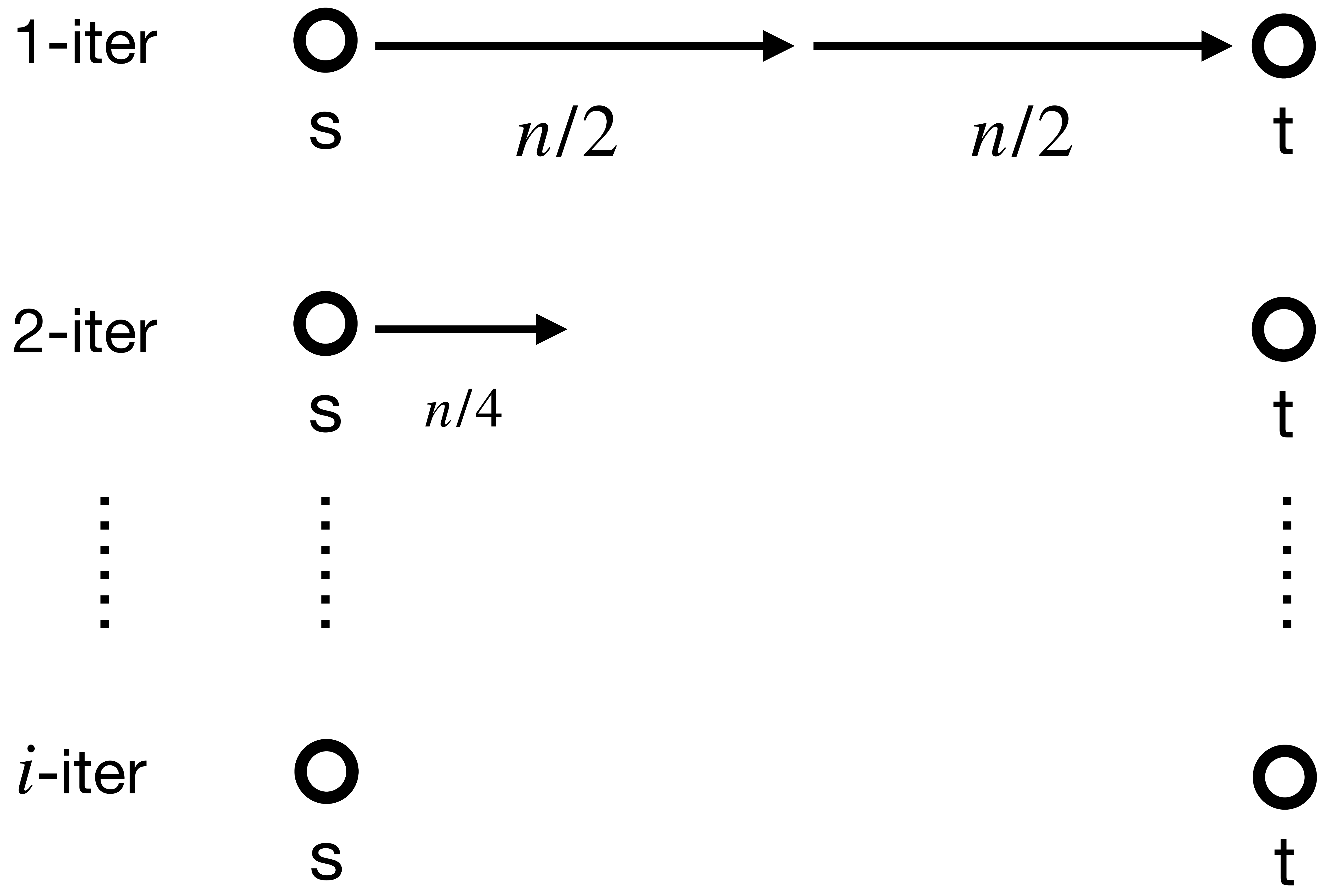

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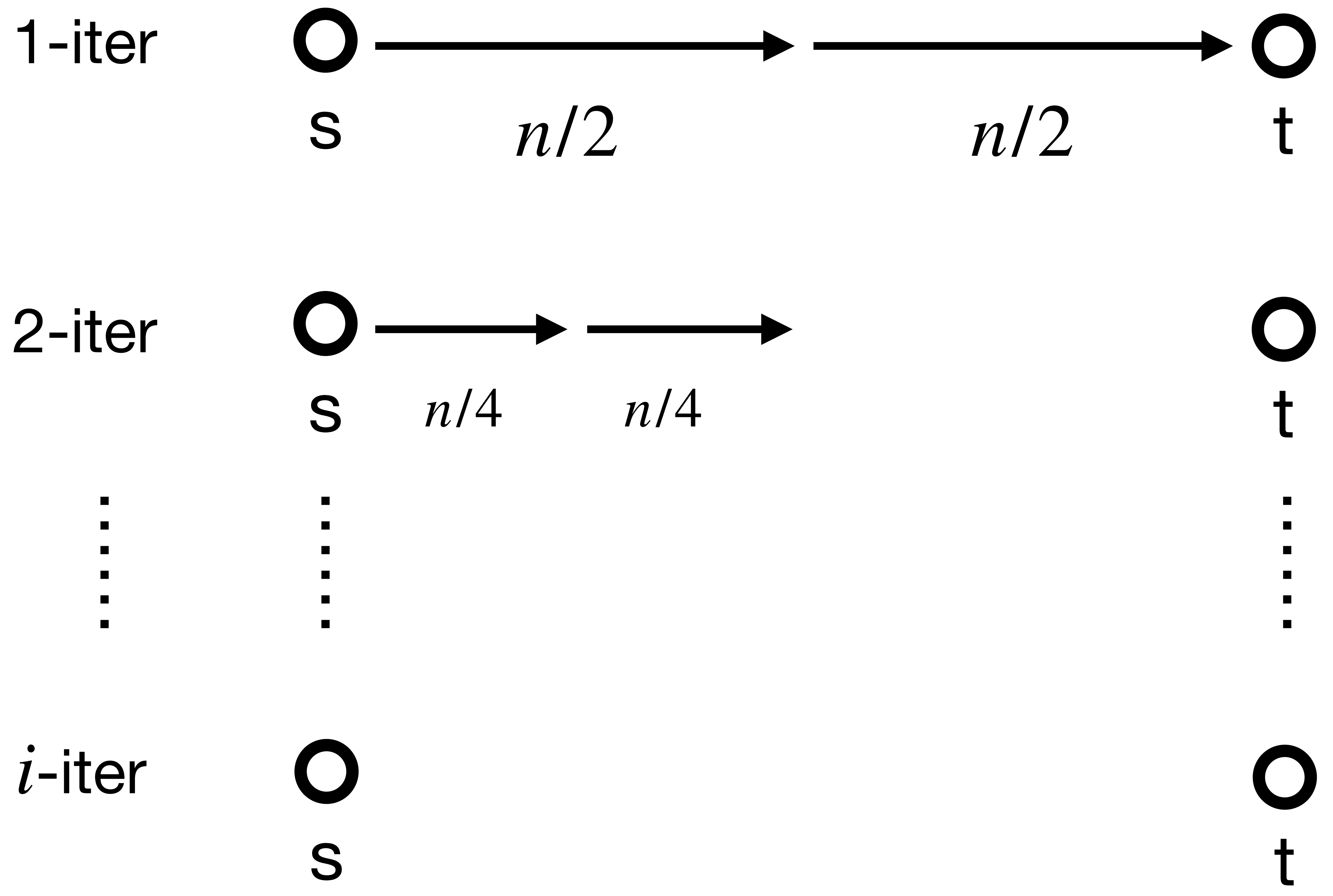

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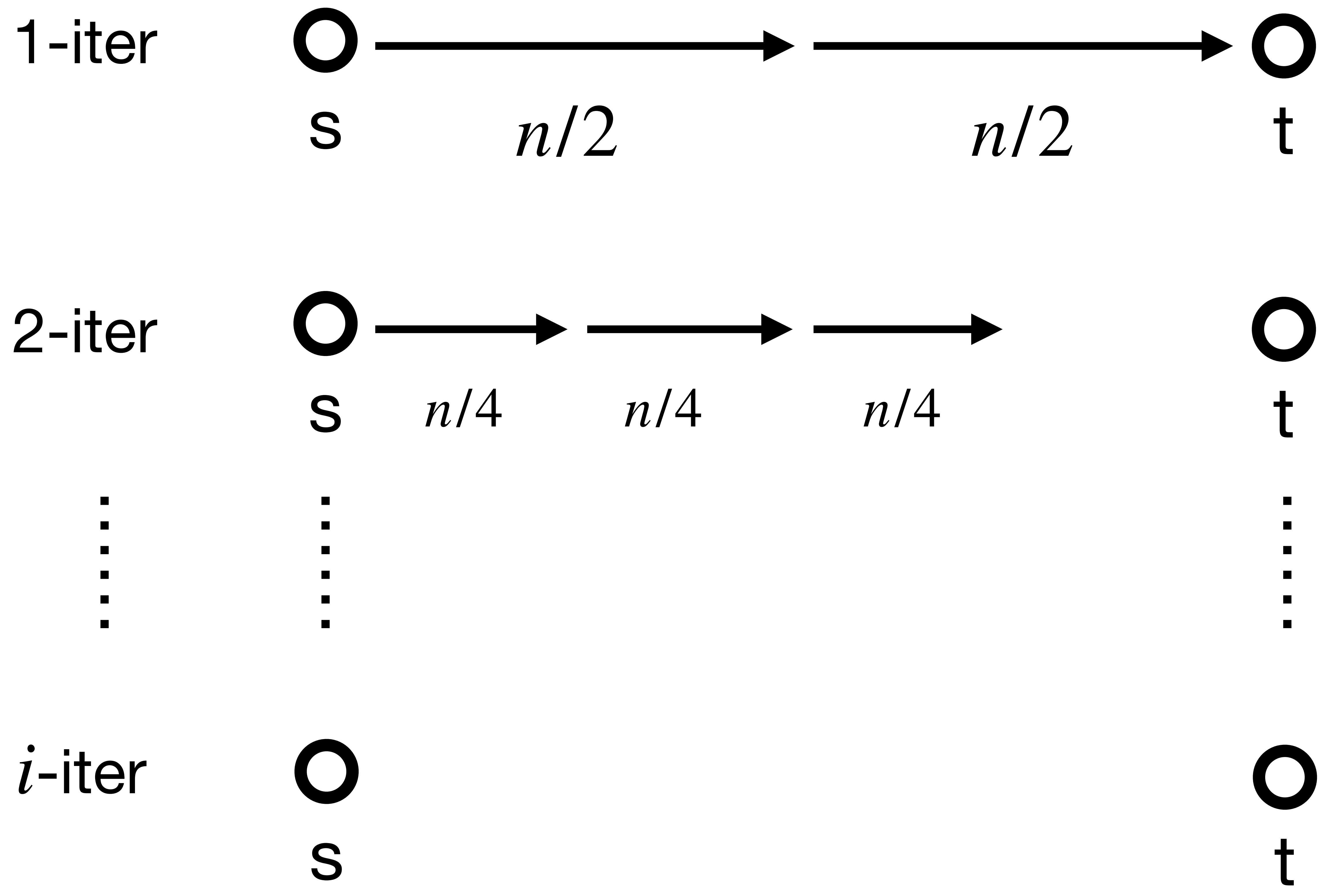

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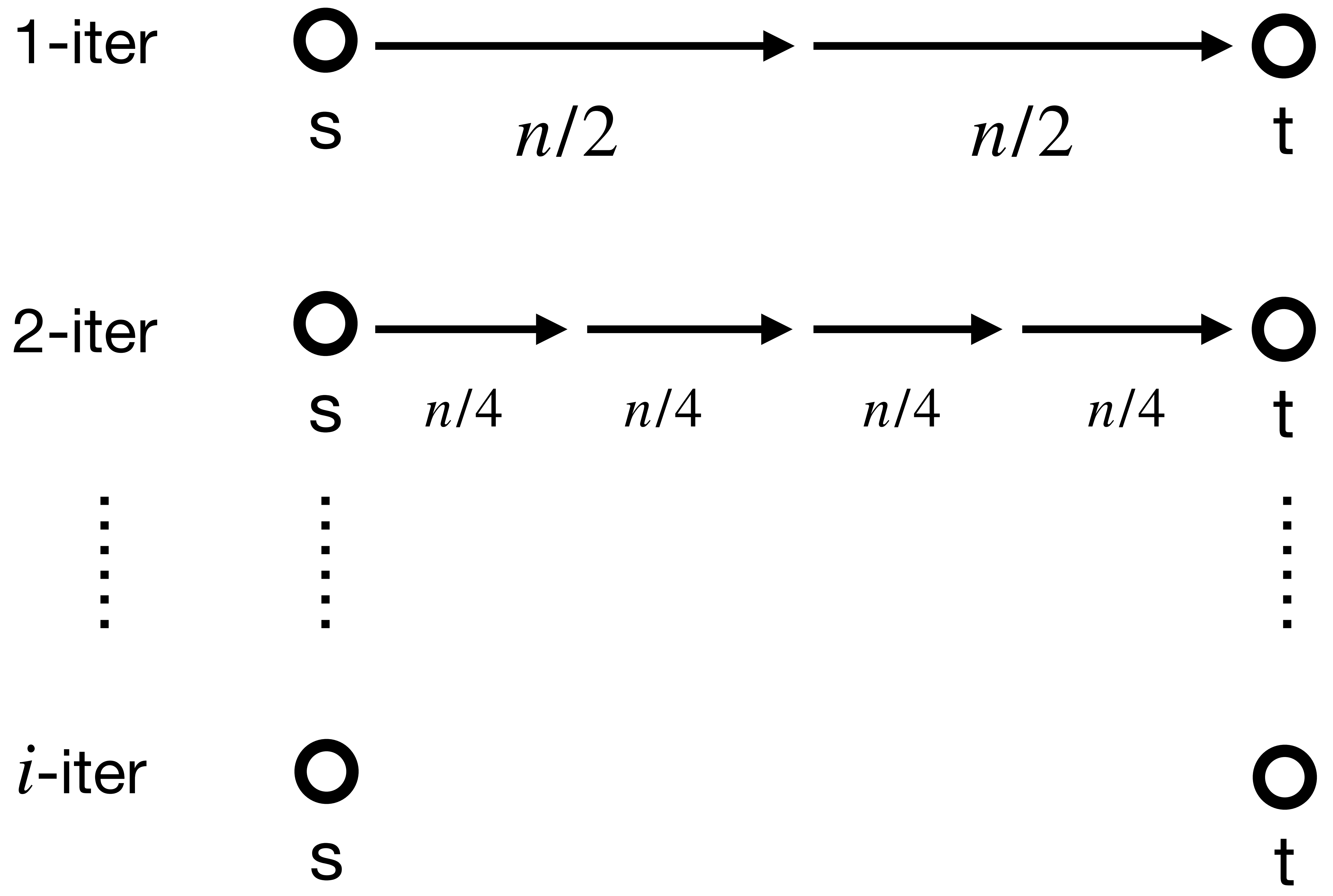

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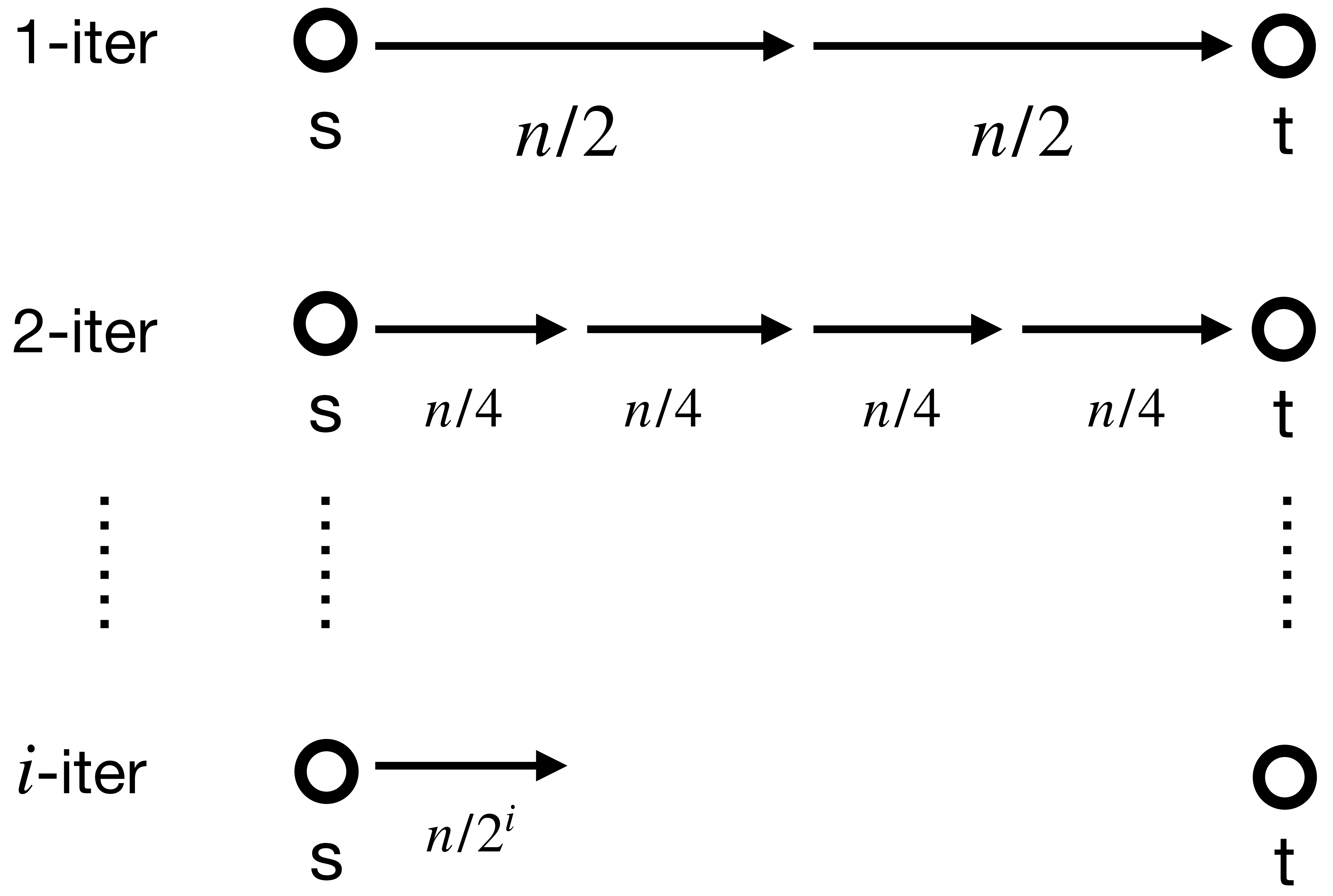

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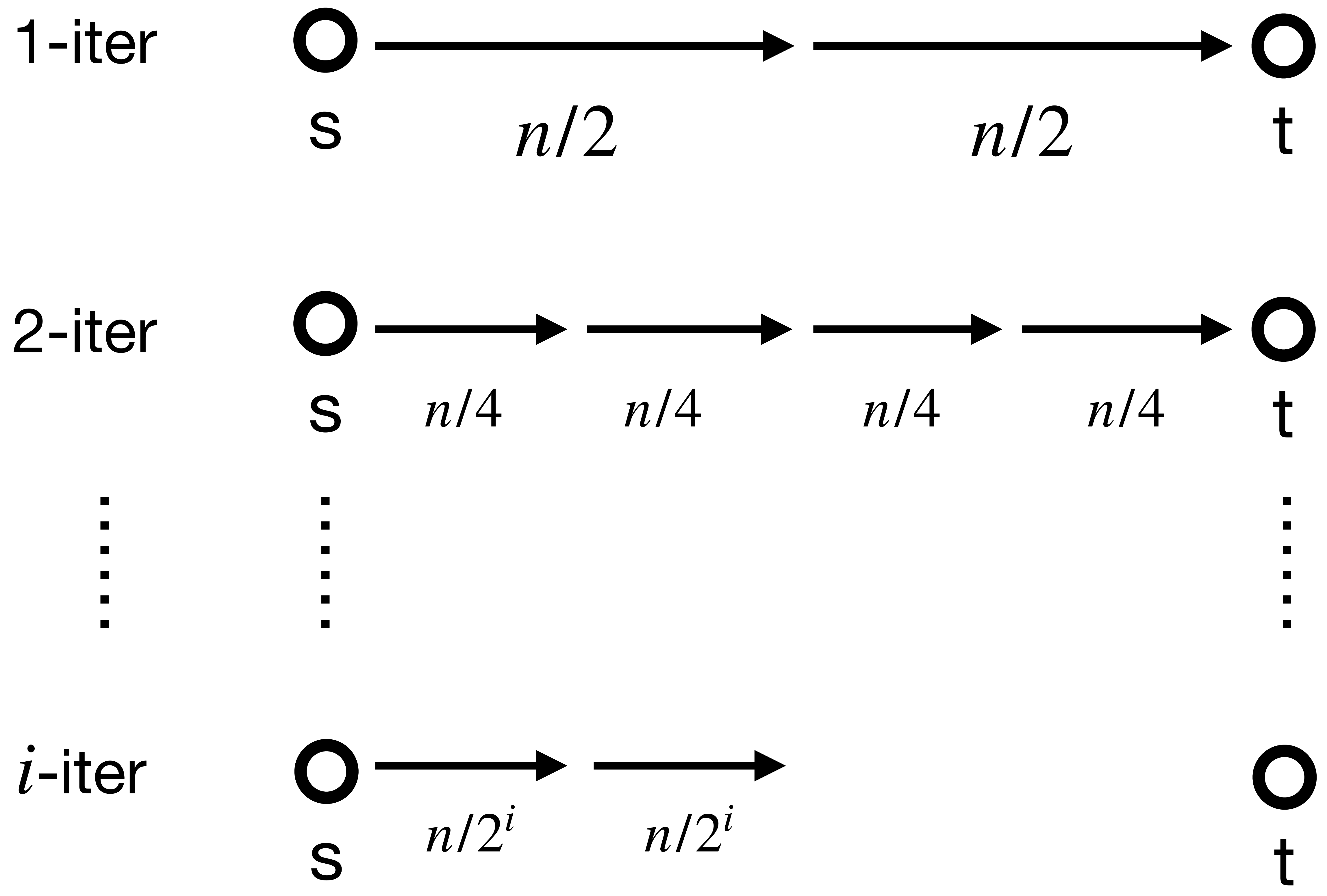

