Vizing's Theorem in Near-Linear Time

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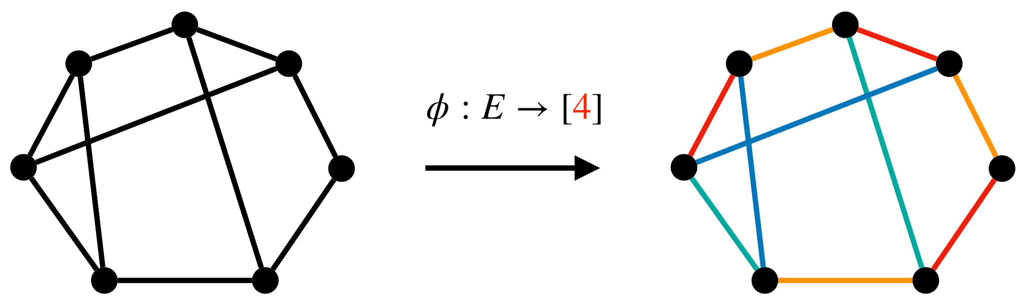
1. Background & Results

In simple graph $G = (V, E)$, compute a mapping (coloring) $\phi : E \to [k]$, such that adjacent edges have different values (colors).



Edge Chromatic Number $\chi_1(G)$ = min #colors

Trivial bound: $\chi_1(G) \geq \Delta$ $\quad \Delta = \max \deg_G(v)$

Vizing's theorem (1964): $\chi_1(G) \leq \Delta + 1$

NP-hard $\chi_1(G) \in \{\Delta, \Delta + 1\}$? [Holyer, 1981]

History of Fast $(\Delta + 1)$-Edge Coloring

m = #edges
n = #vertices

- $O(mn)$ [Vizing, 1964]
- $\tilde{O}(mn^{1/2})$, $\tilde{O}(m\Delta)$ [Arjomandi, 1982] [Gabow et al, 1985]
- bipartite Δ-edge color $O(m \log \Delta)$ [Cole, Ost, Schirra, 2001]
- $\tilde{O}(mn^{1/4})$ [Bhattacharya et al, 2024 & 2025]
- $\tilde{O}(n^2)$ [Assadi, 2025]

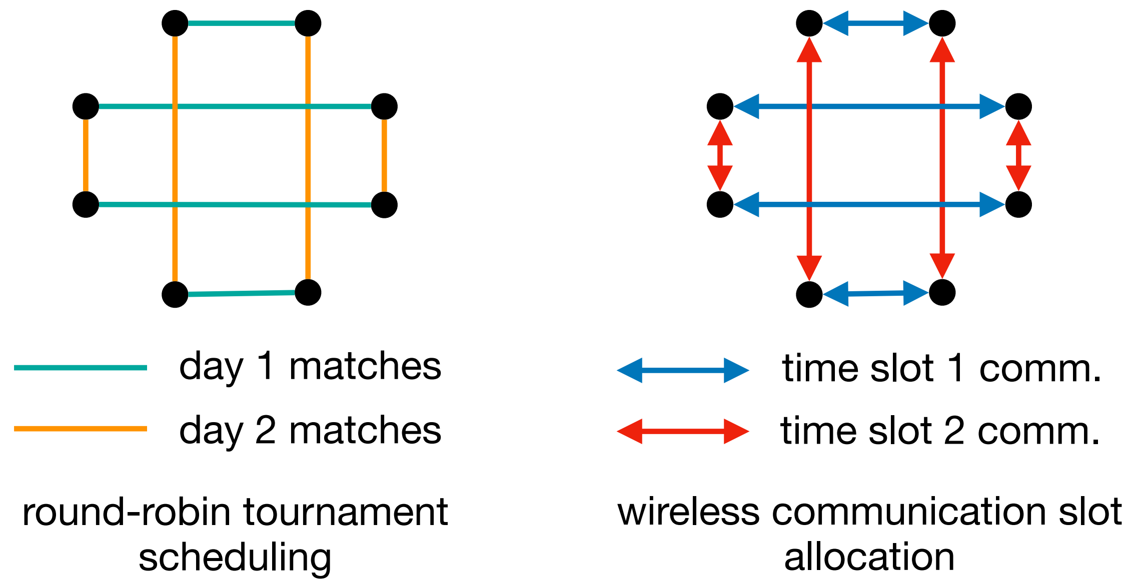
Main Result
$(\Delta + 1)$-edge coloring in $O(m \log \Delta)$ time whp

Other Result

$\lfloor 3\Delta/2 \rfloor$-edge coloring in $O(m \log \Delta)$ time whp in multi-graphs (Shannon's theorem)