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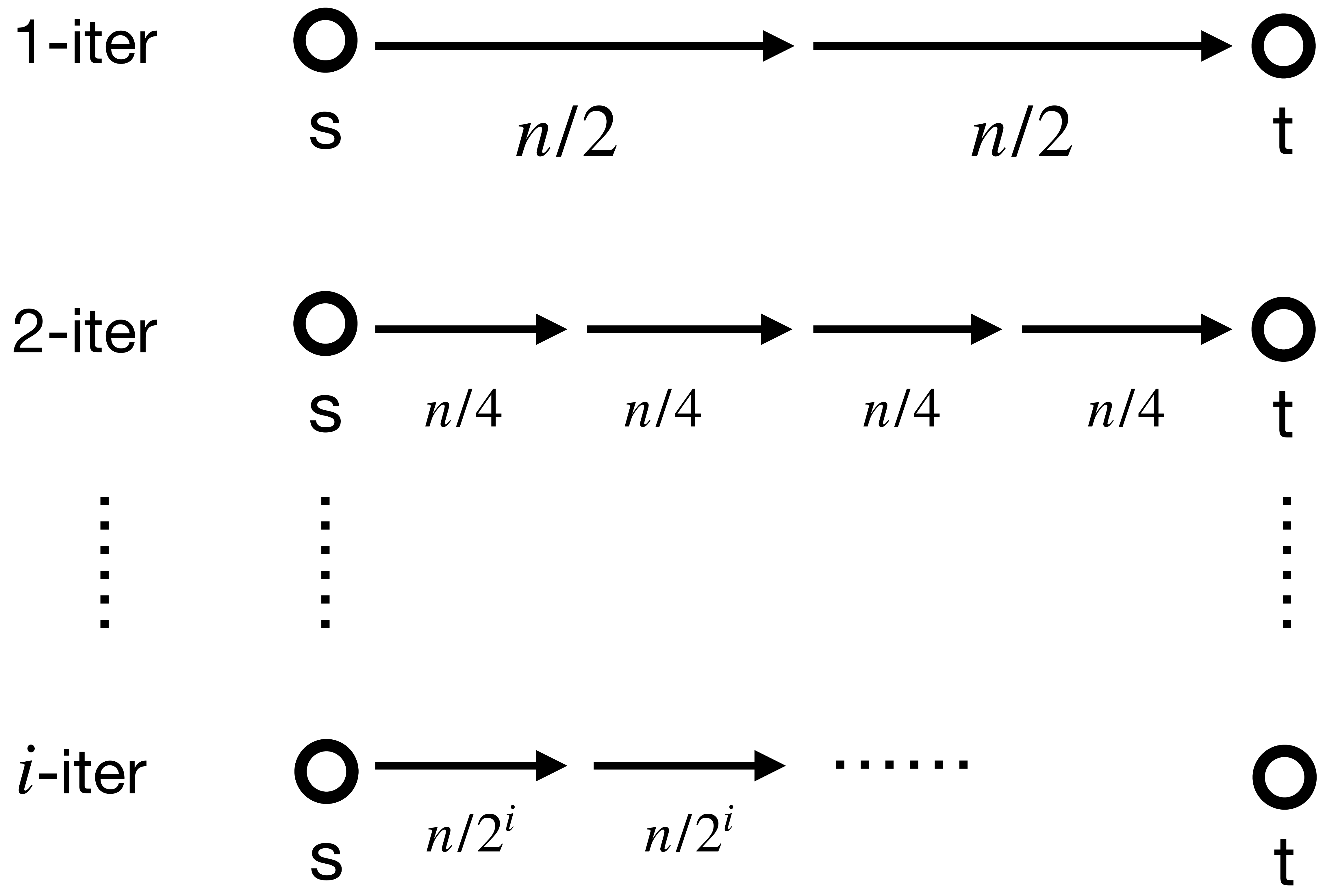

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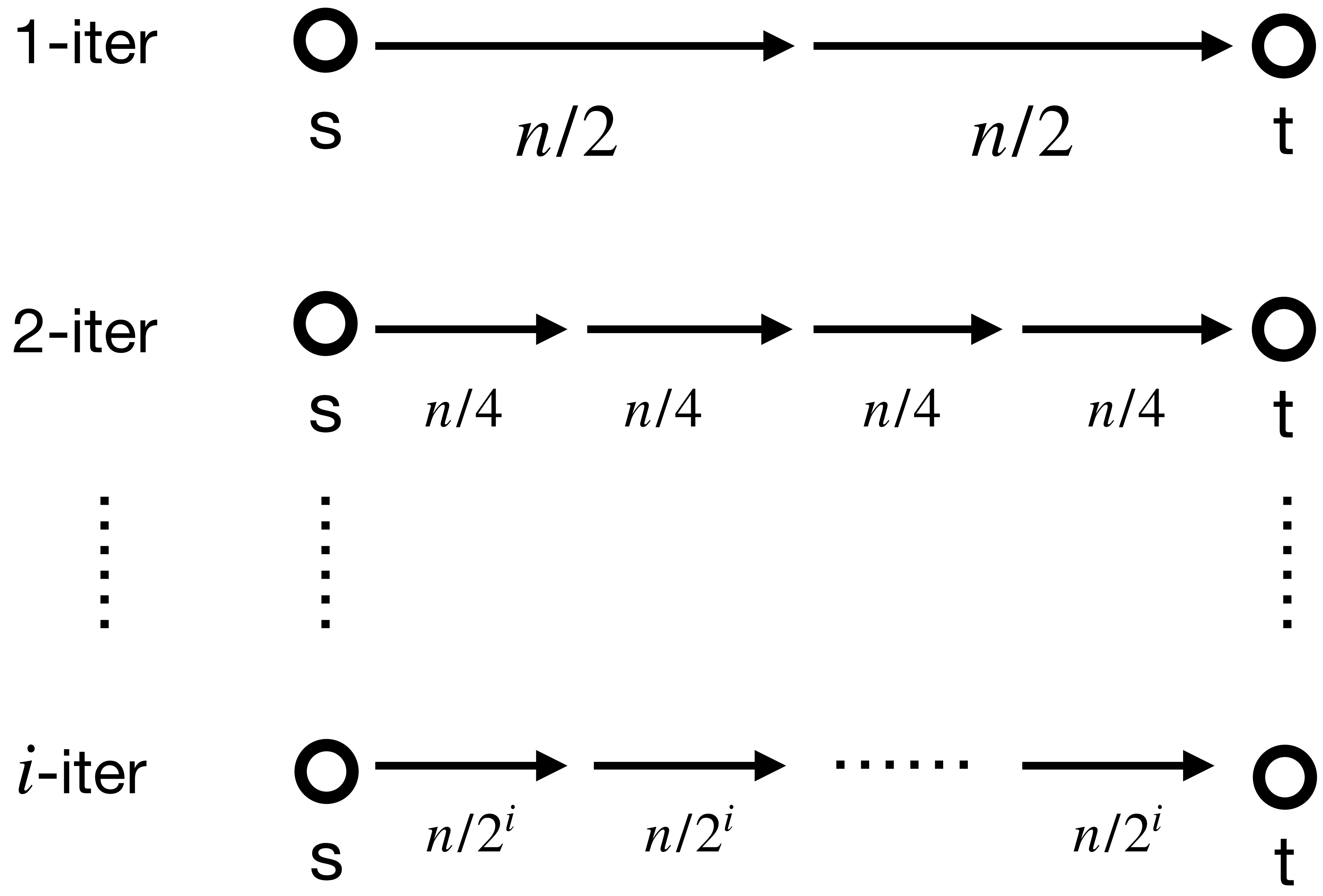

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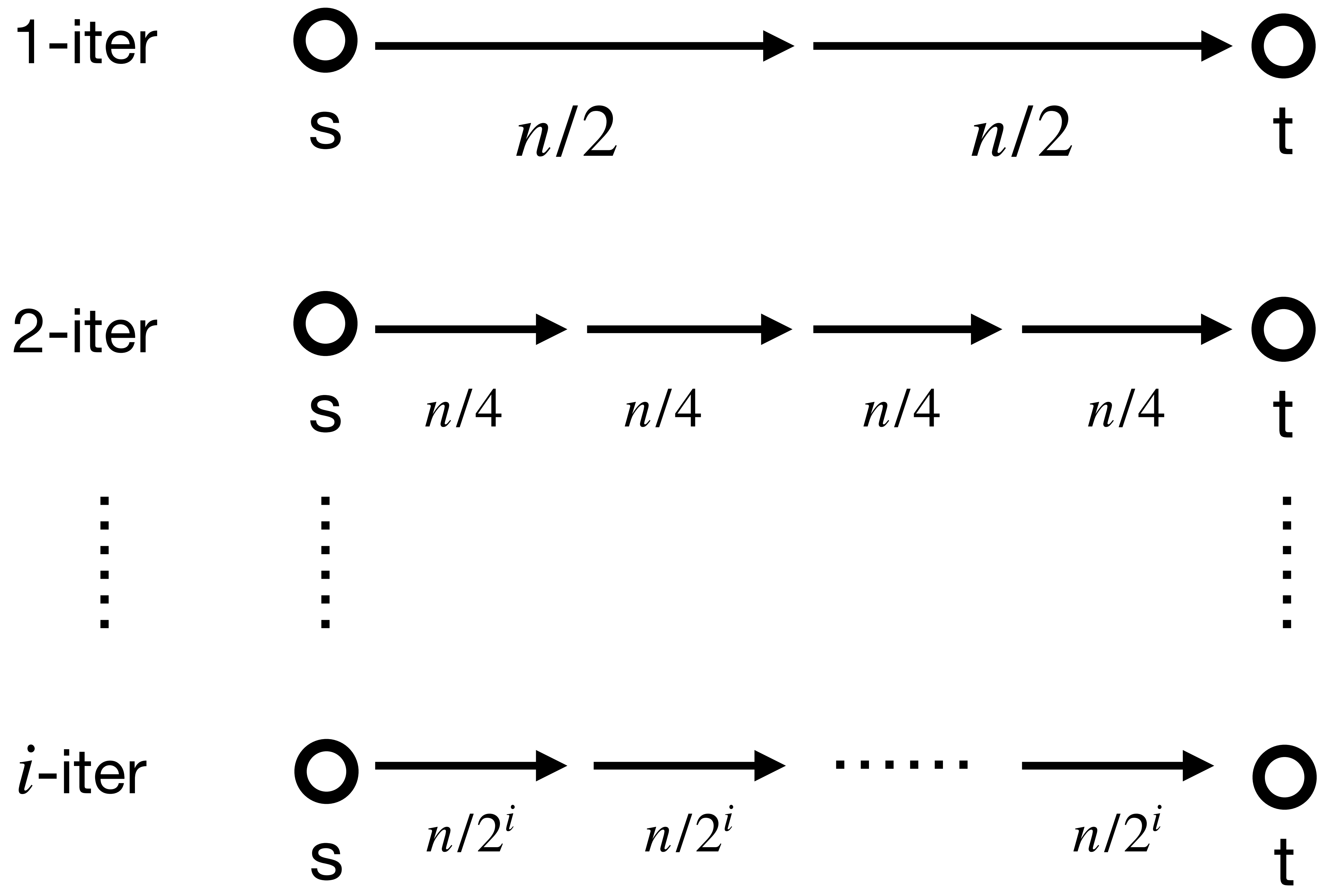
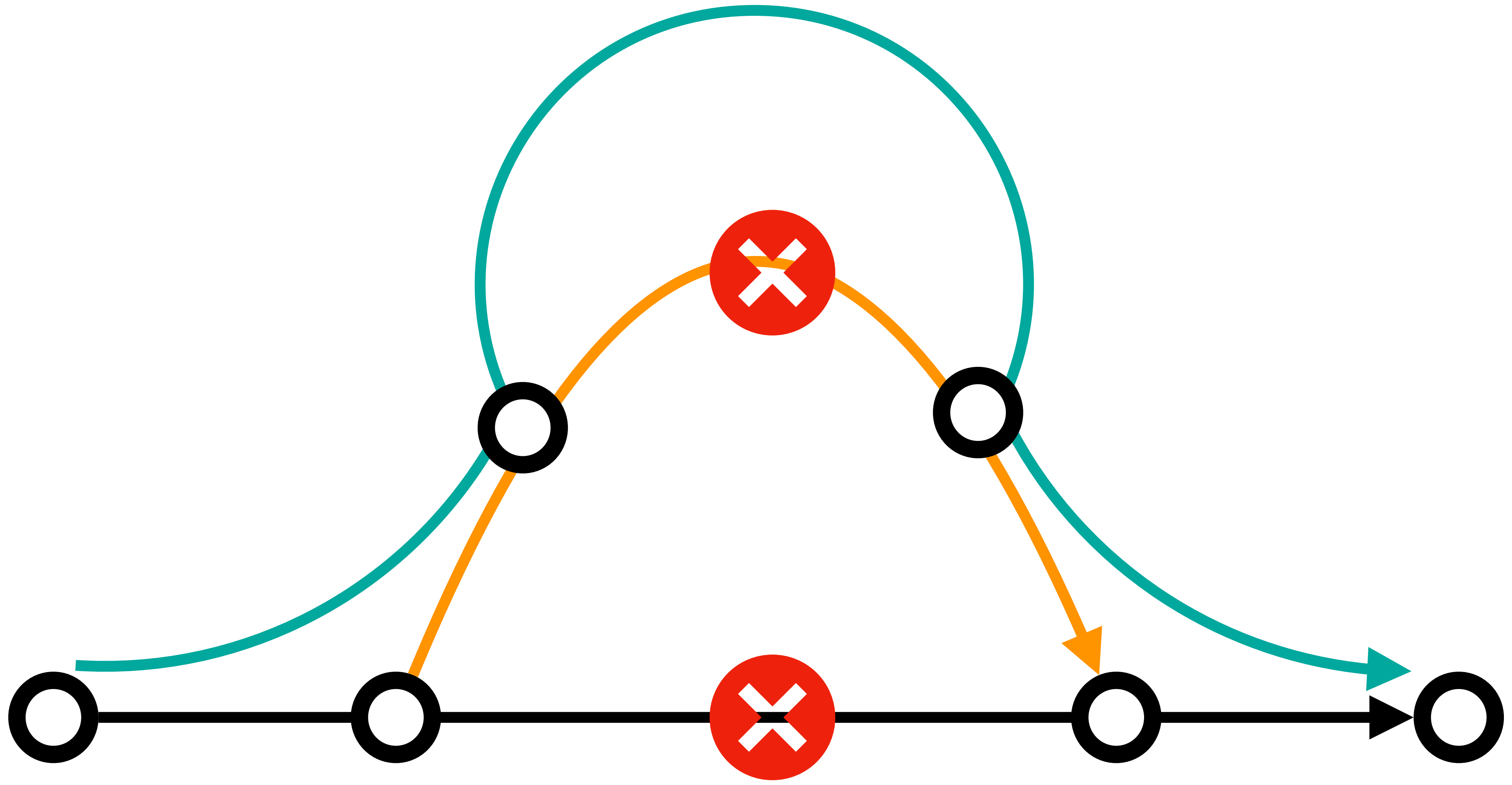


$(1+\epsilon)$ factors **accumulate $\log n$ times** [Bern'10]

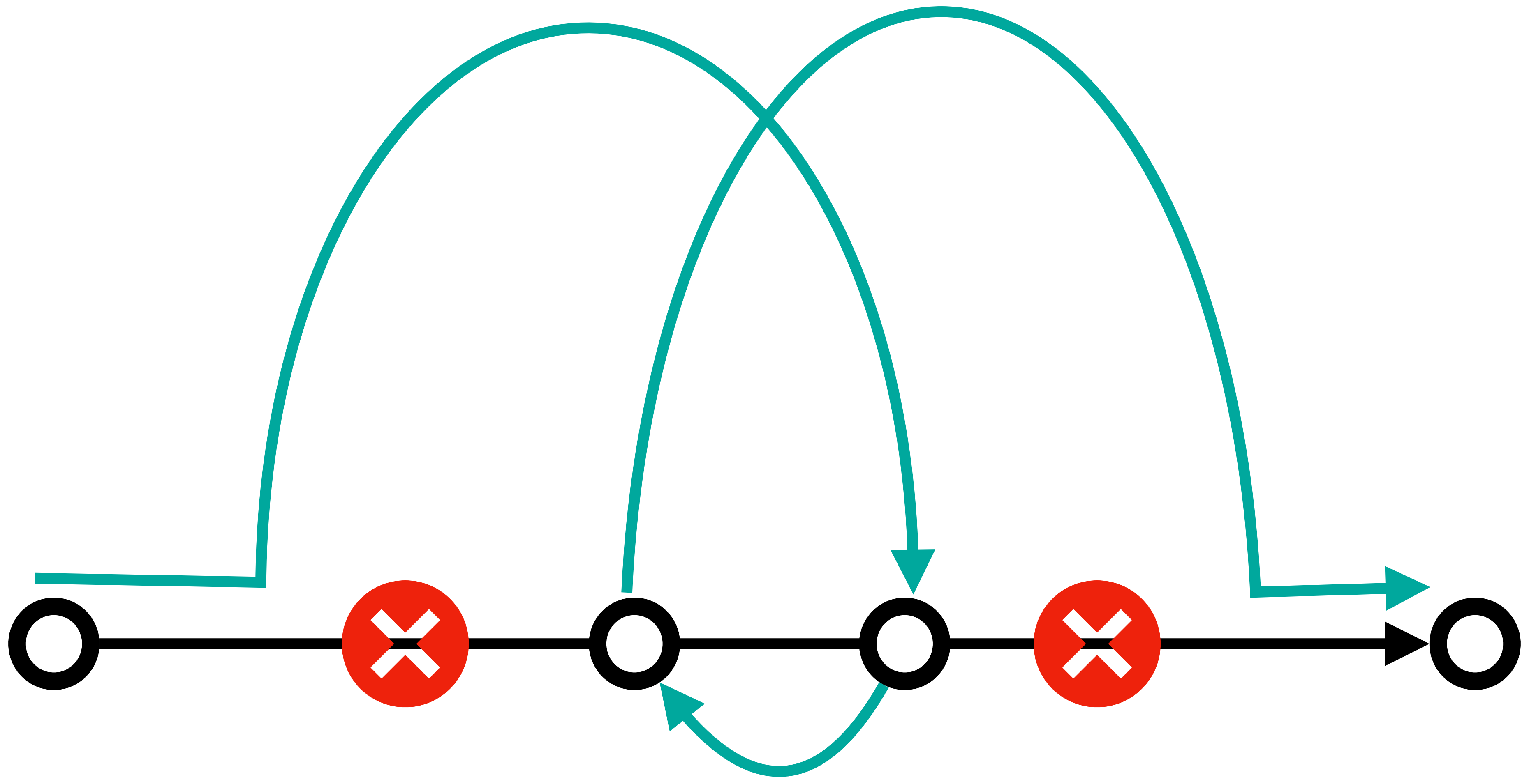
Dual-Failure Approx-RP

Two main cases

One failure on st-path



Both failures on st-path



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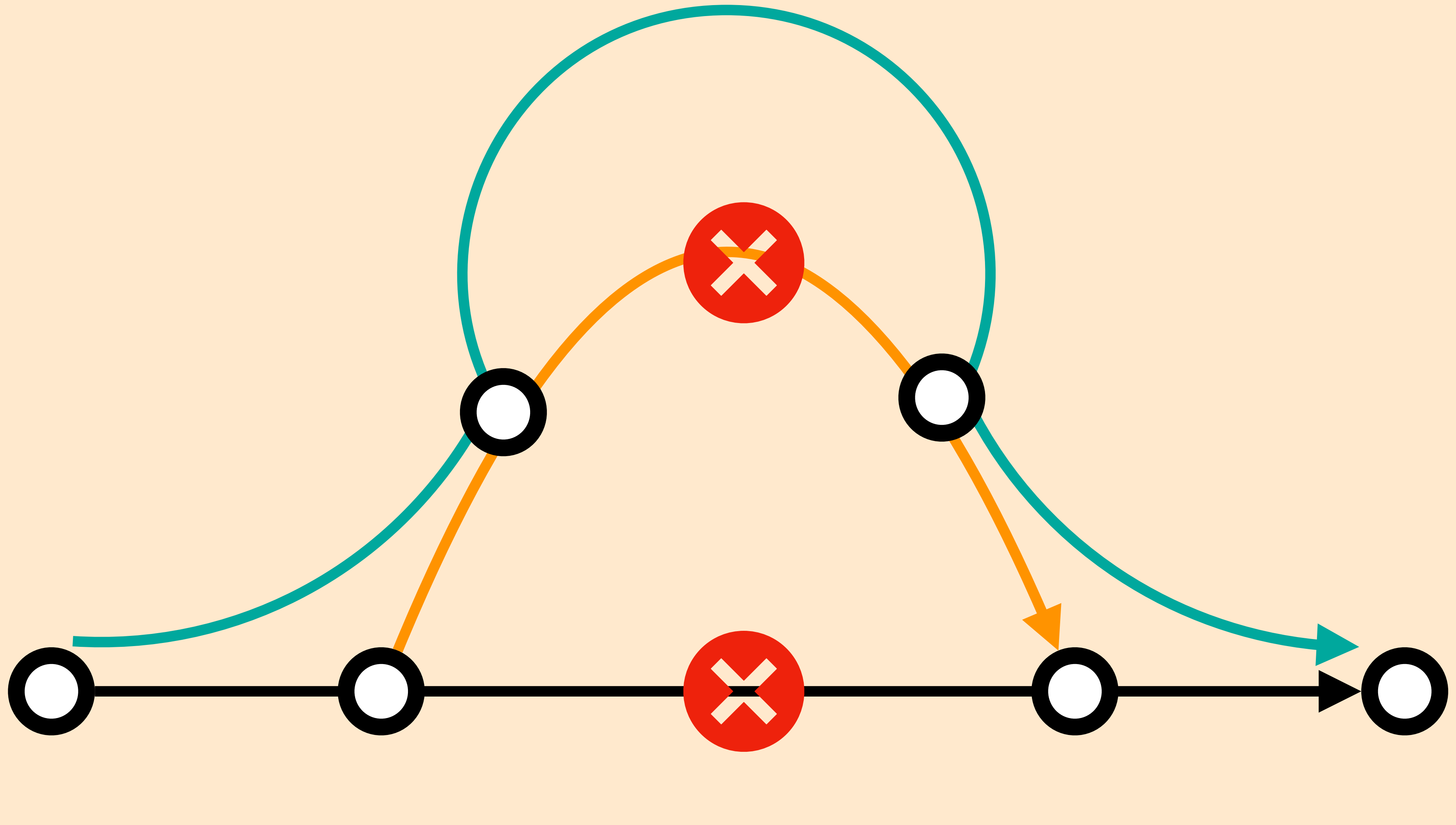

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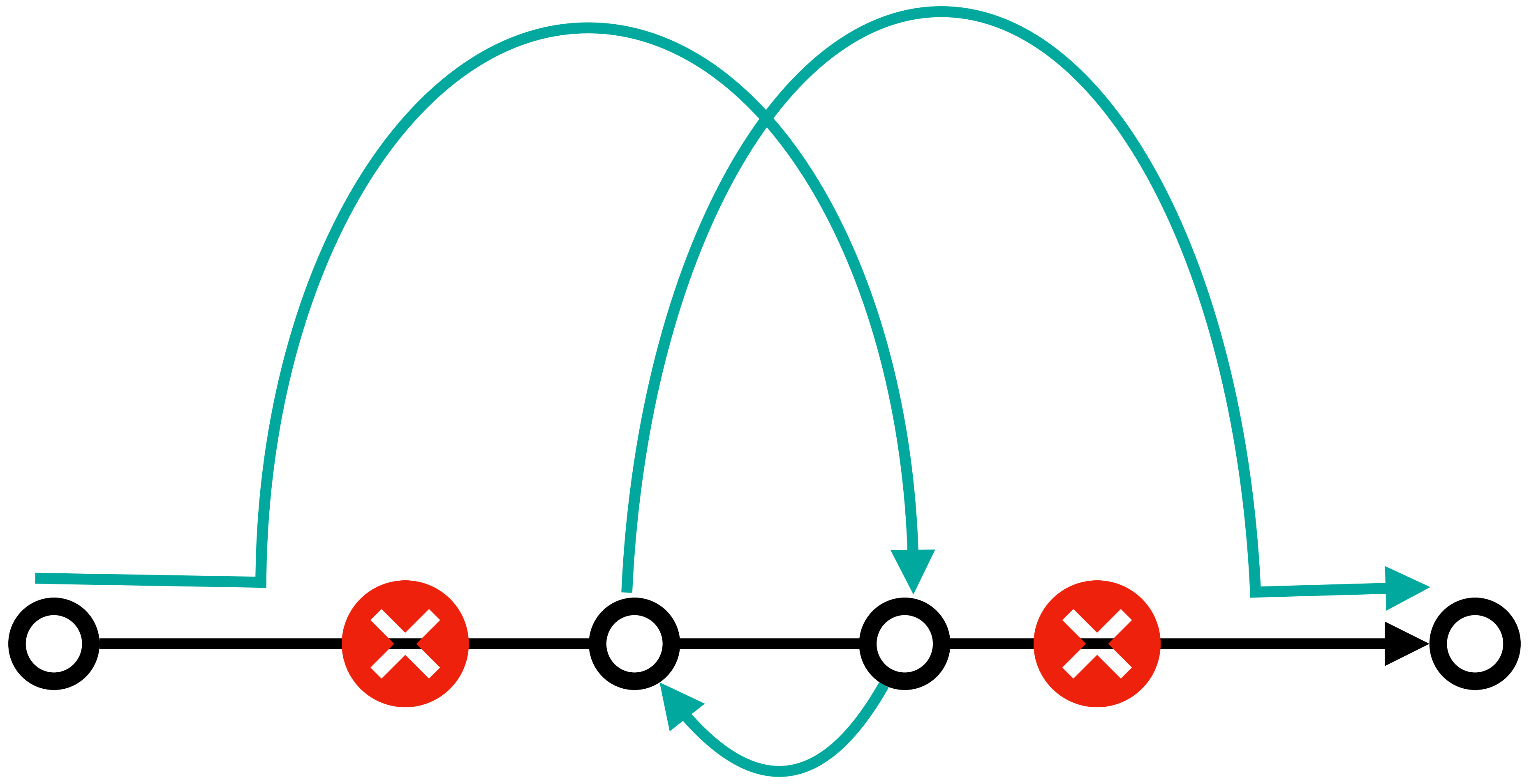
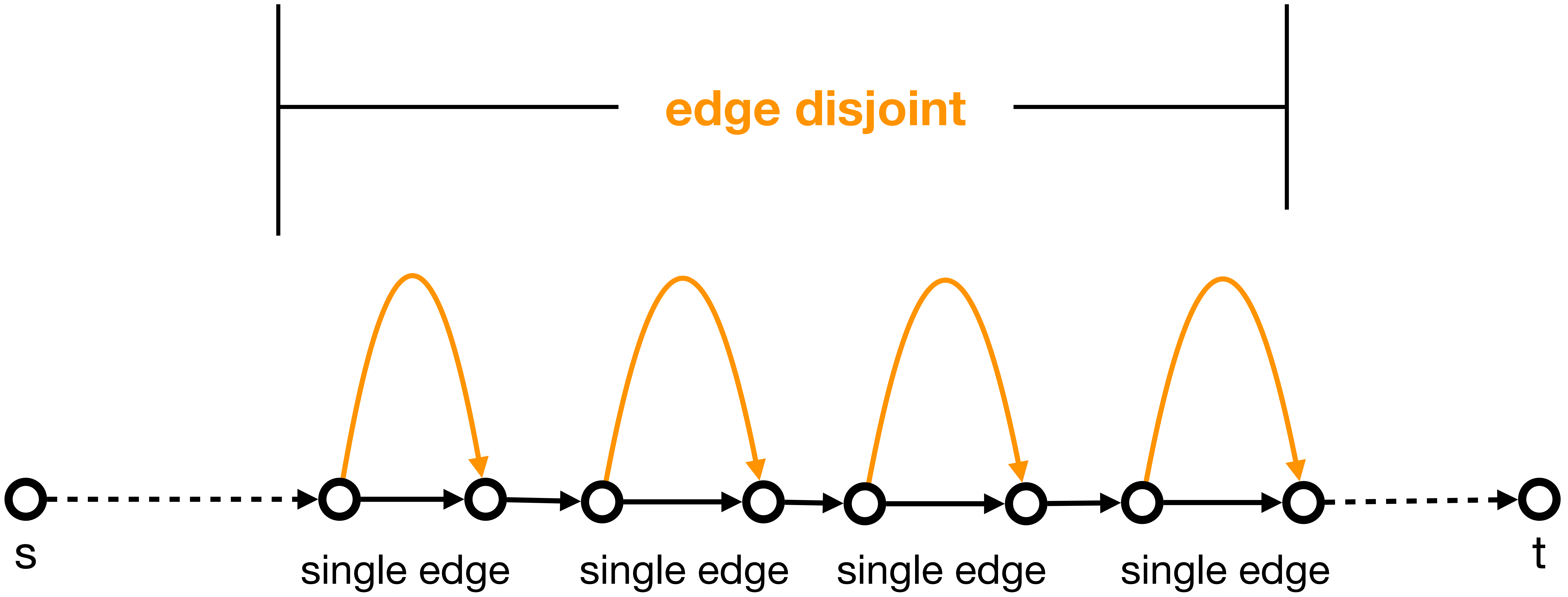


Wishful thinking



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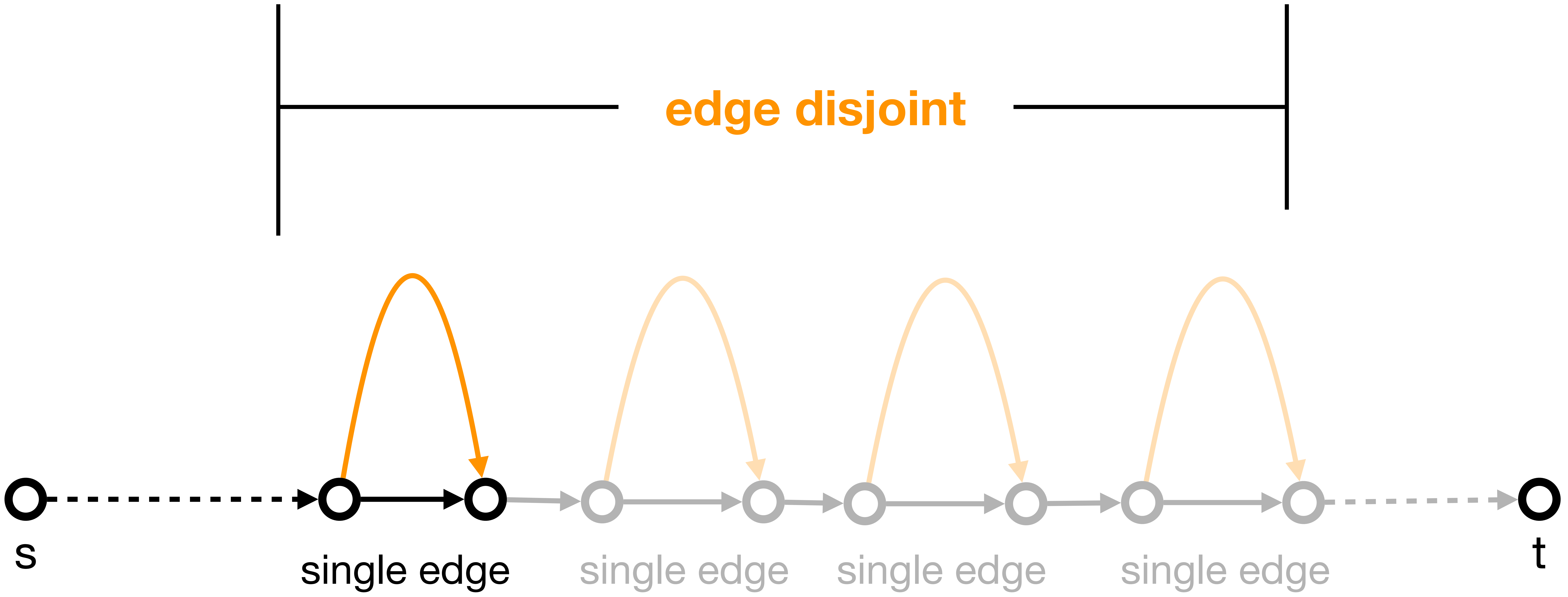

Add edges & detours one by one using progressive Dijkstra

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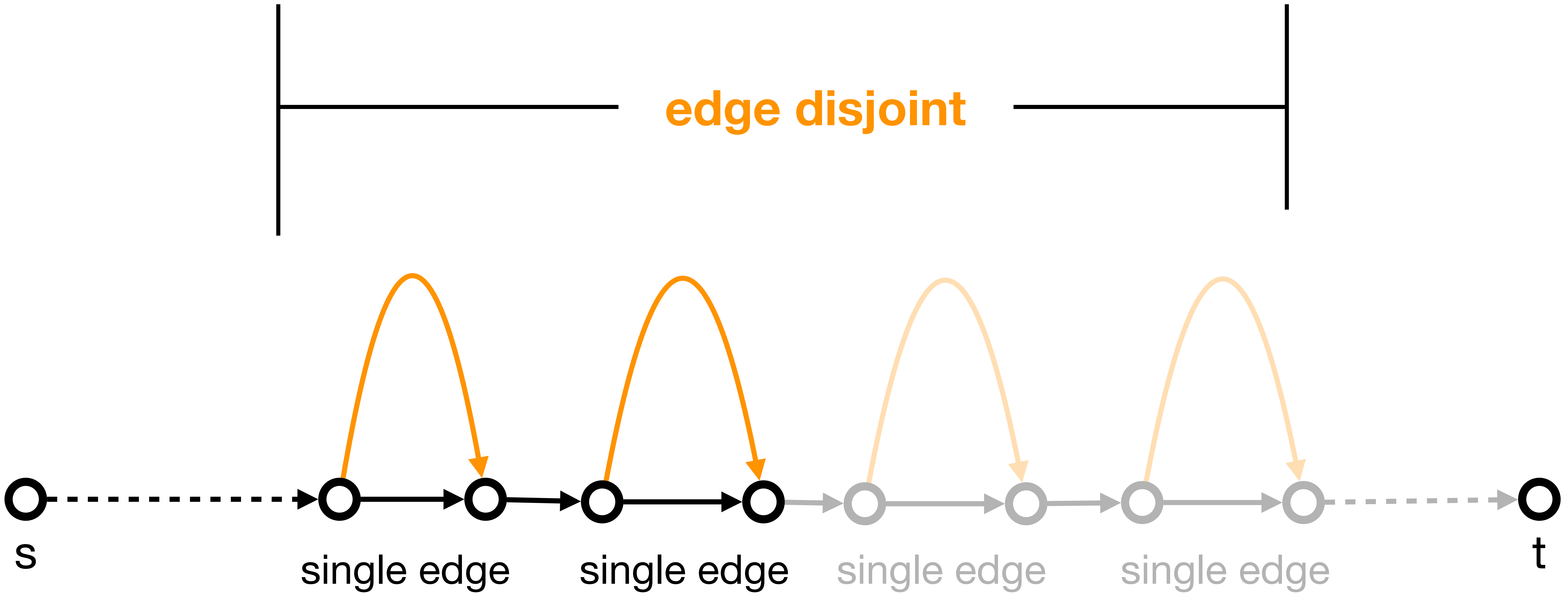

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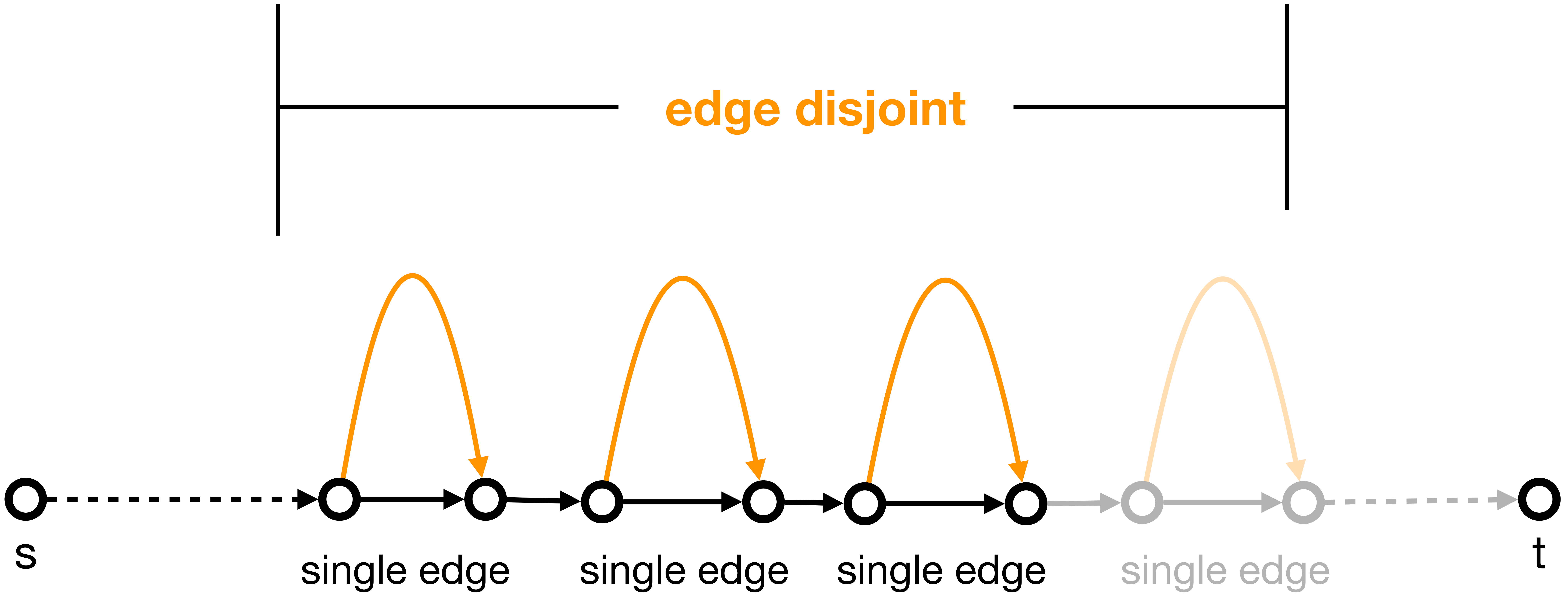

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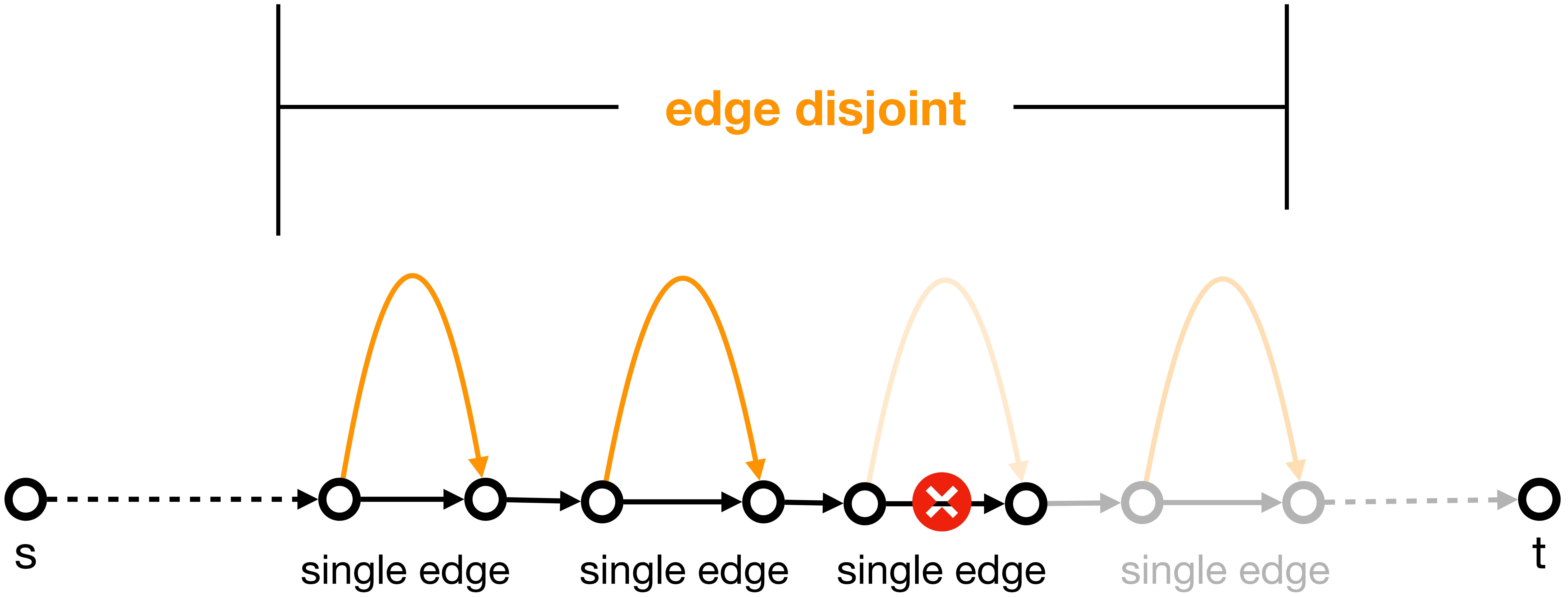

Add edges on detours one by one, also using progressive Dijkstra

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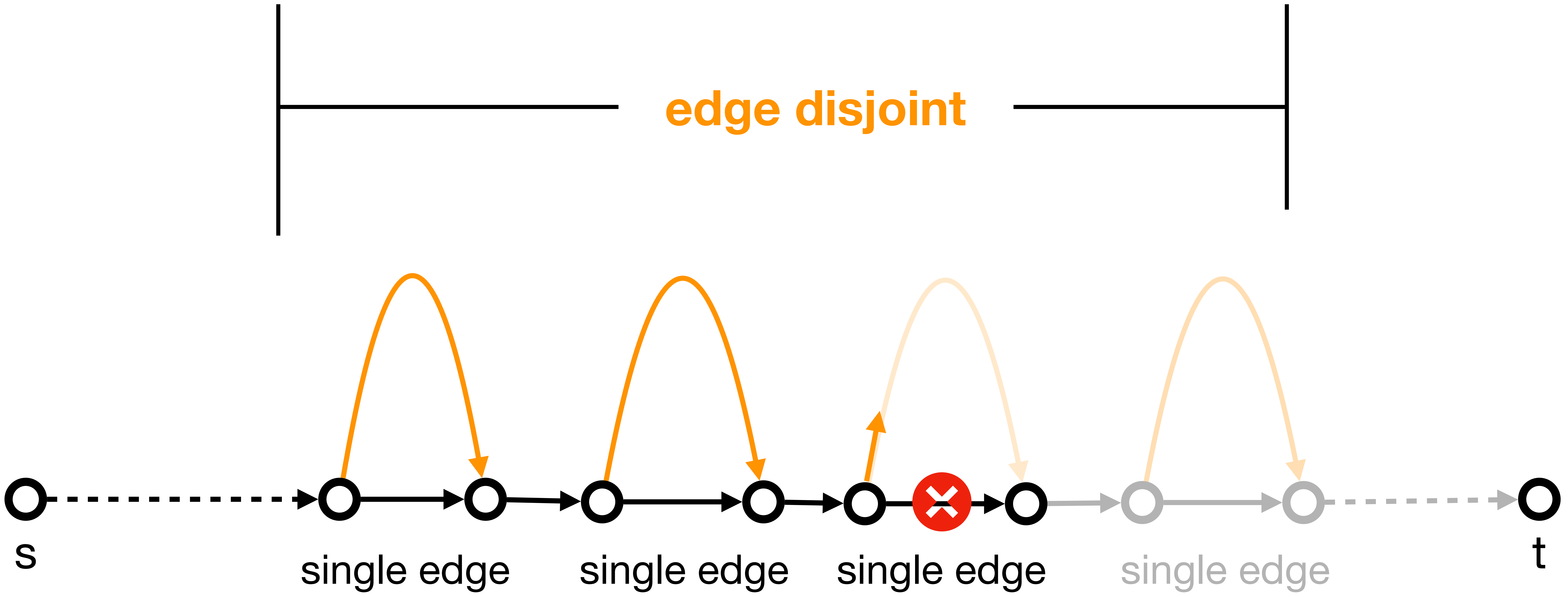

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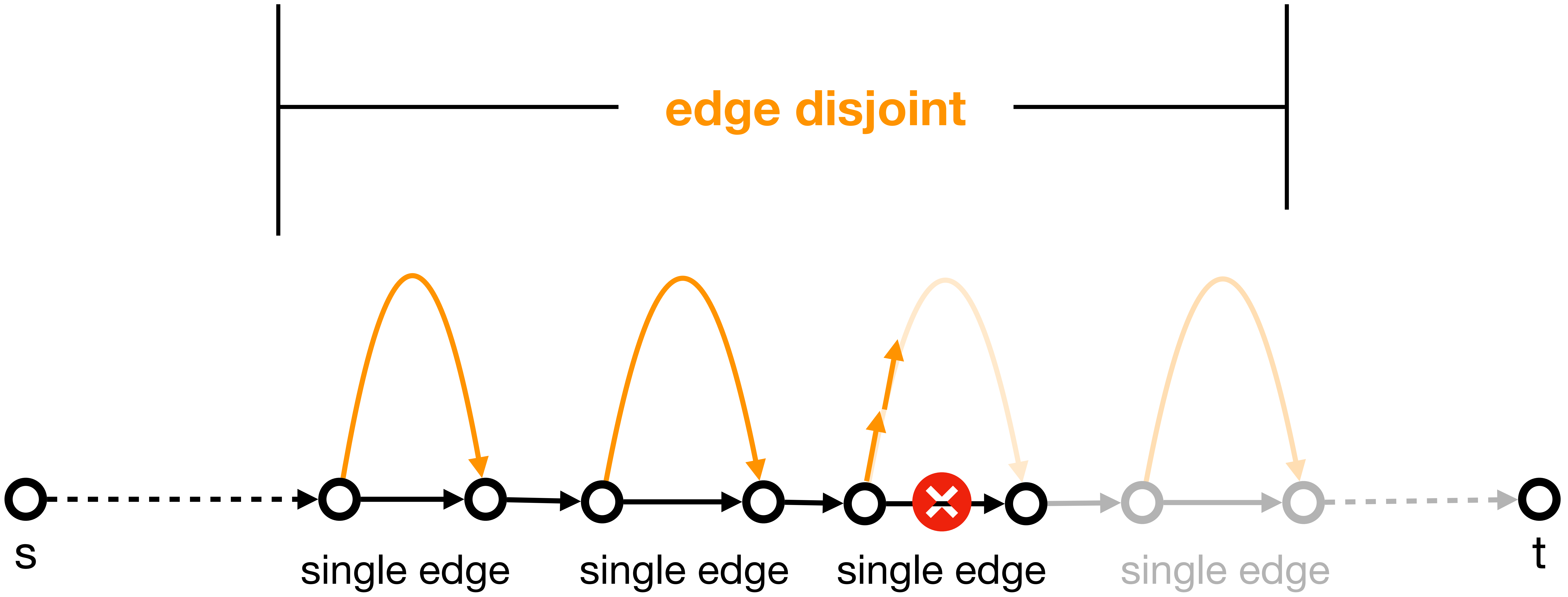

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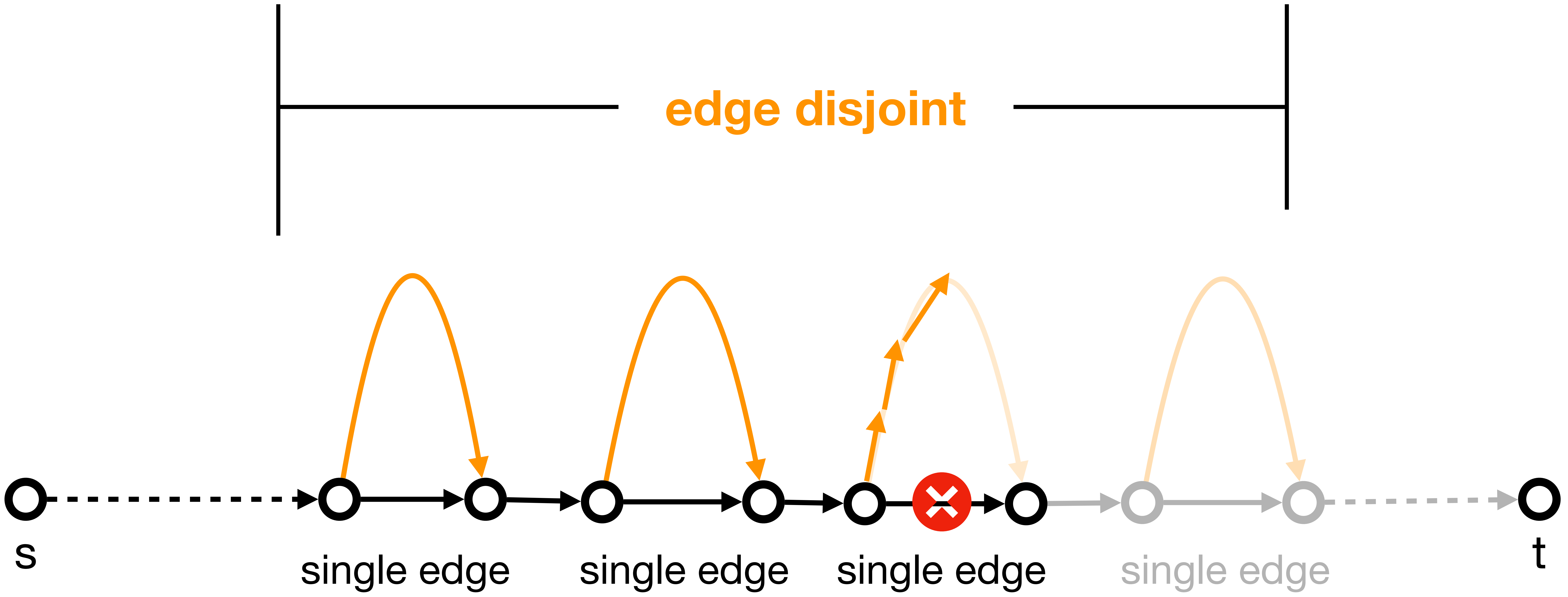