Previous runtime of $\lfloor 3\Delta/2 \rfloor$-edge coloring:
$O(mn + m\Delta)$ [Shannon, 1949]
$O(n \cdot \mathrm{poly}(\Delta))$ [Dhawan, 2024]

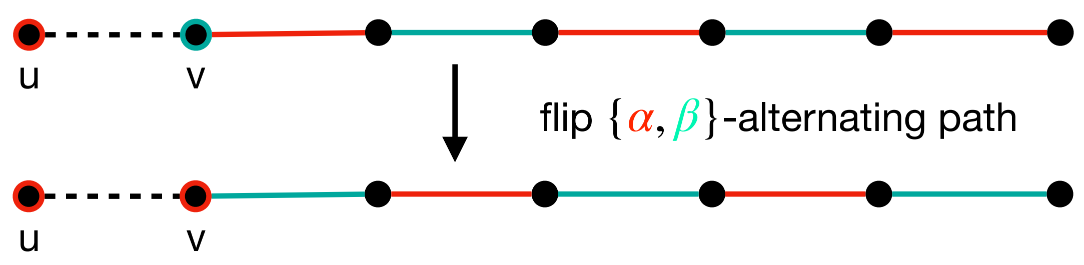
Practical Applications according to wikipedia



2. Color Extension in Bipartite Graphs

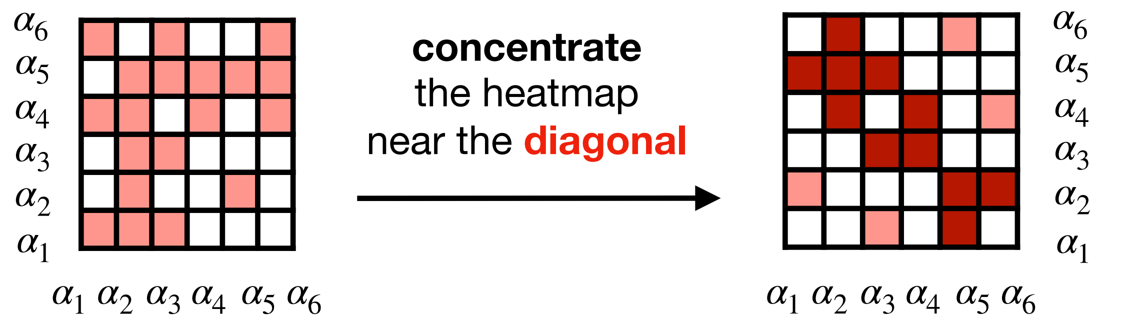
The Color Extension Problem
Partial coloring $\phi : E \to [\Delta + 1] \cup \{\perp\}$ with at most $O(m/\Delta)$ uncolored edges, how to extend ϕ to all edges in the graph?
Remark: Solving color extension in near-linear time is sufficient for the original problem

Alternating Paths in Bipartite Graphs
Uncolored edge (u, v) with colors α, β available around u, v, flip the $\{\alpha, \beta\}$-alternating path at v and then assign α to edge (u, v)

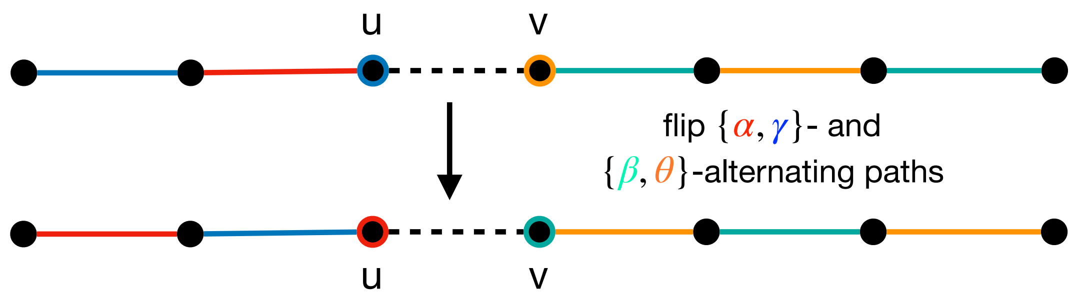


Alternating Paths in Batches
Fix a color type $\{\alpha, \beta\}$, we can flip all the $\{\alpha, \beta\}$-alternating paths in graph in time $O(n)$
—> Total runtime = $O(m\Delta)$ for Δ^2 color types