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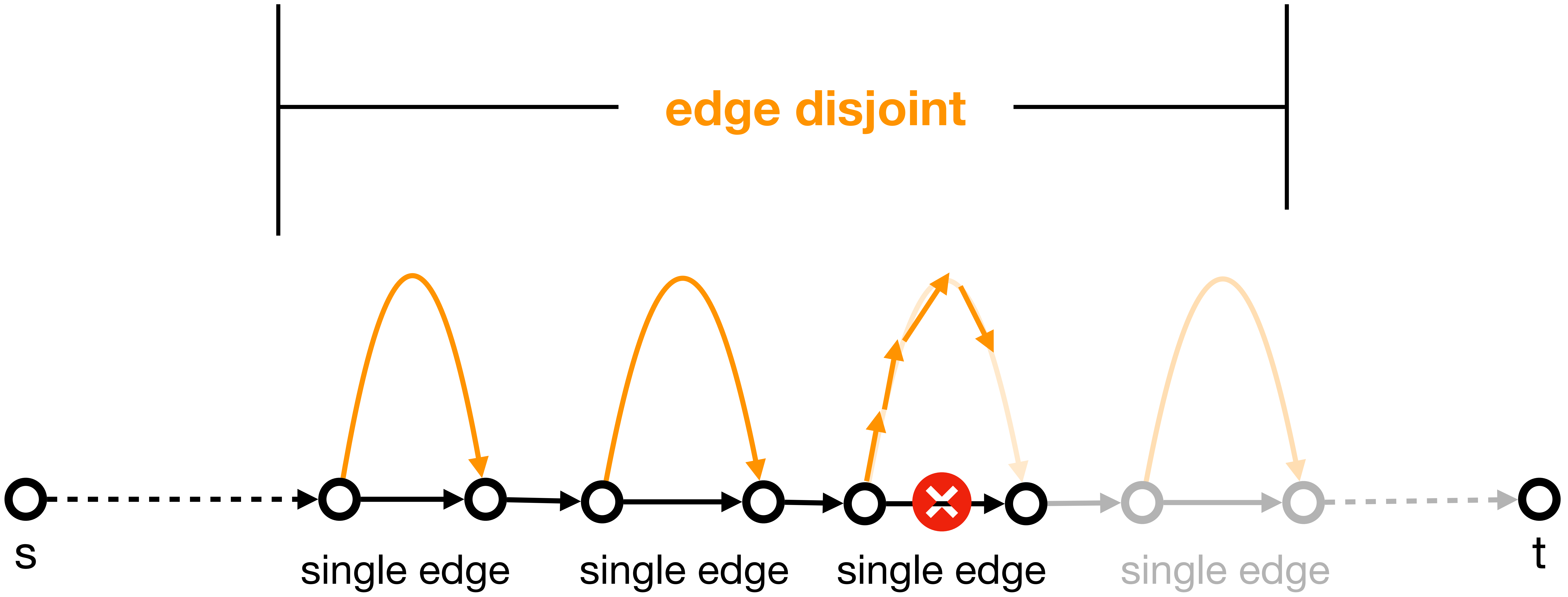

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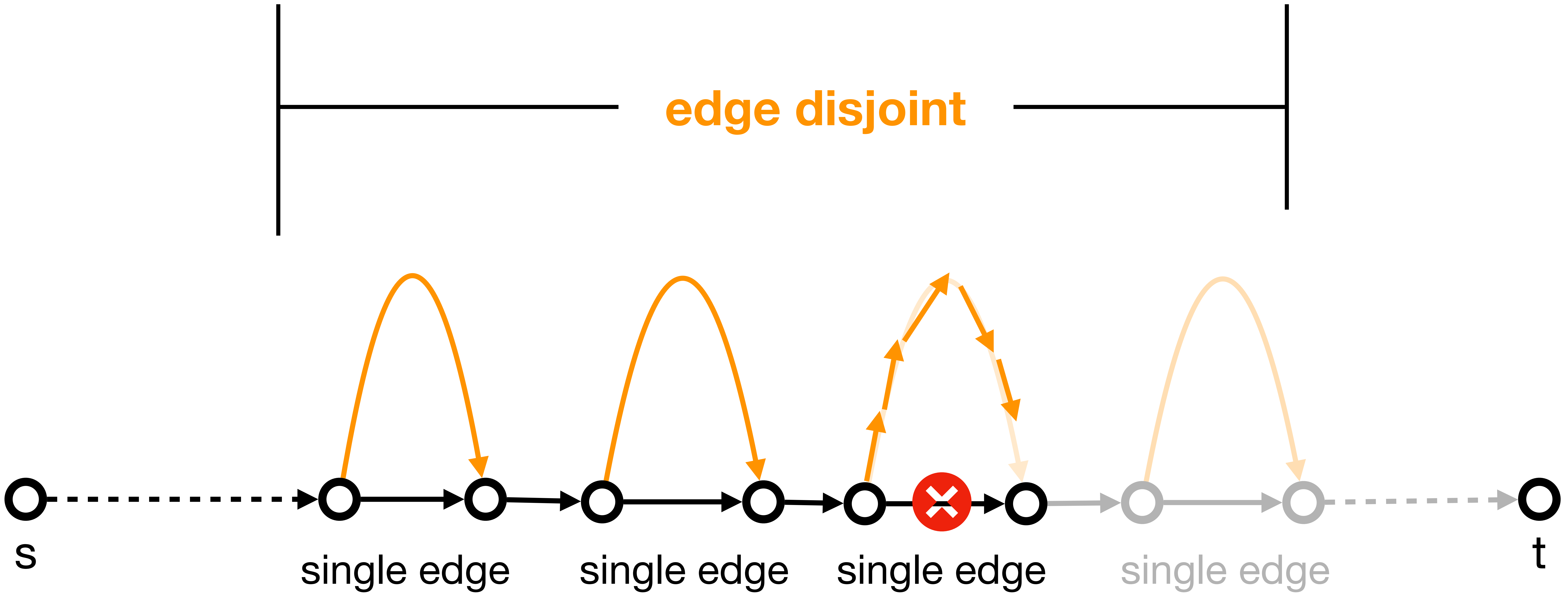

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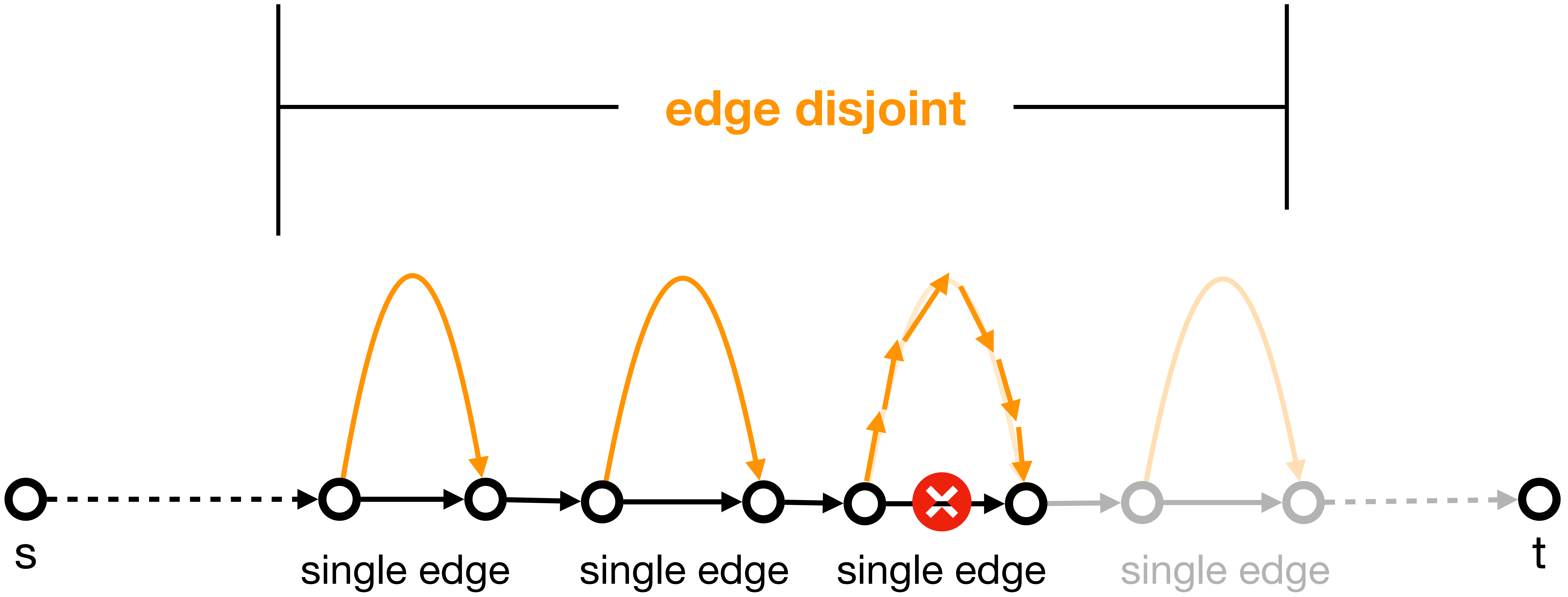

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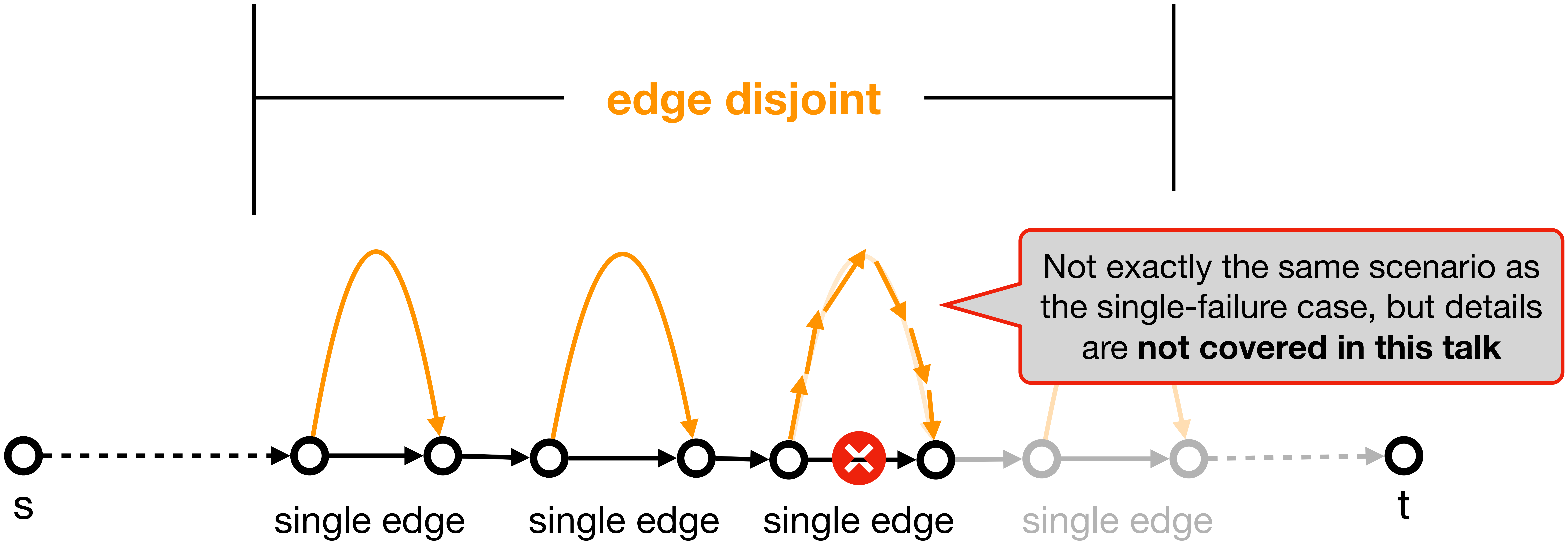

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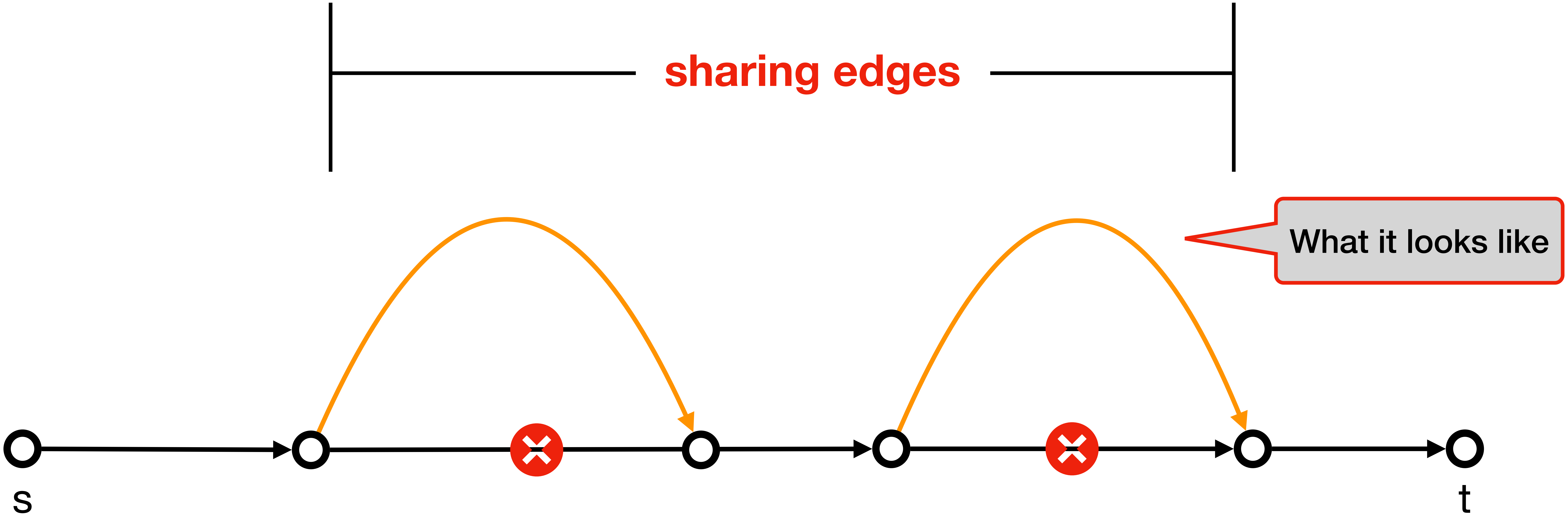
Wishful thinking: technical issues

<u>Main issue:</u> Detours may intersect



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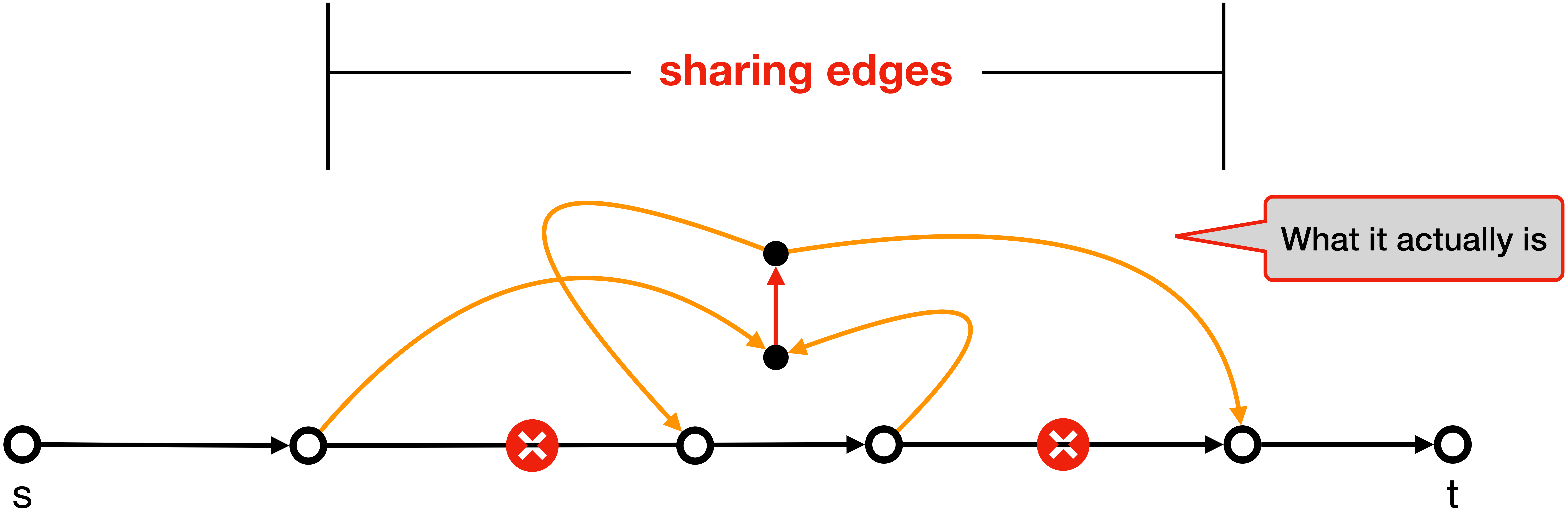

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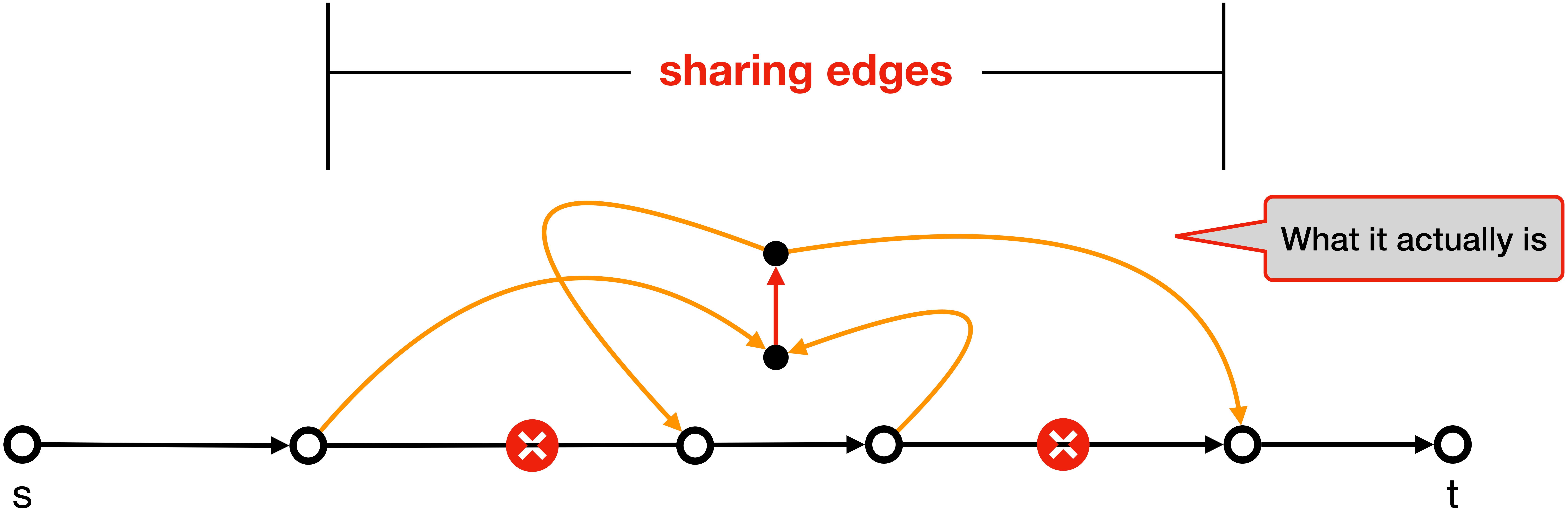

Consequence: cannot use progressive Dijkstra

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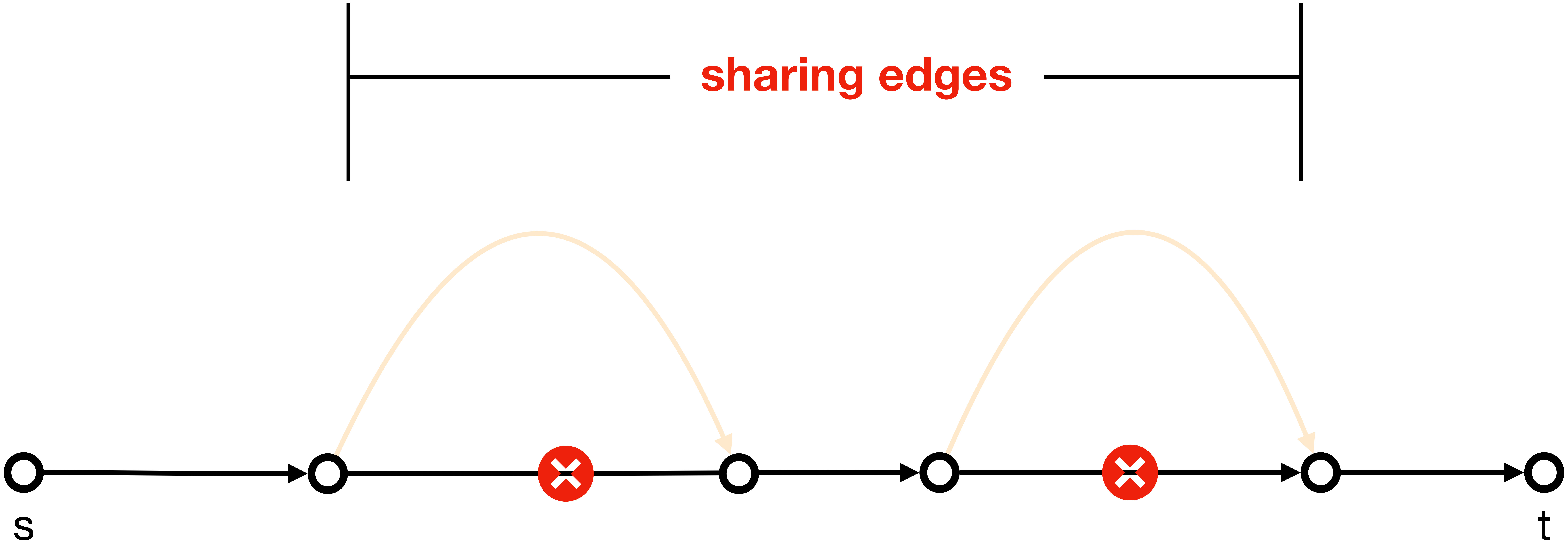

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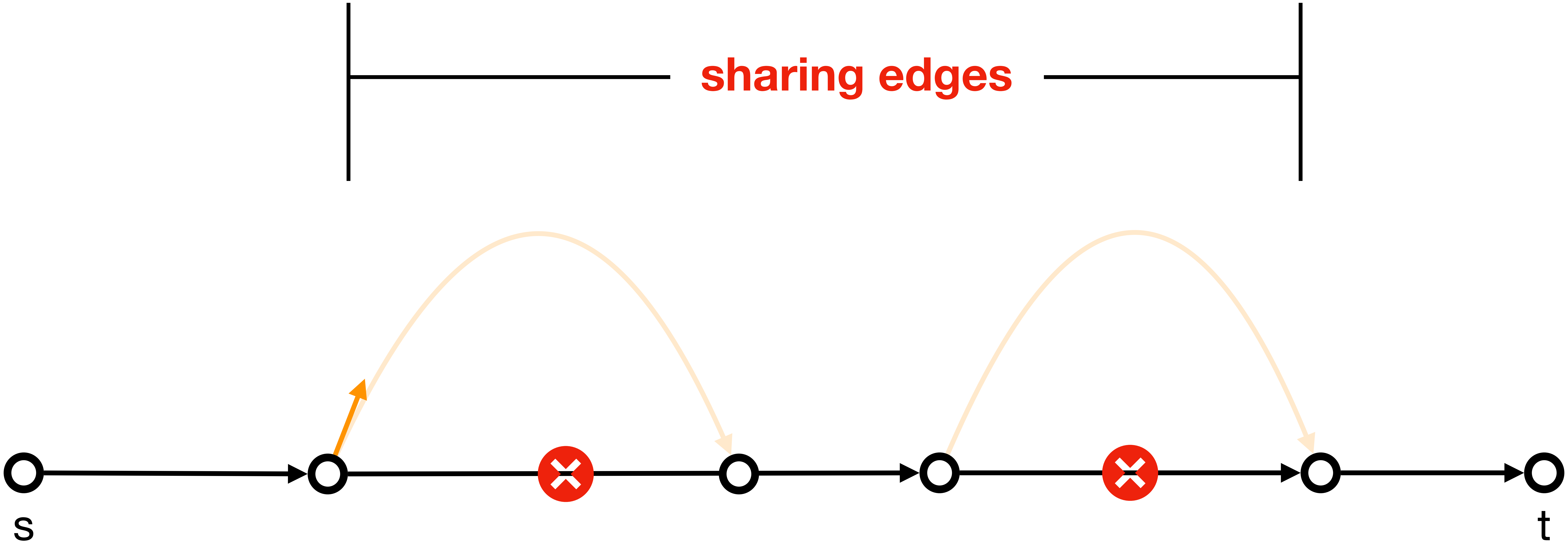

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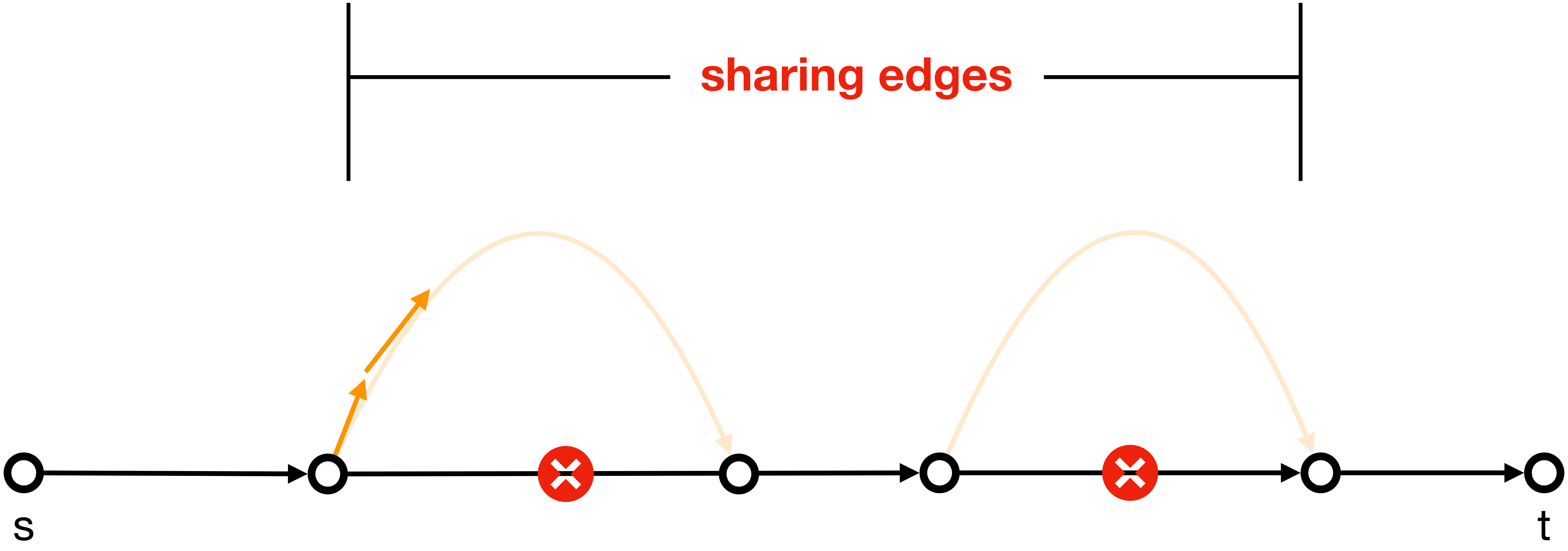

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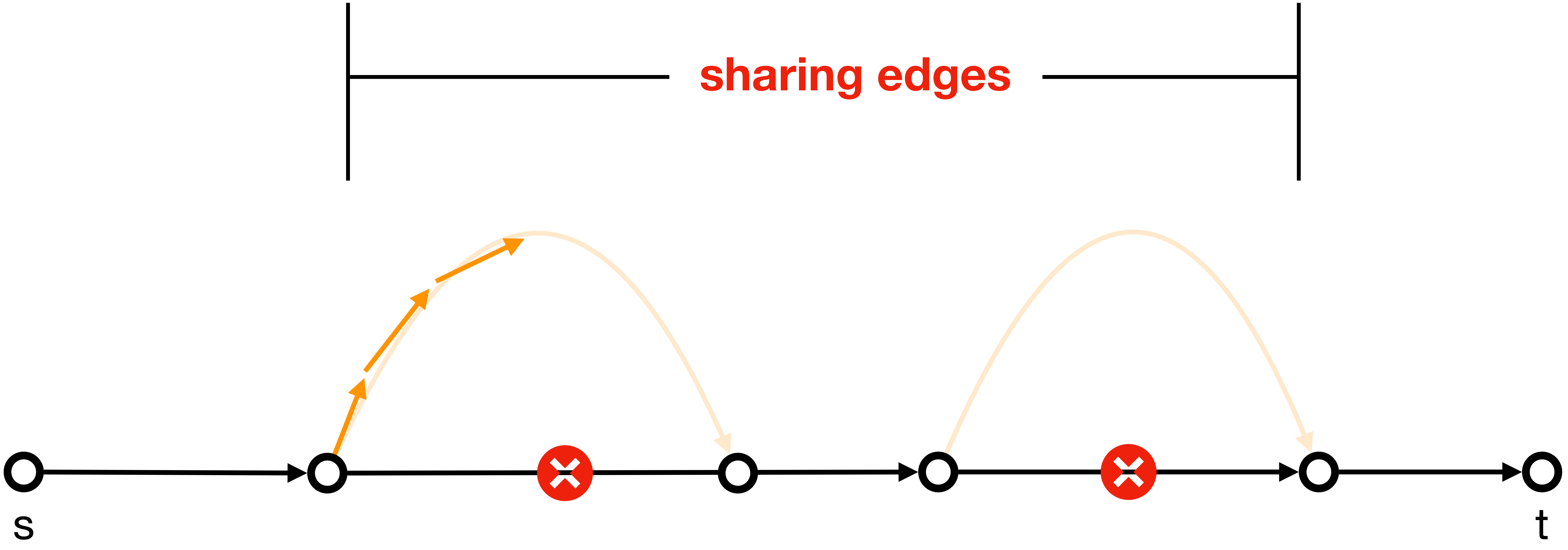

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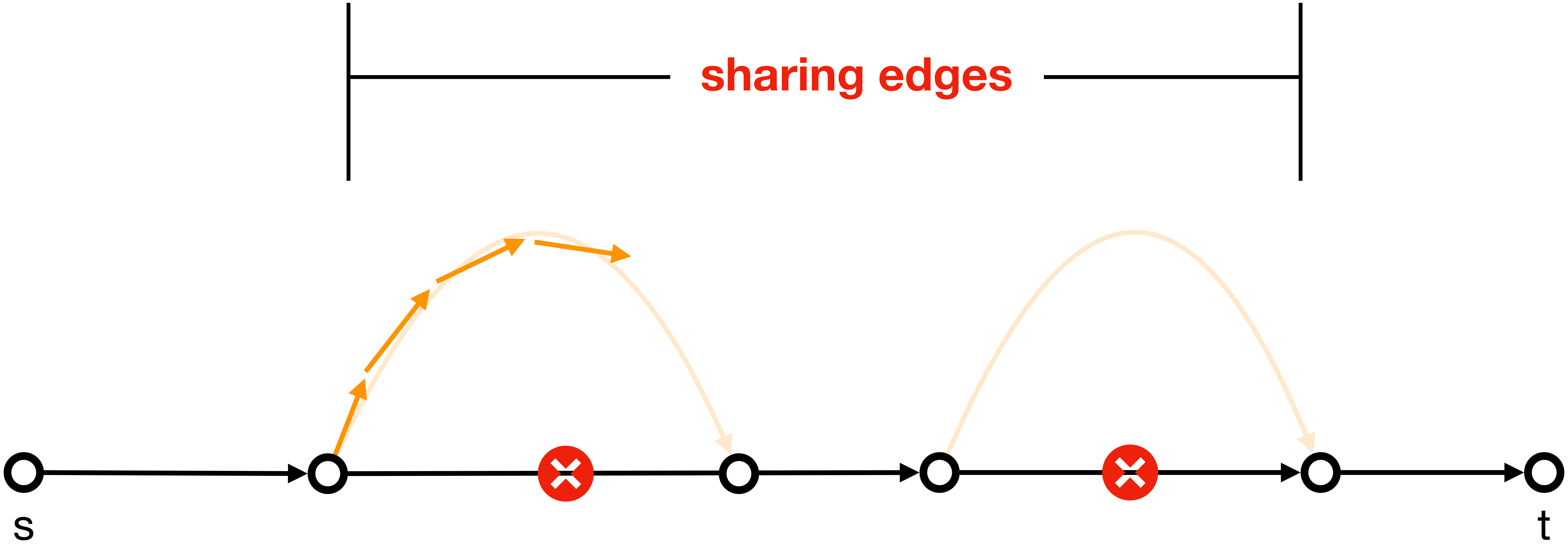

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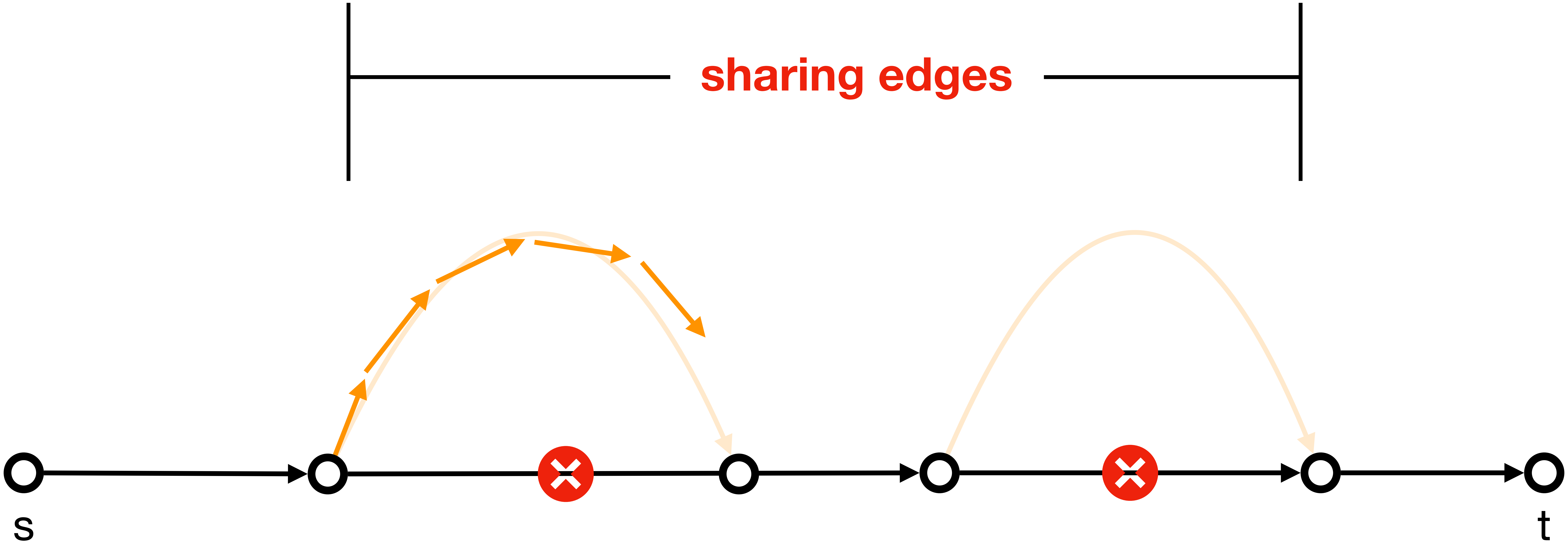

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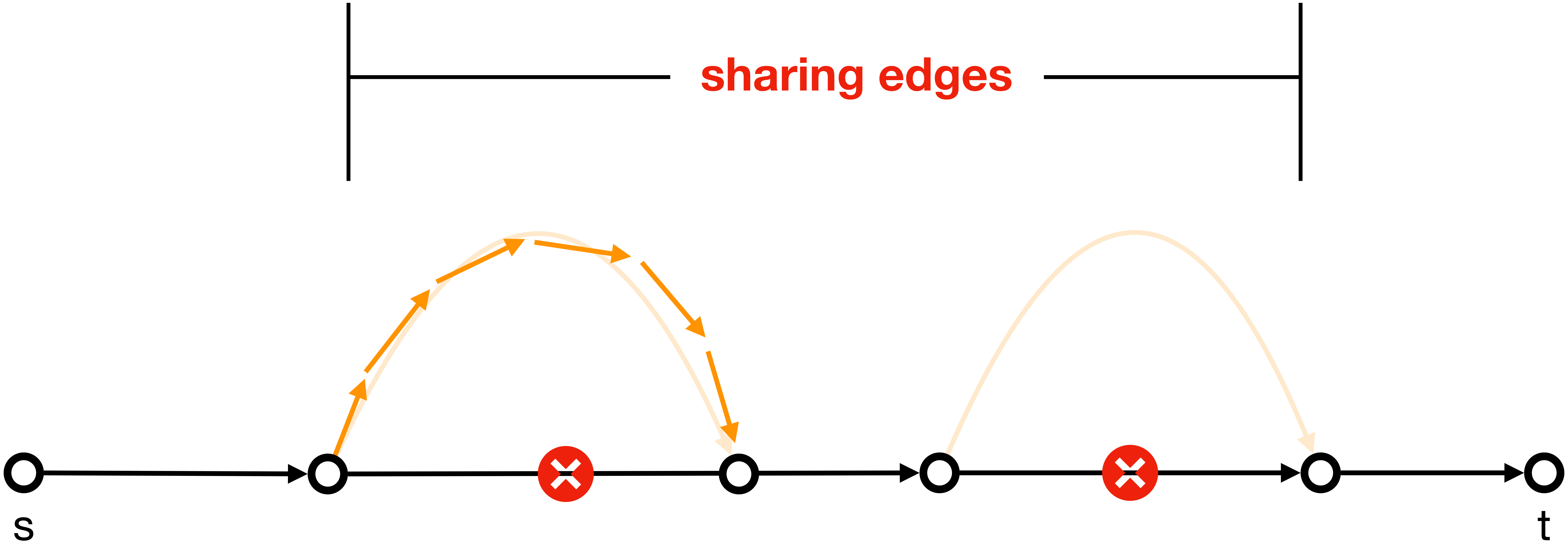

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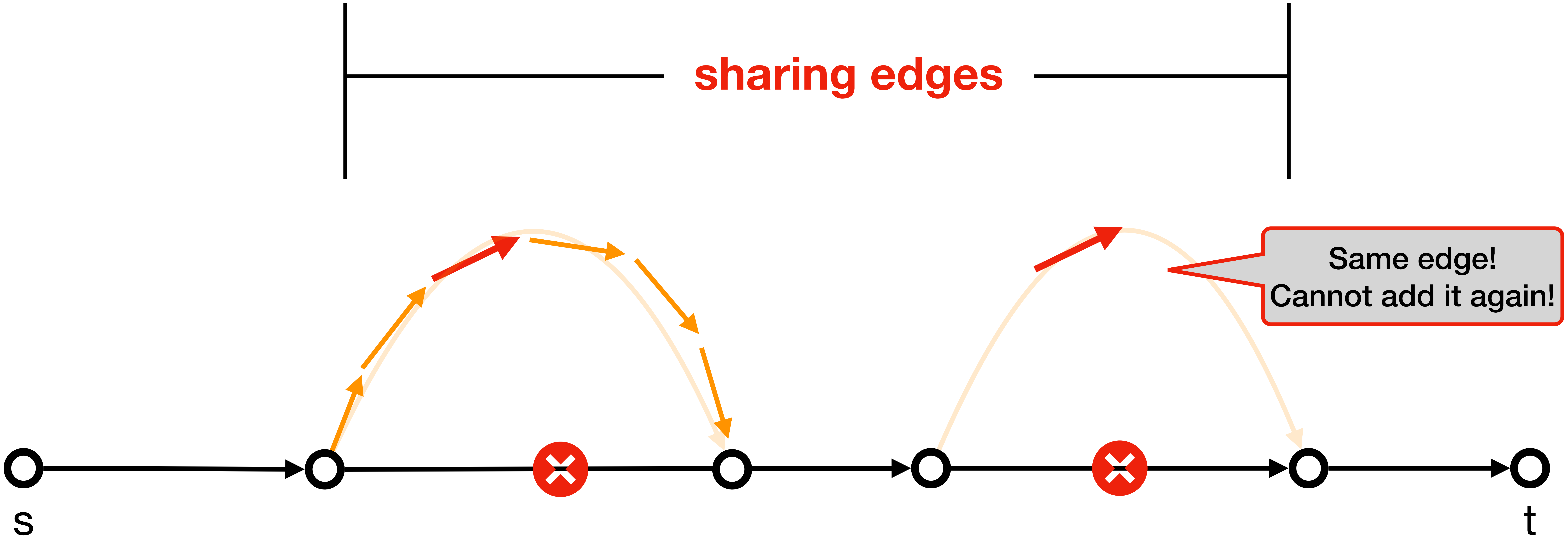

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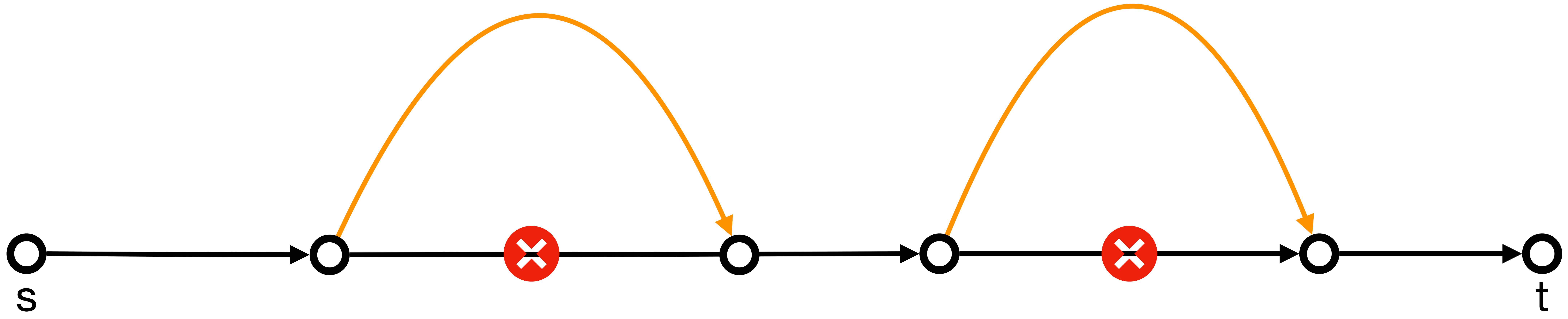
Dealing with detour intersections

Main issue: Detours may intersect

Solution: **Concat**enate detour while **doubling** the span on st-path



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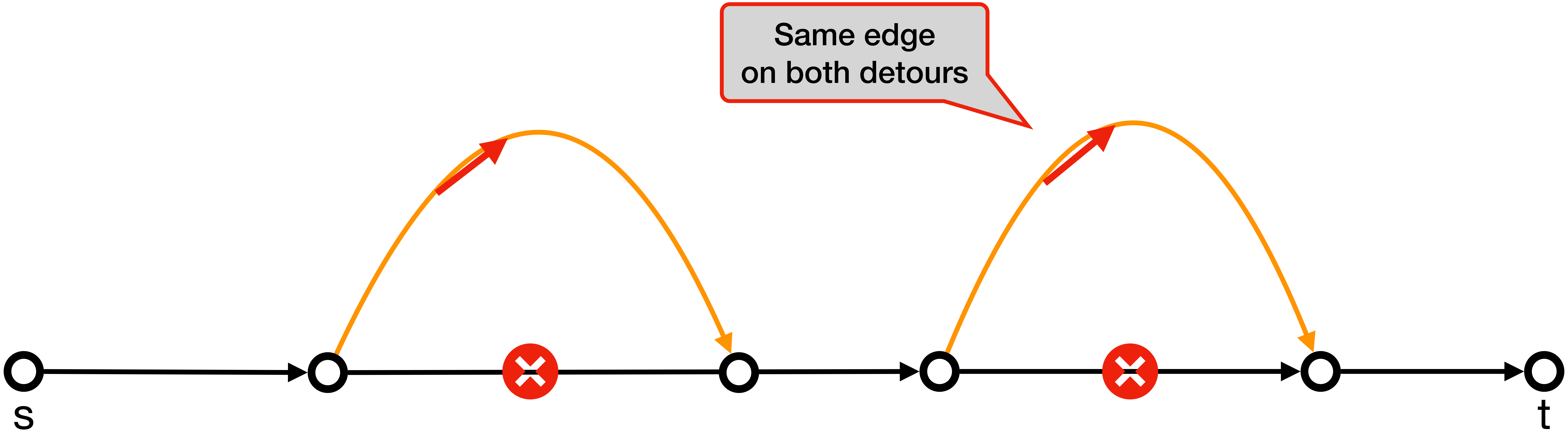

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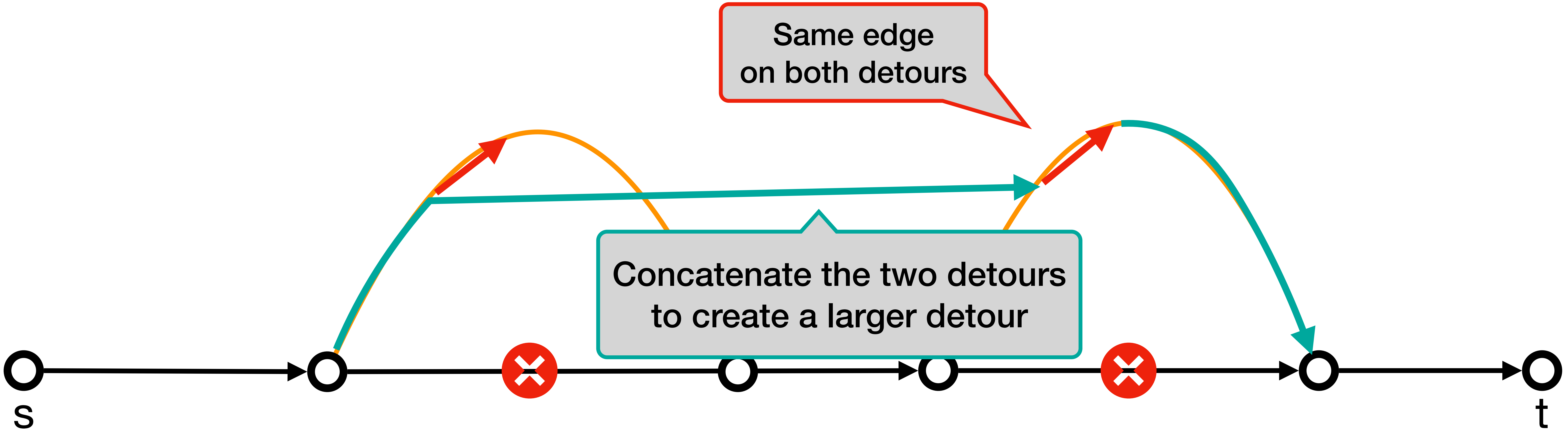