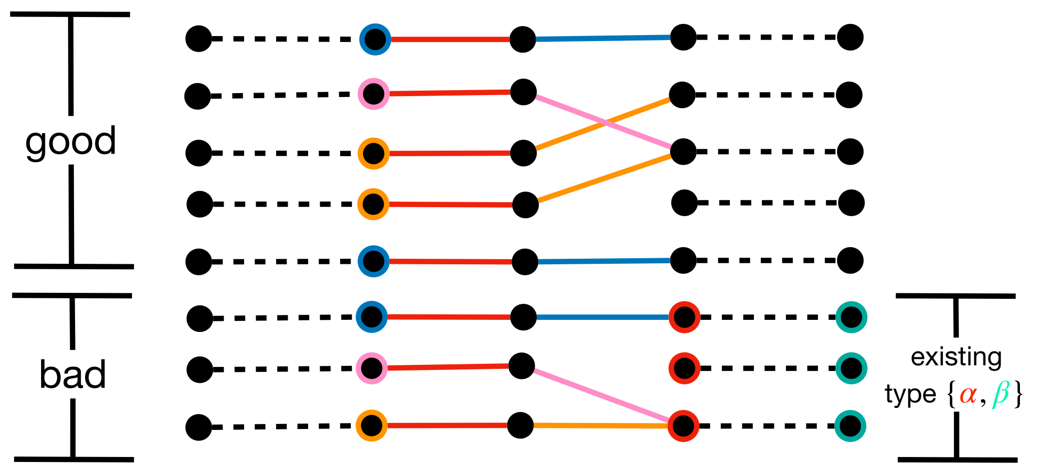
Key Idea: Color Type Concentration
Pre-flip to **concentrate** most uncolored edges among $O(\Delta)$ types —> From Δ^2 to Δ



Path Flipping for Type Concentration
Uncolored edge (u, v) with type $\{\gamma, \theta\}$, flip the $\{\alpha, \gamma\}$- and $\{\beta, \theta\}$-alternating paths at u, v, so that edge (u, v) now has type $\{\alpha, \beta\}$.

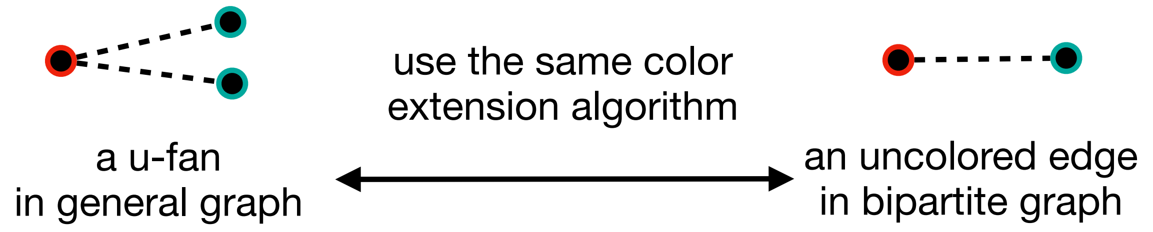


Main Technical Challenge
Path flips might destroy existing types $\{\alpha, \beta\}$
Key argument: Most path flips are good



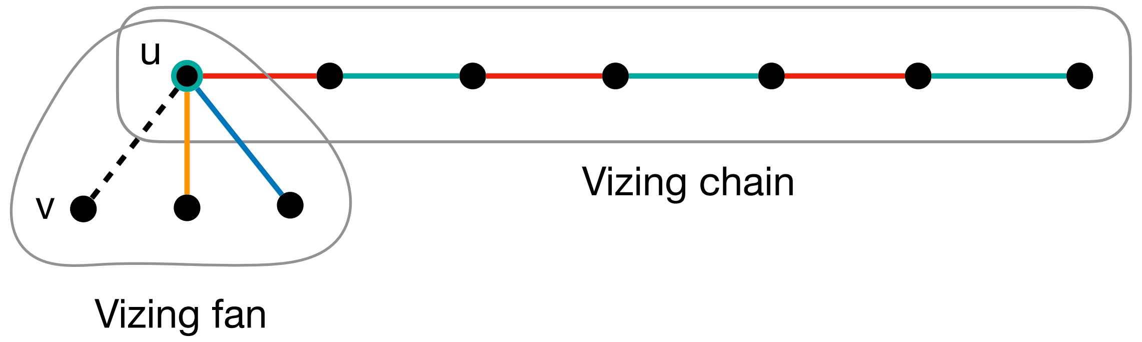
3. Color Extension in General Graphs

U-Fans [Gabow et al, 1985]
A pair of uncolored edges $\{(u, v), (u, w)\}$ is called a **u-fan** if v, w share an available color
Observation: Type conc. generalizes to u-fans.



Vizing fans and chains [Vizing, 1964]
Given an uncolored edge (u, v), can extend ϕ to (u, v) by modifying two structures

1. **Vizing fan:** some edges around u
2. **Vizing chain:** an alternating path from u



A Win-Win Argument
Either Vizing chains are short and vertex-disjoint, or pair uncolored edges to create u-fans.