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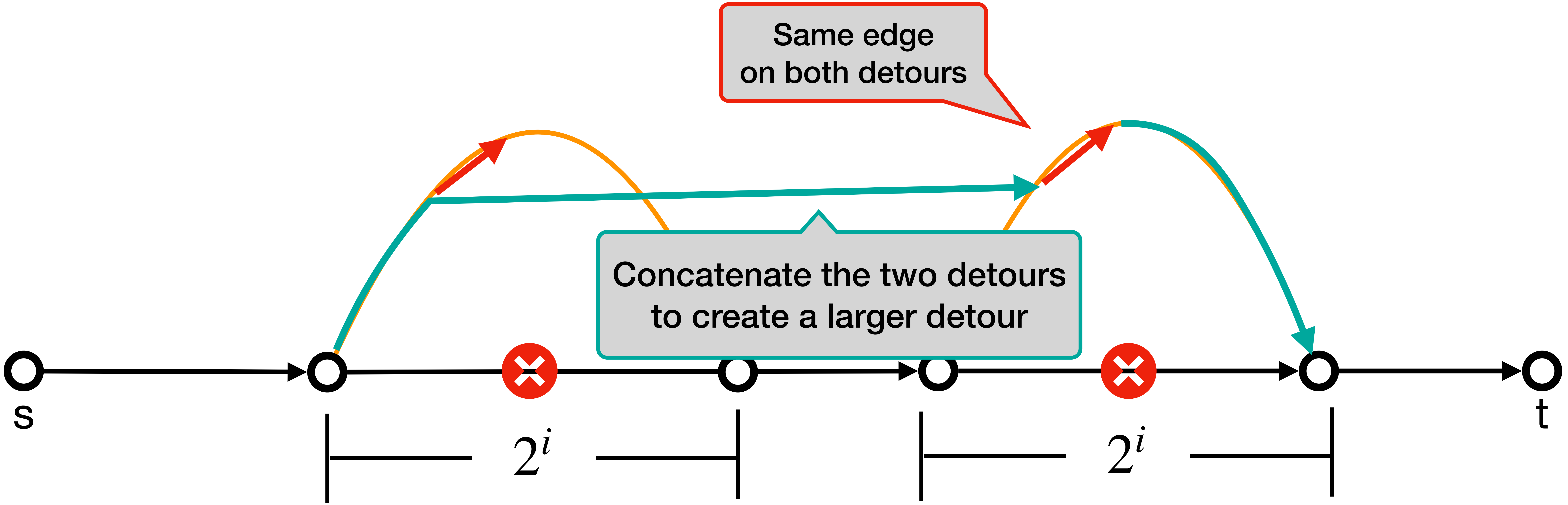

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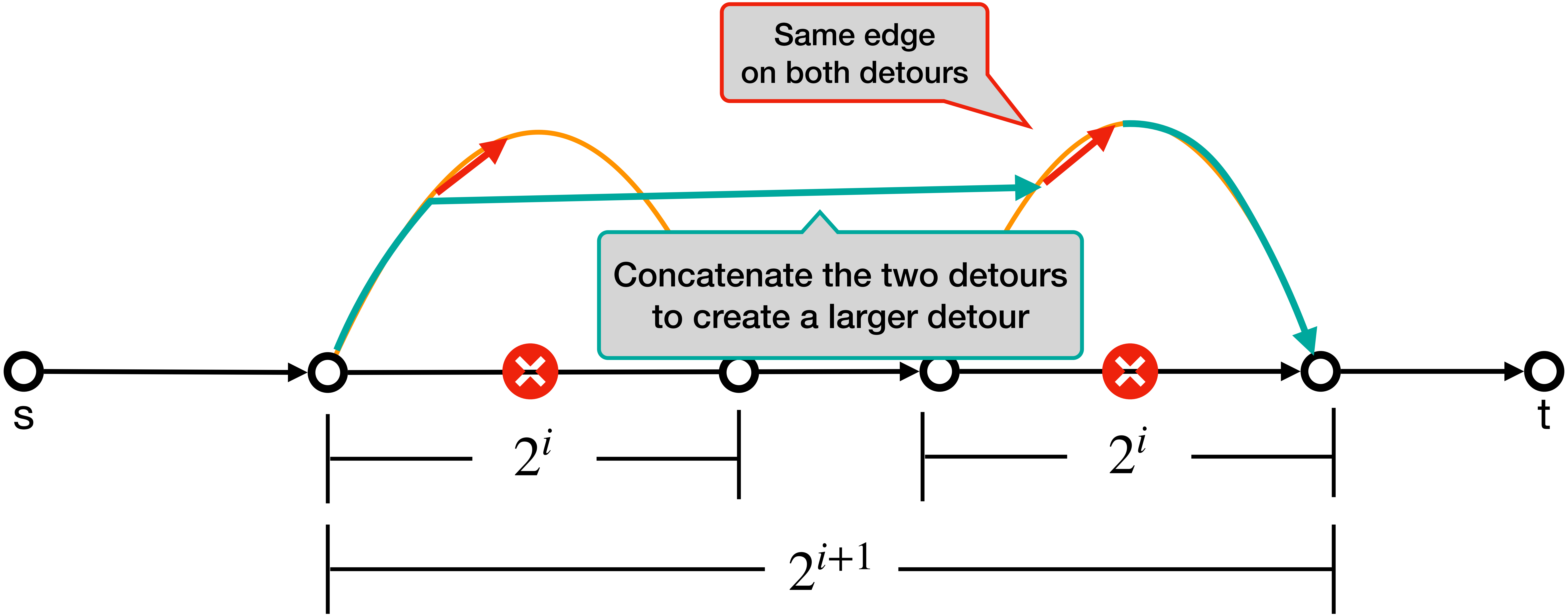
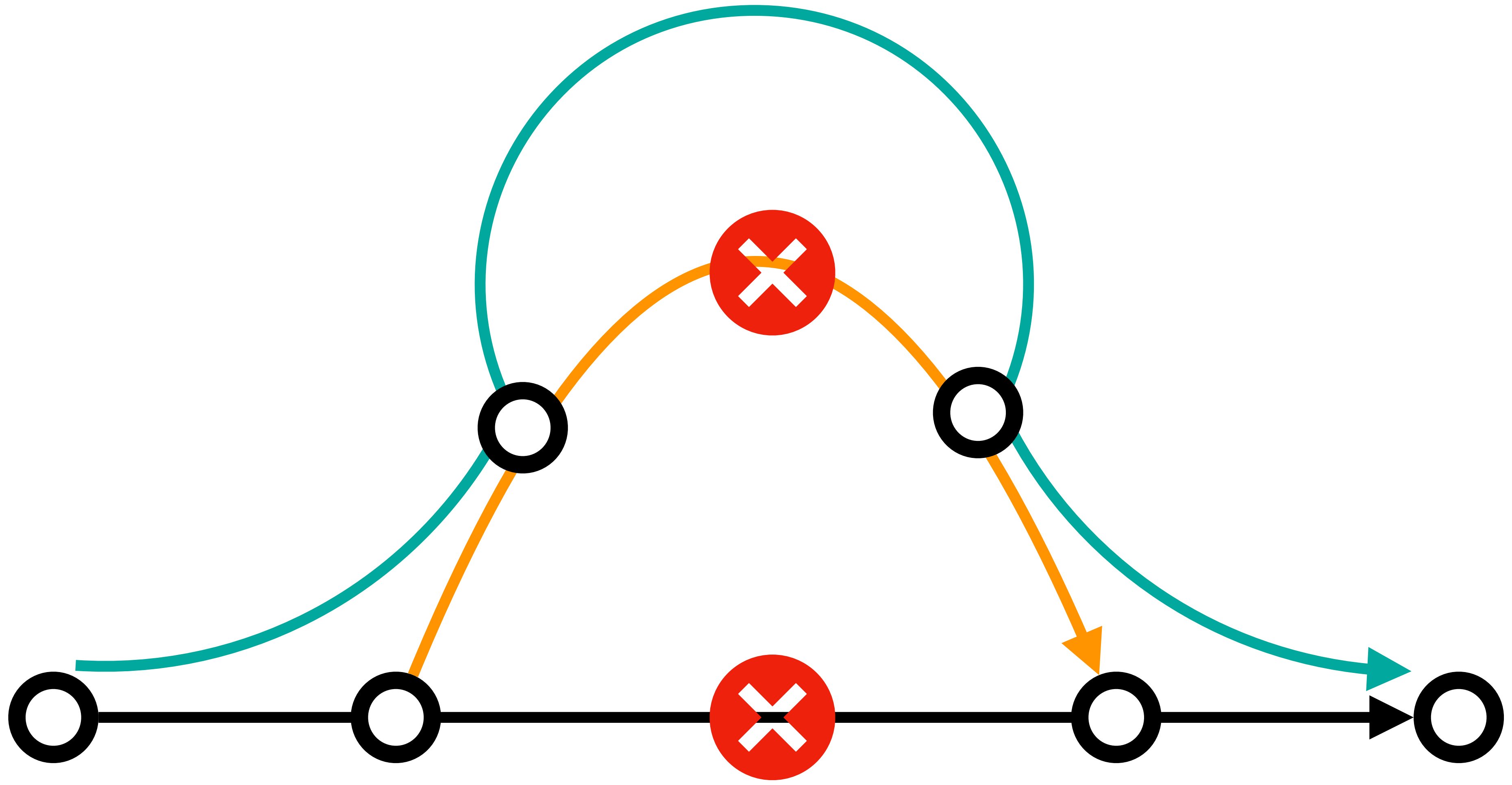


Two main cases

One failure on st-path



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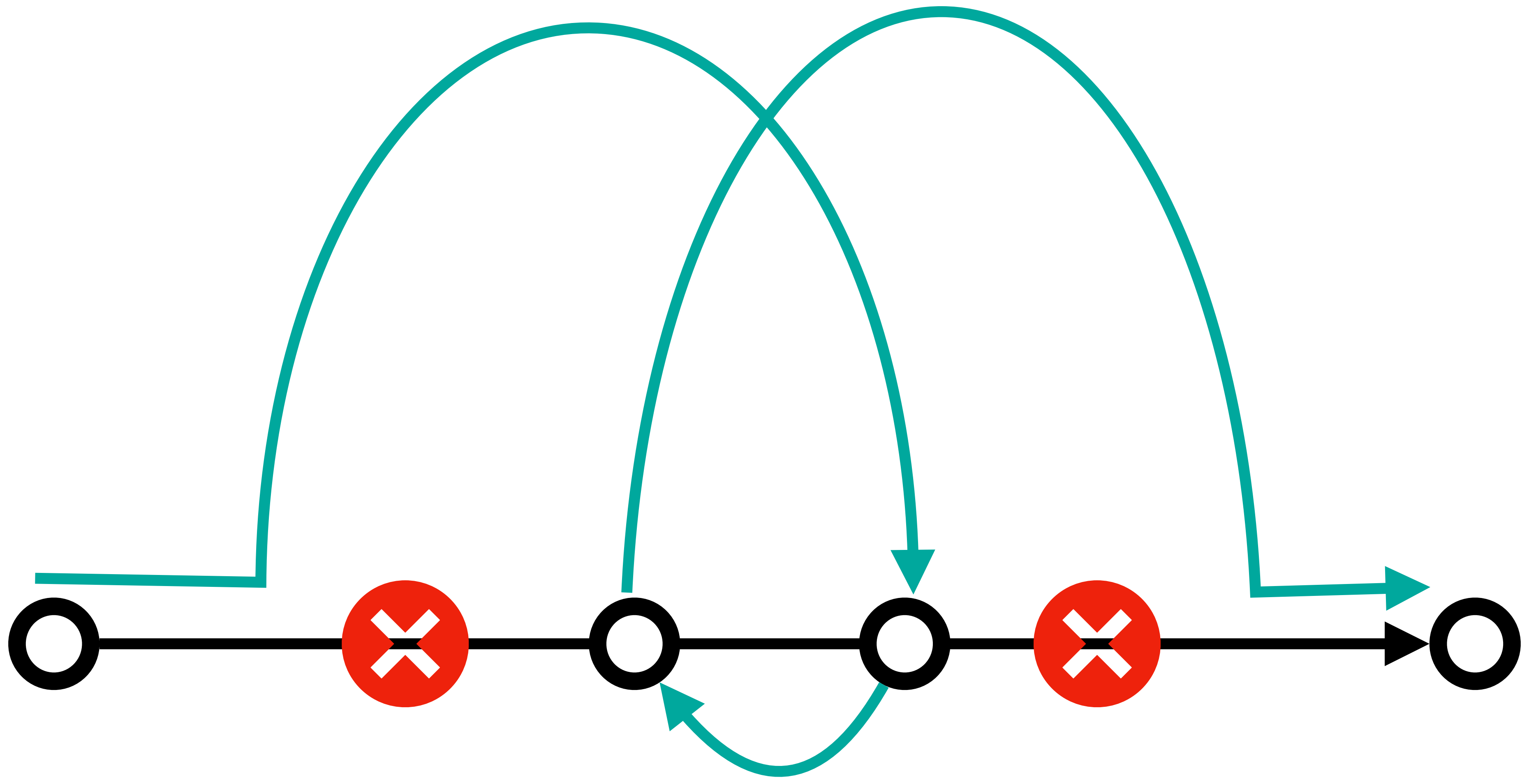

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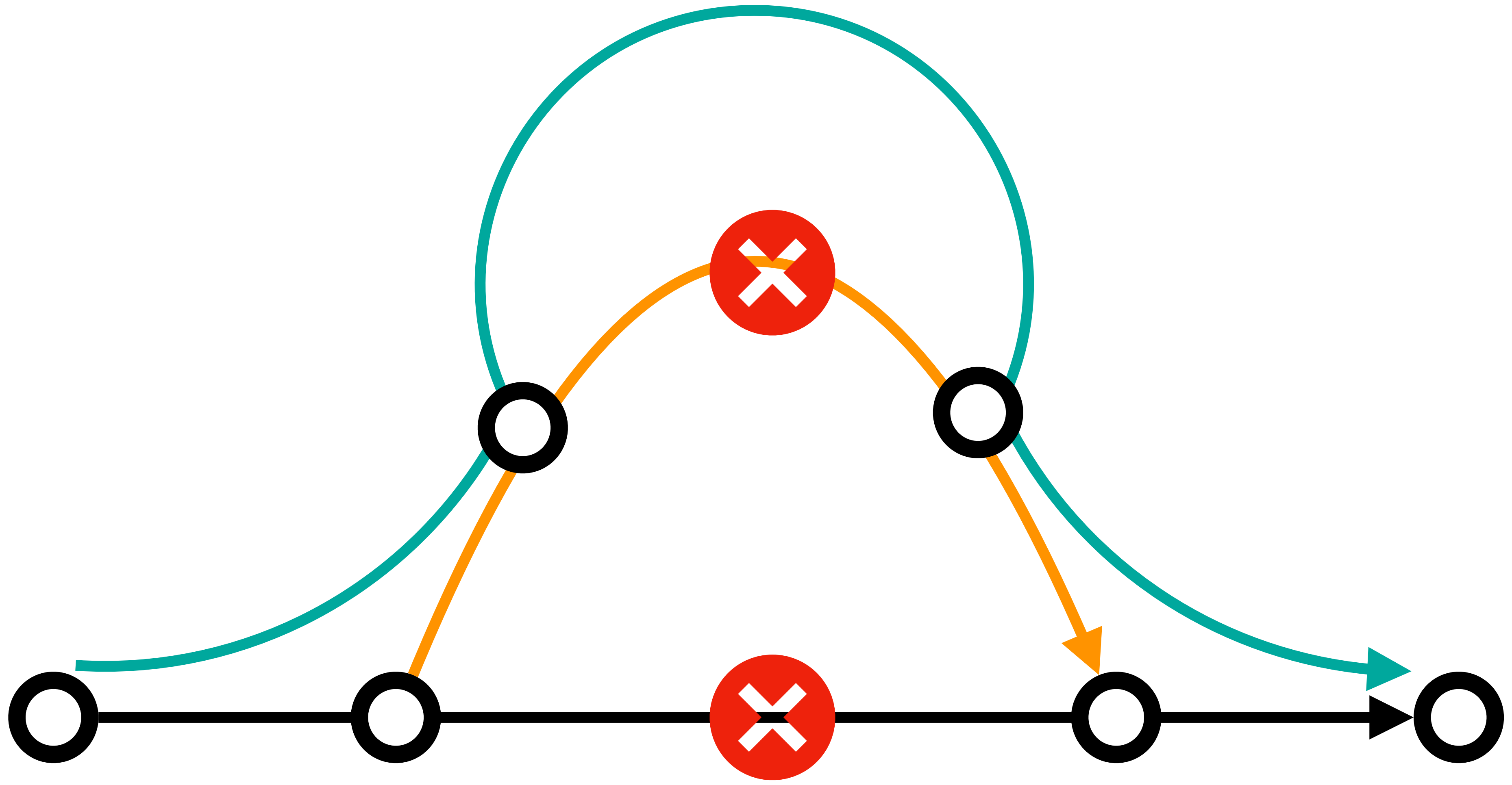

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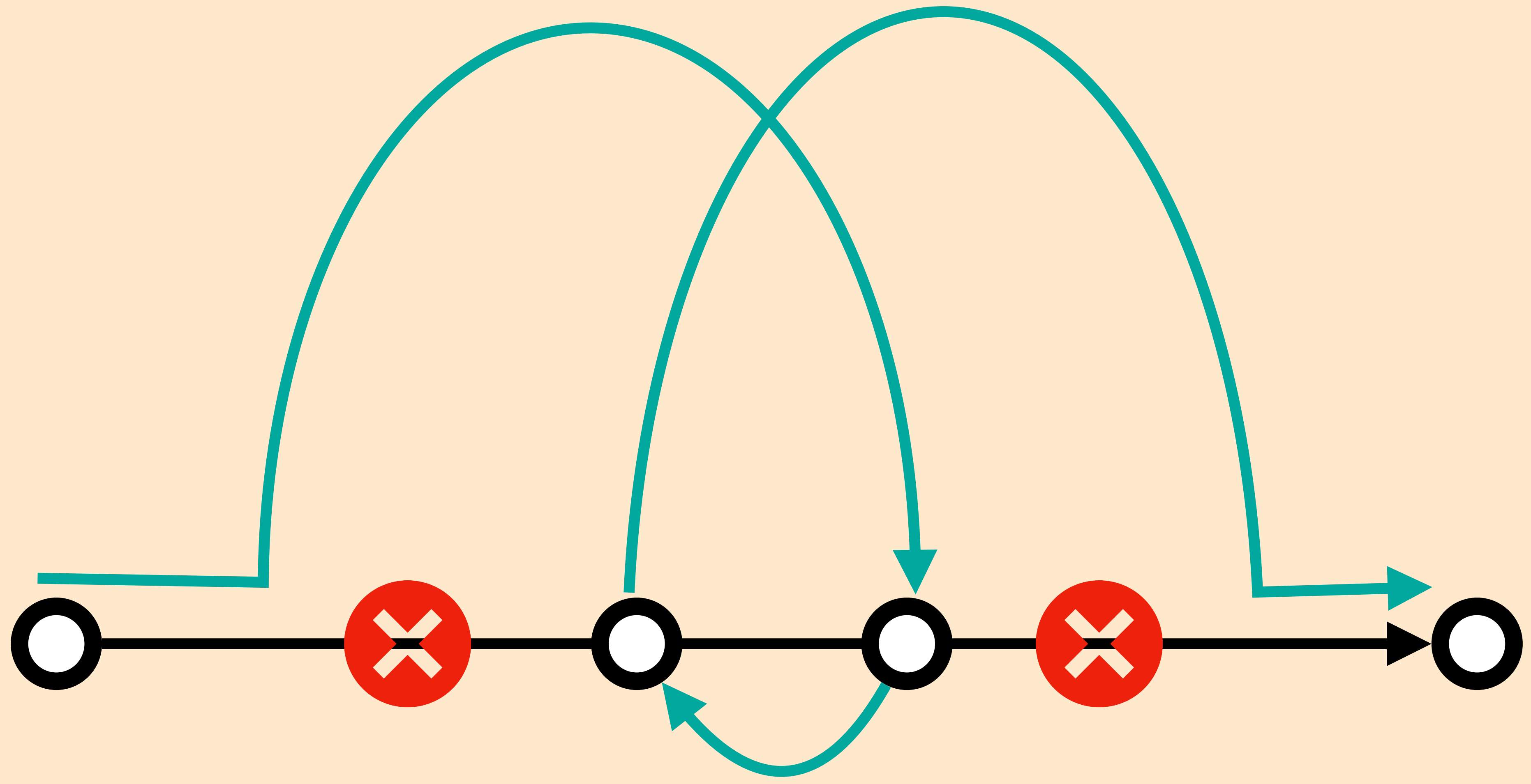
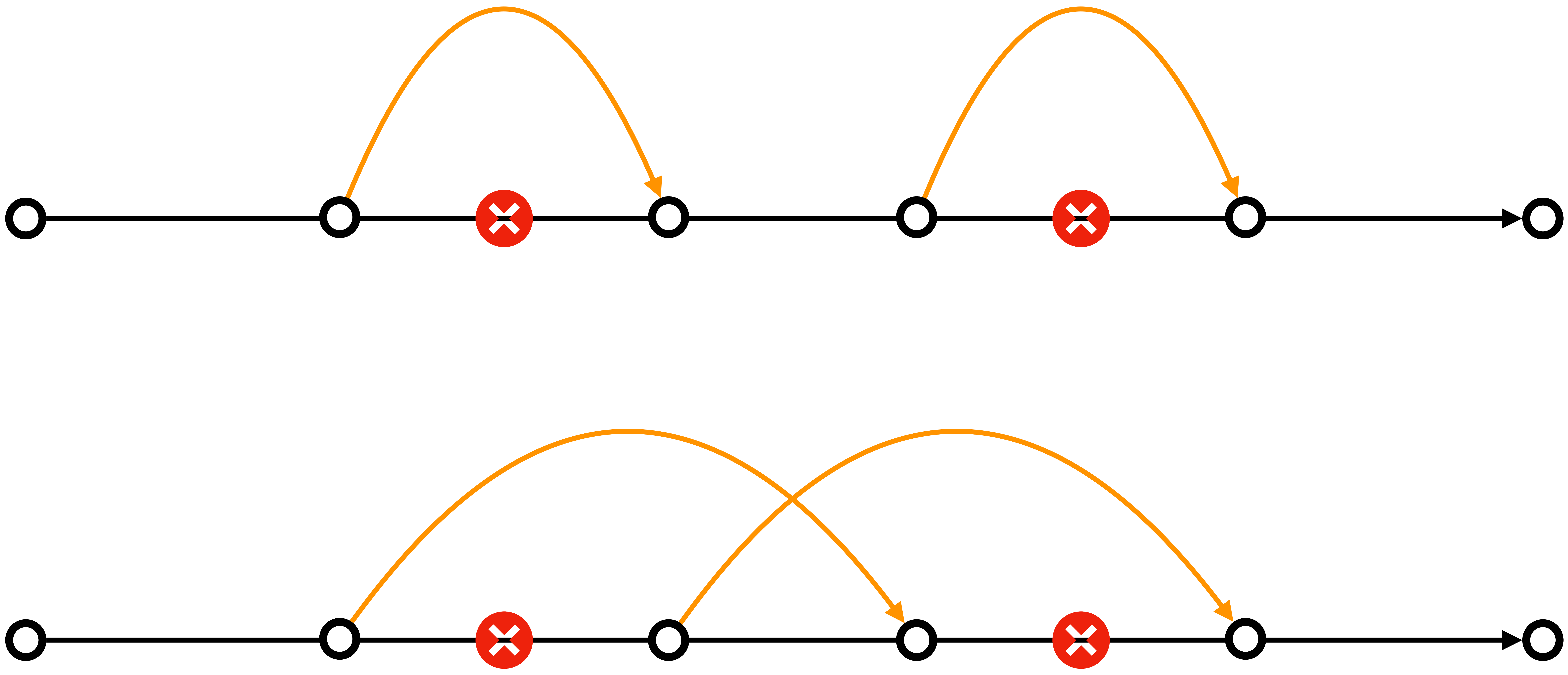


Easy & Hard Cases



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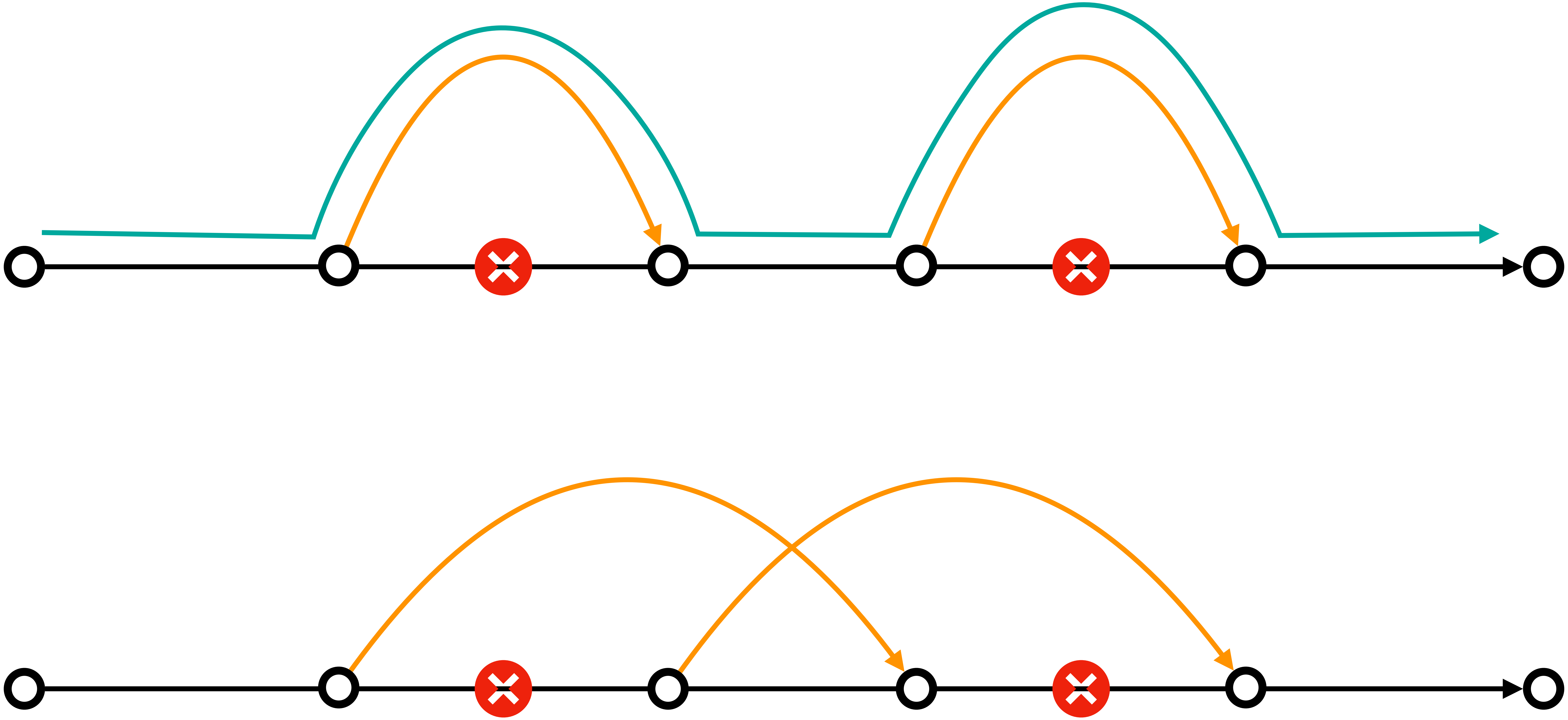

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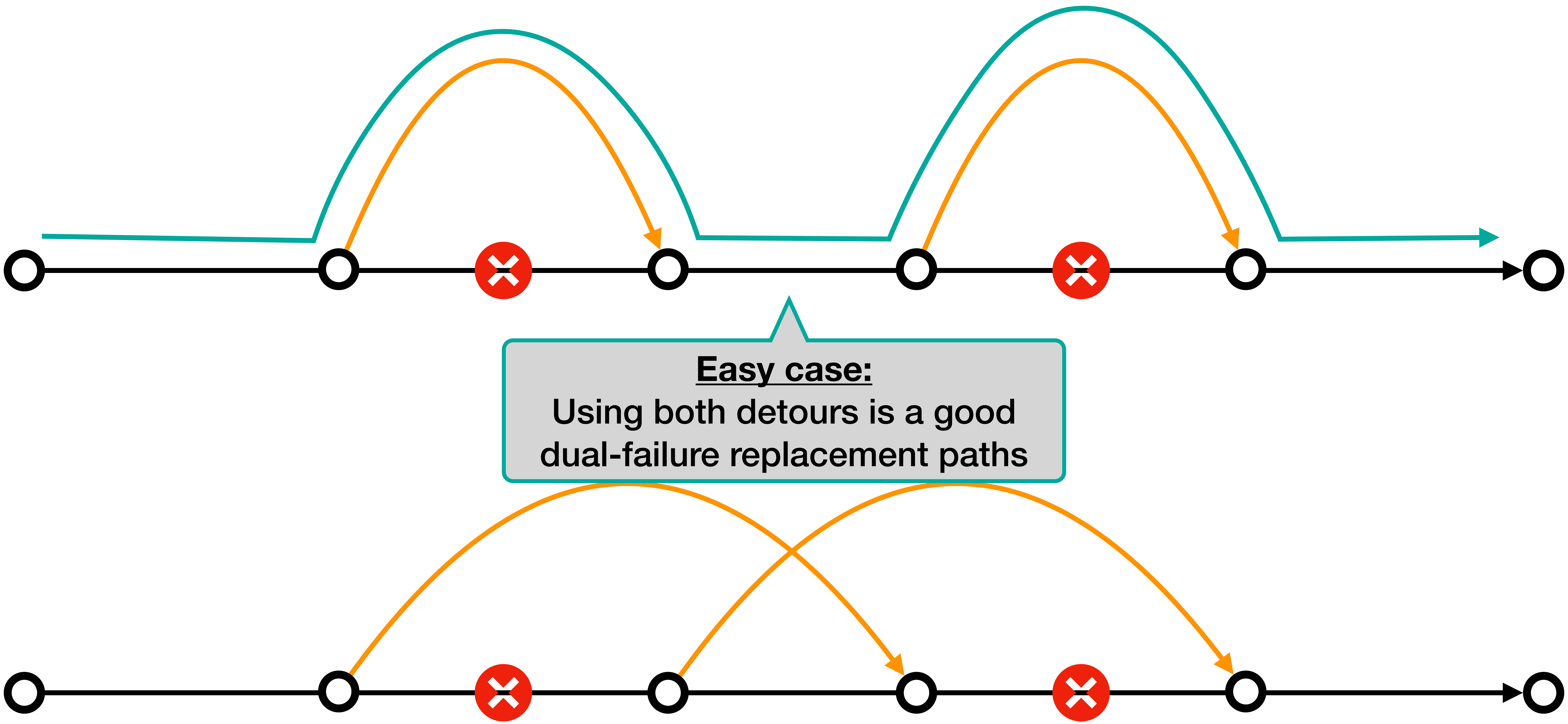
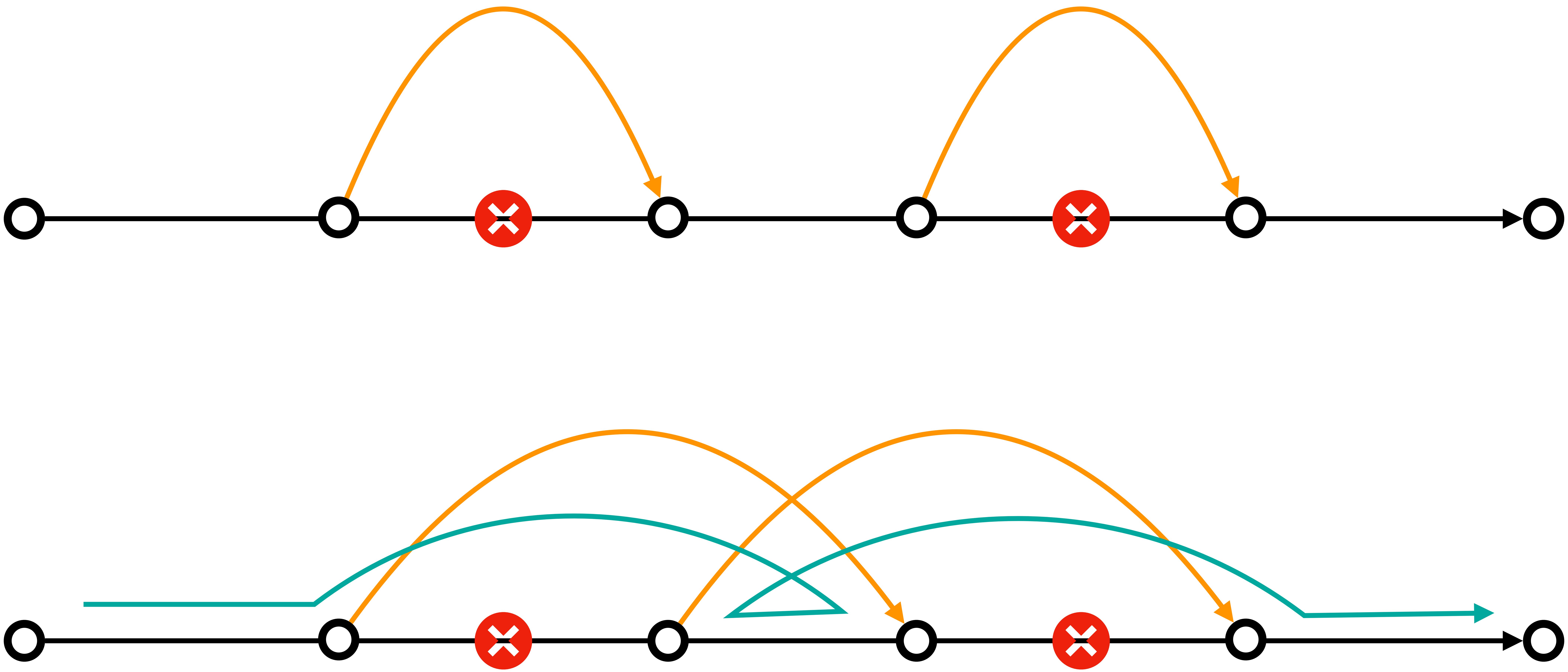


<u>Easy case:</u>
Using both detours is a good dual-failure replacement paths

Easy & Hard Cases



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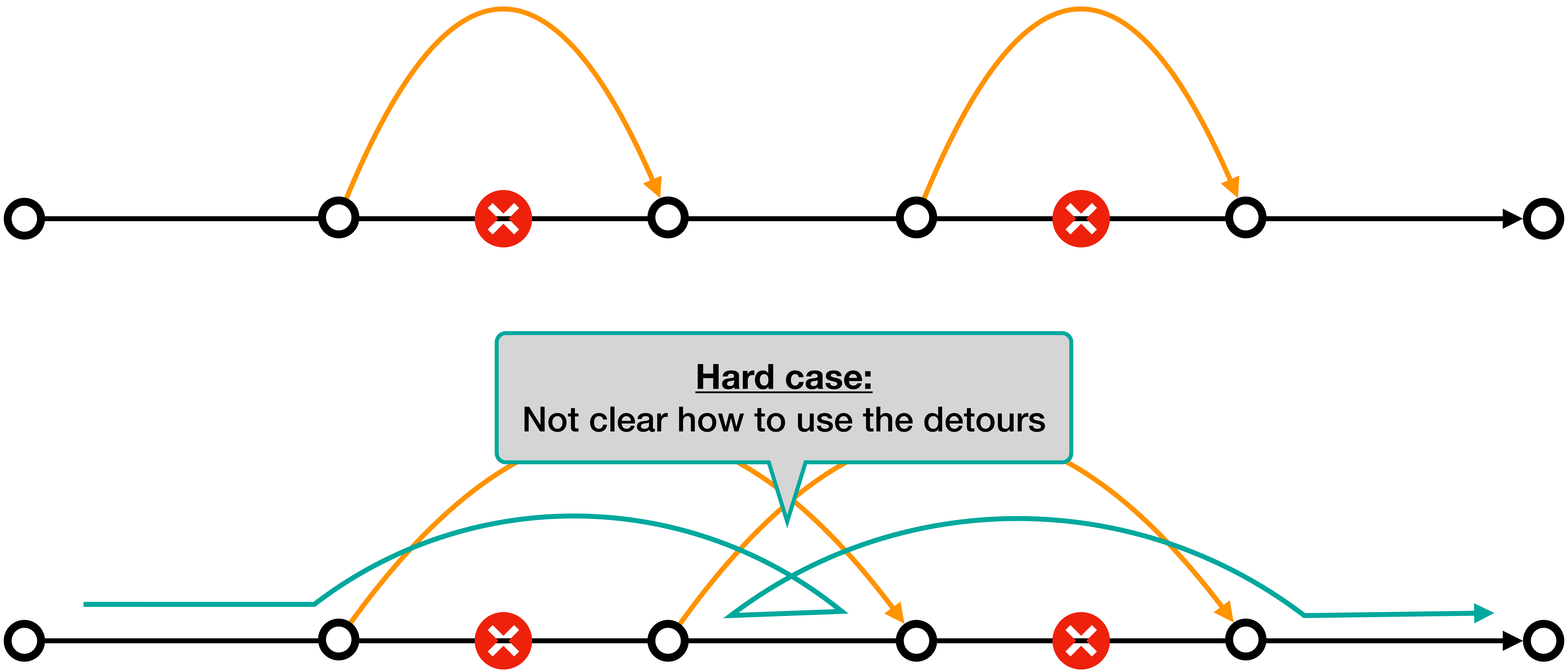
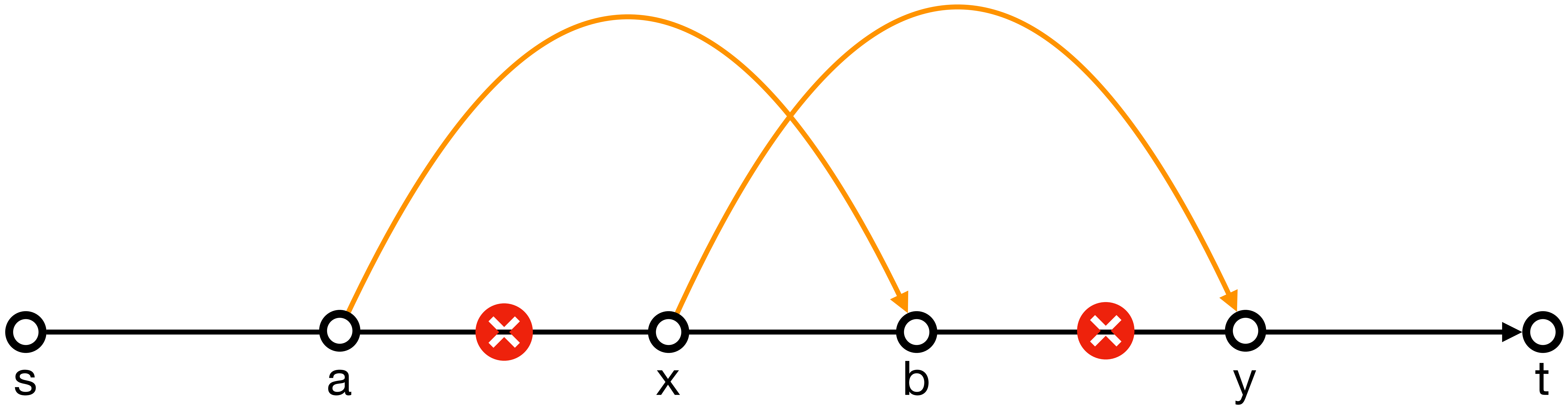
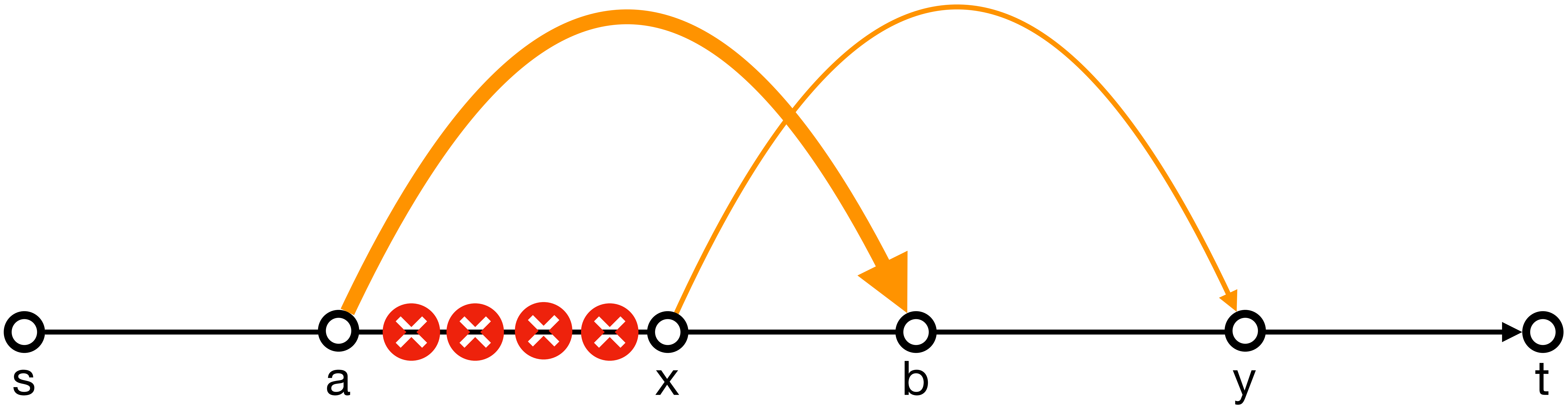


A Simplified Setting



Two overlapping detours, two failures on two intervals

A Simplified Setting



Simplifying assumption:

All failures in the interval share the same single-failure replacement path

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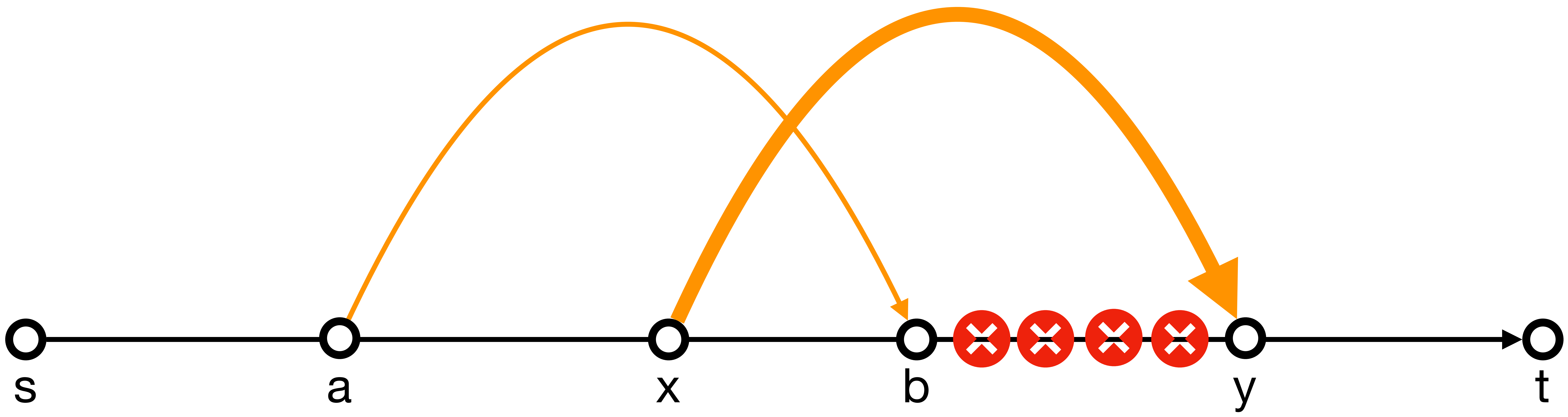

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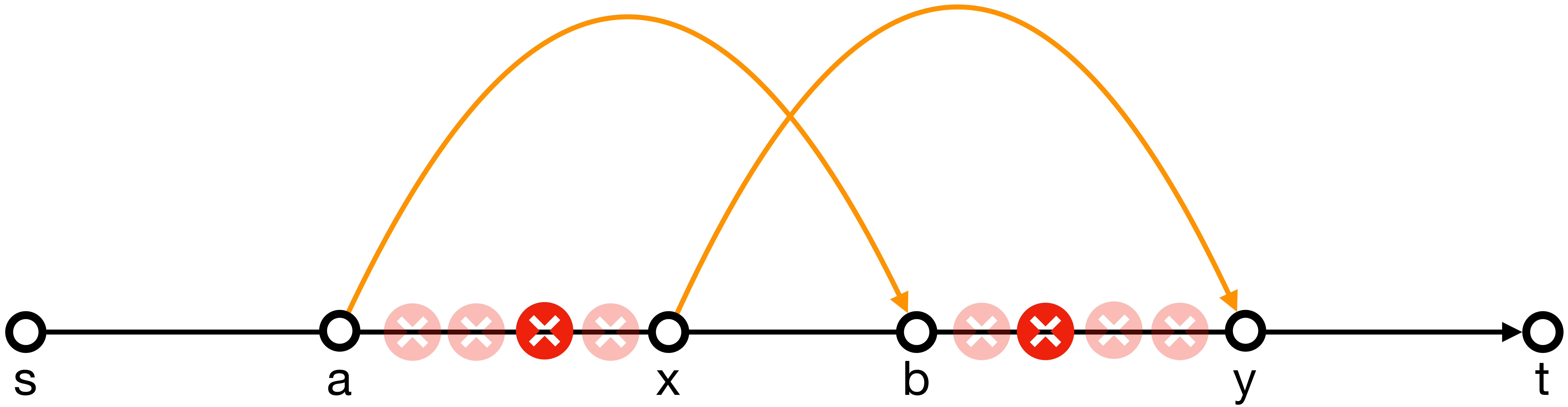


<u>Simplified goal:</u>

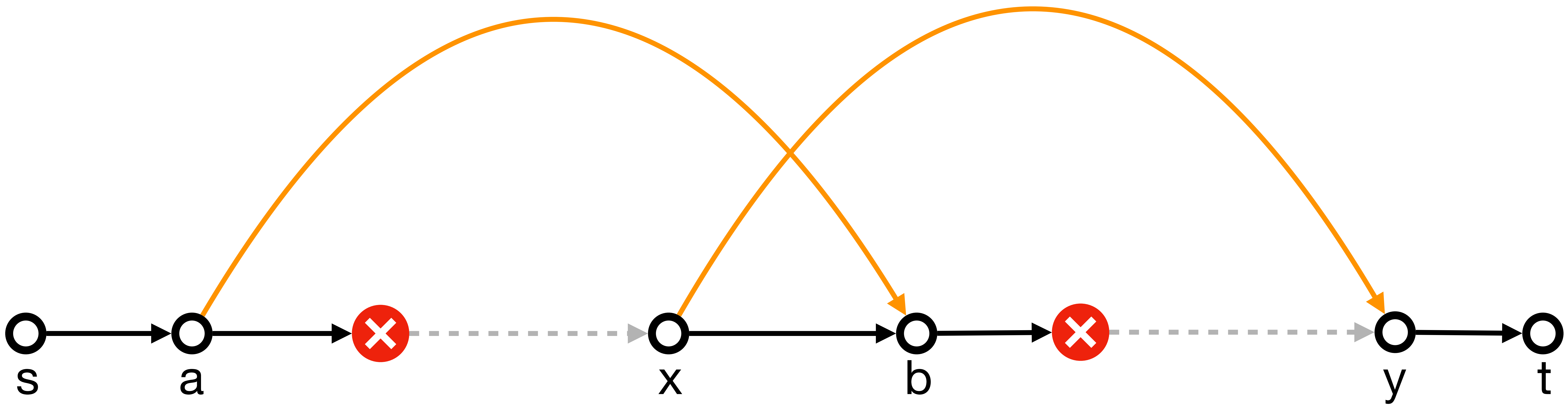
Enumerate **pairs** of failures in both intervals, and compute dual-failure **RP**

Technical difficulties

Main issue with progressive Dijkstra:

Impossible to **order all the pairs** so that the graph is **monotonically growing**

For example, alphabetic order does not work



Using an alphabetic order of the failure pairs, the graph is **not monotone**

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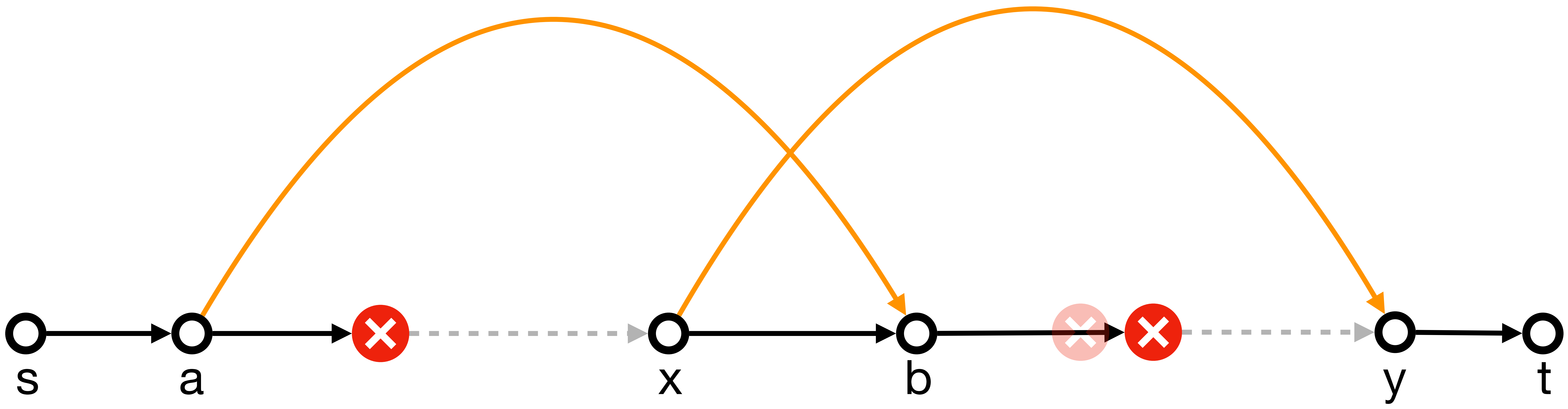

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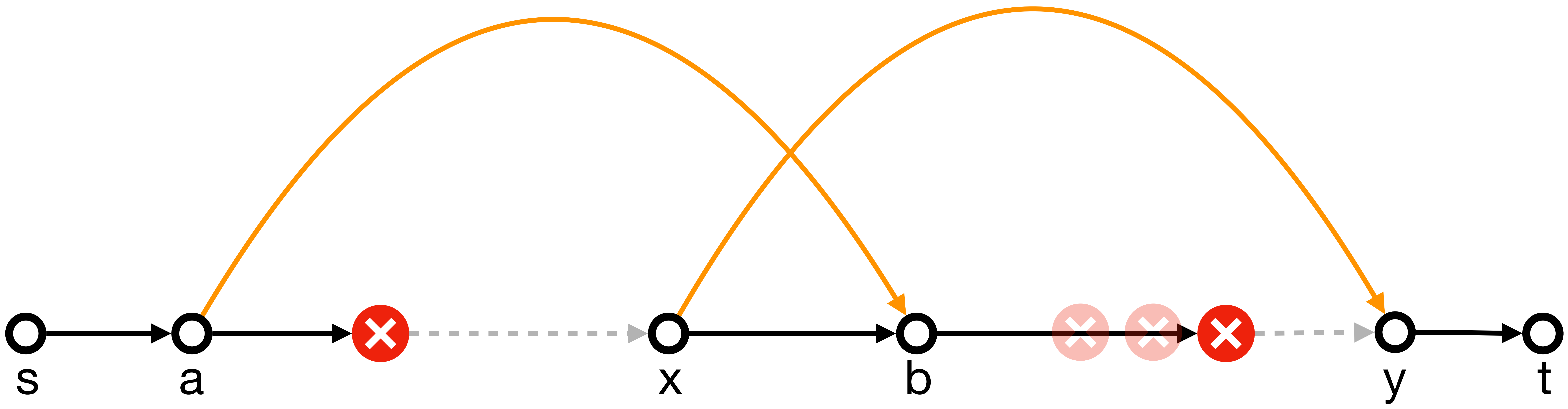

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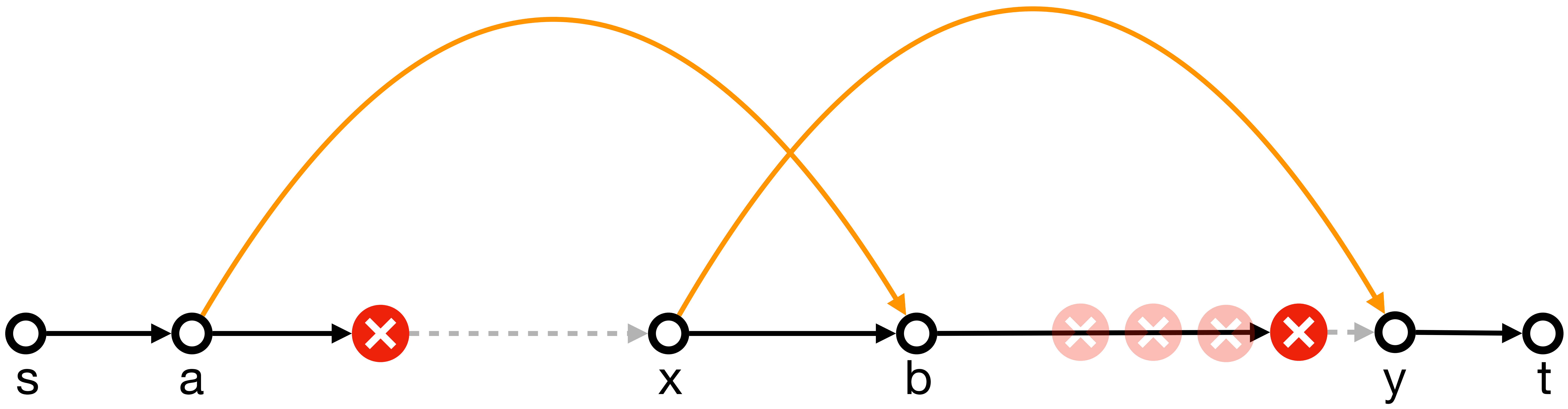

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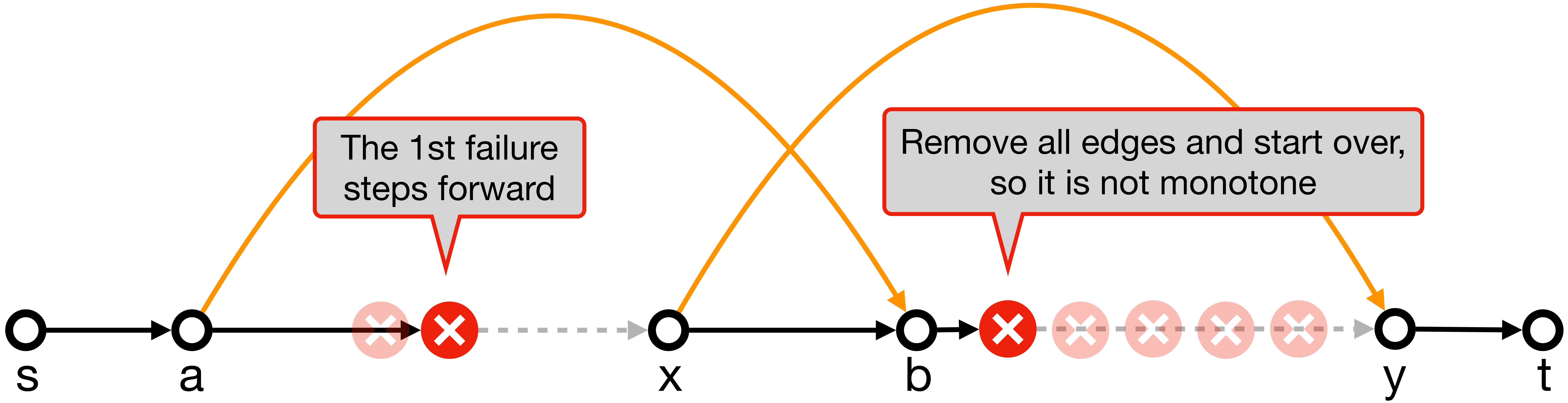

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Decoupling dual-failures

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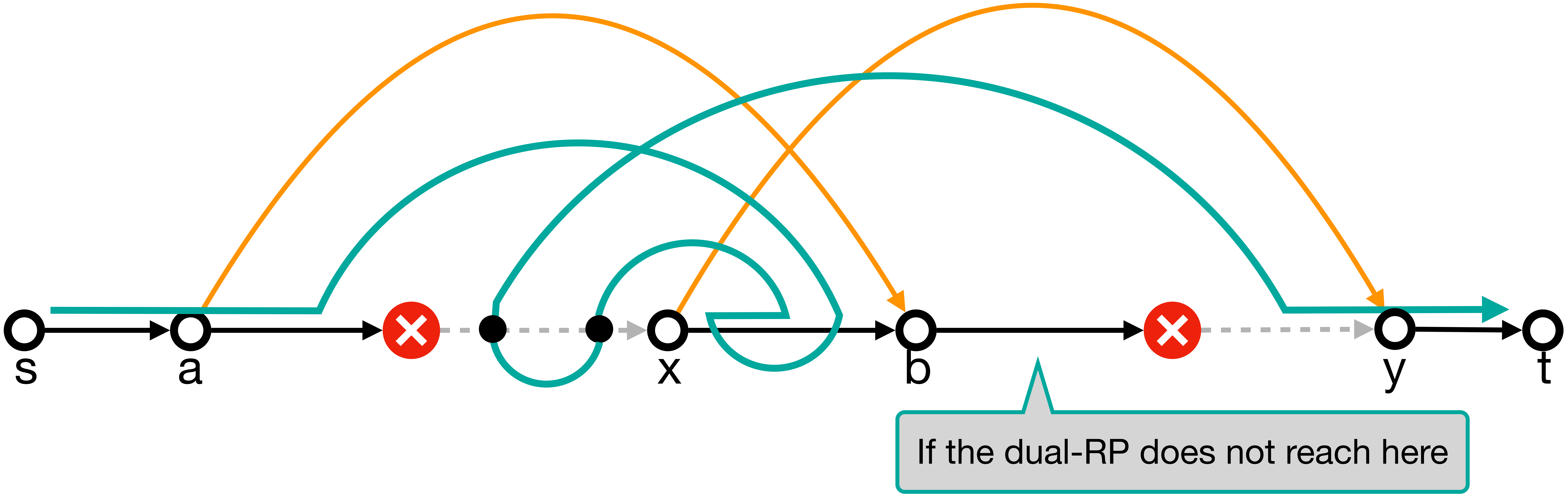
Solution: **Decouple** the two failures and use progressive Dijkstra separately



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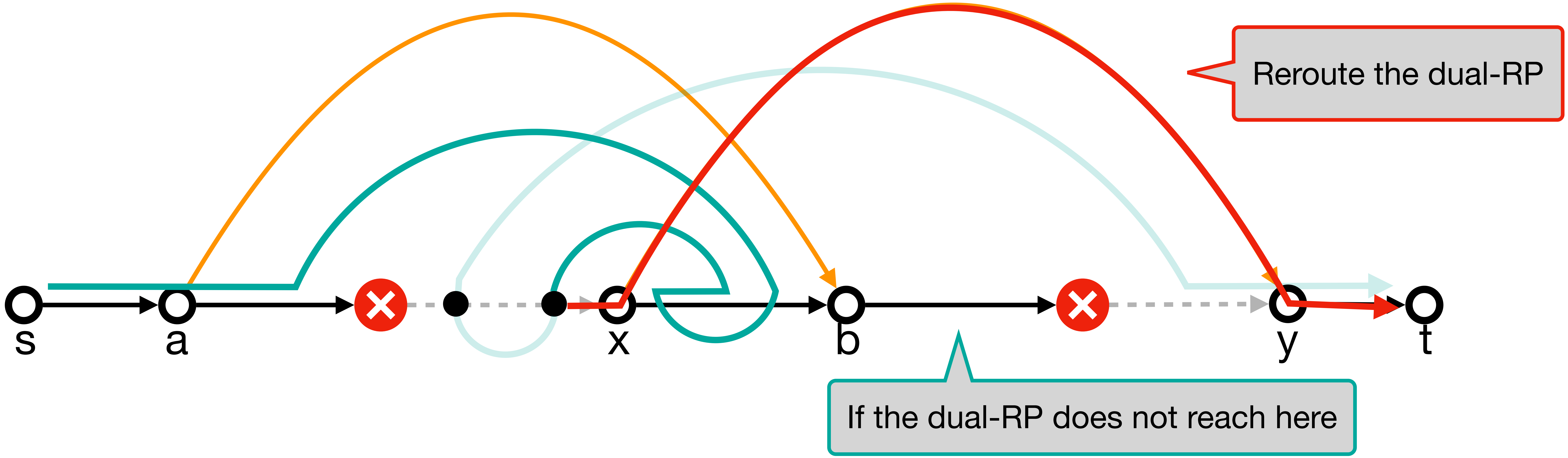

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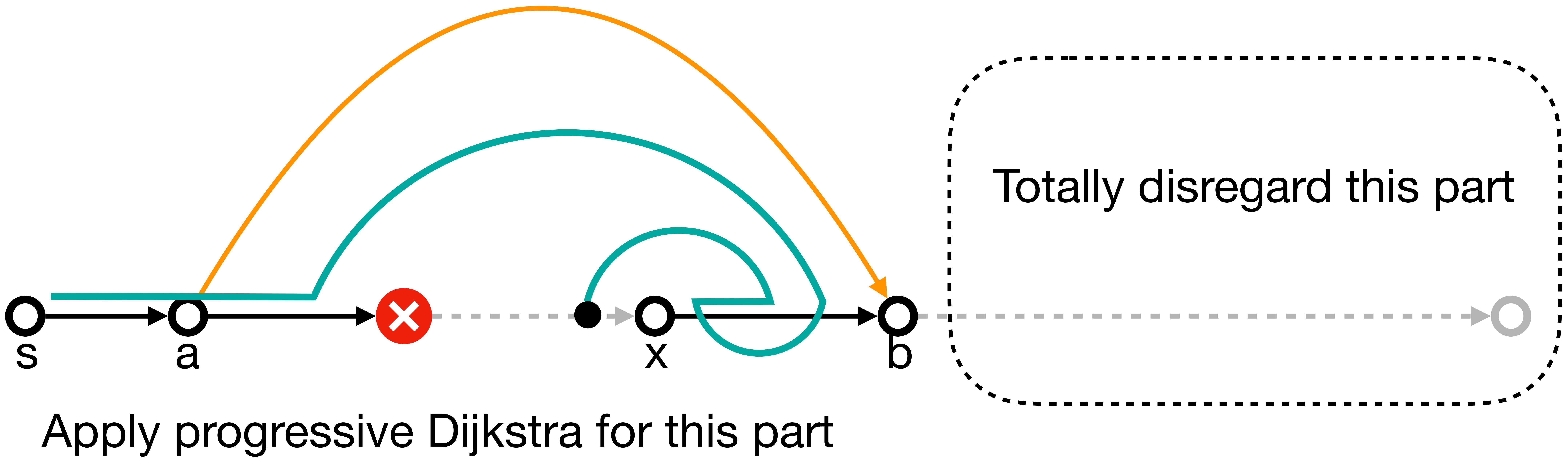

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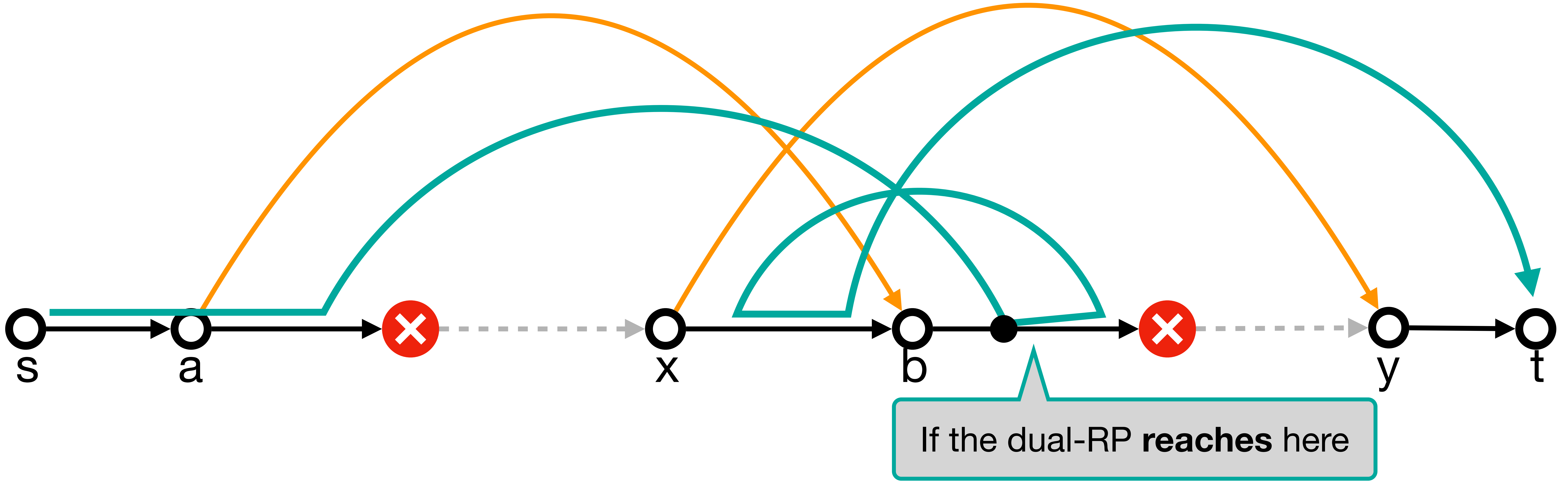

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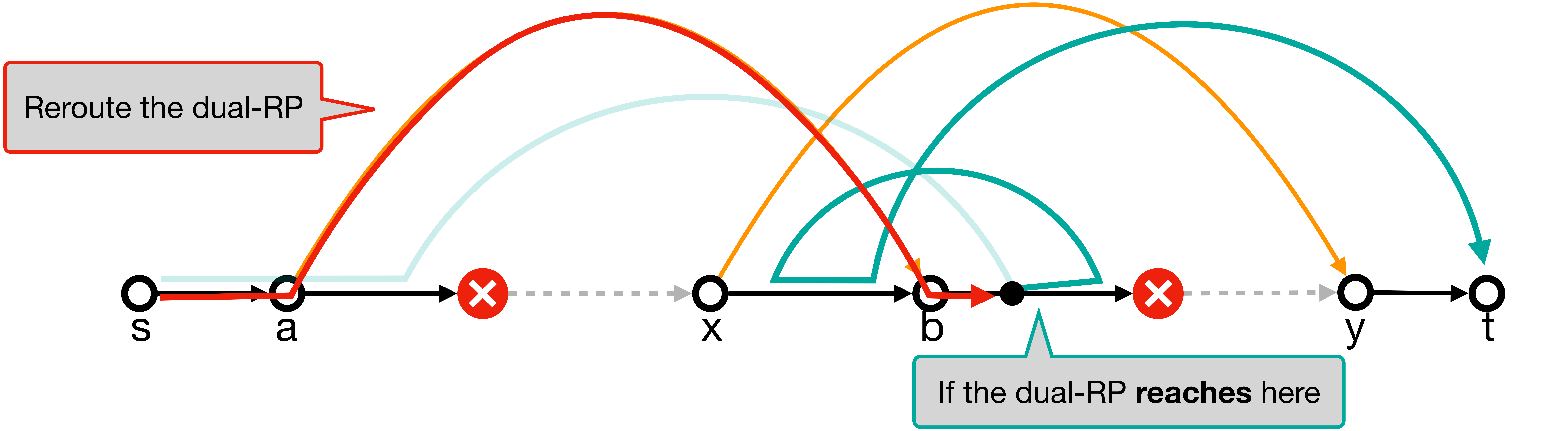

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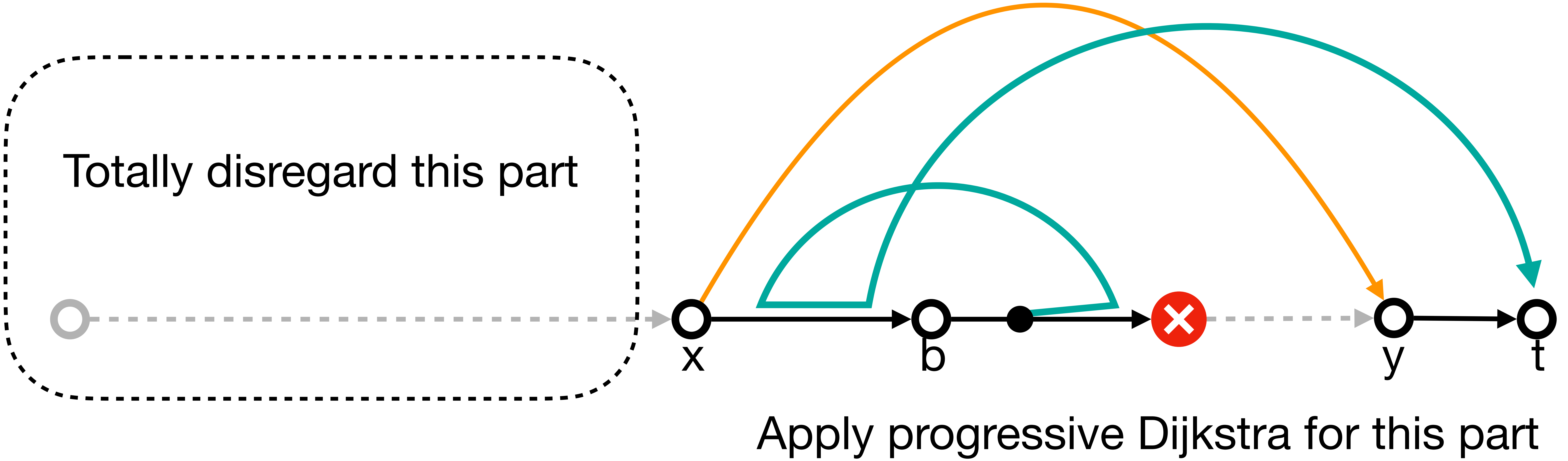

Conclusion

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- Quadratic time for approximate dual-failure st-shortest paths
- How about approximate **single-source** RP?
- Approximate single-failure single-source RP in linear time?